On the homotopy type of p-subgroup posets

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Simple groups, representations and applications
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We study their topological properties via their order-complexes.

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Theorem (various authors). If G is p-solvable and $O_p(G) = 1$, then G satisfies $(\mathcal{QD})_p$. Thus (H-QC) holds for p-solvable groups.

On Quillen's conjecture: the Aschbacher-Smith result

Aschbacher-Smith's Theorem. (H-QC) holds for G if $p > 5$ and:

(H2U). If $L \cong \mathrm{PSU}_n(q)$, $p \mid q+1$, q odd, is a component of G, then p-extensions of $\mathrm{PSU}_m(q^e)$ satisfy $(\mathcal{QD})_p$, $\forall m \leq n$ and $e \in \mathbb{Z}$.

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- A p-extension of L is a split extension of L by some $B \in \mathcal{A}_p(\mathrm{Out}(L)) \cup \{1\}$.

$$1 \to L \to LB \to B \to 1.$$

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5. Every remaining component has a 2-elementary Robinson subgroup, and thus G satisfies (H-QC). Works for $p > 3$.

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Methodology: replace strongly CFSG-dependent steps by more combinatorial arguments.

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Remark. The CFSG is only invoked to guarantee that the list in (4) is complete.

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Proposition.

$$X_G(HK) \simeq W_G(H, K) := (\mathcal{A}_p(H) \underline{*} \mathcal{A}_p(K)) \cup \mathcal{F}_G(HK).$$

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Variant! $\mathcal{A}_p(G) \simeq W_G^{\mathcal{B}}(H, K) = (\mathcal{B}_p(H) \underline{*} \mathcal{A}_p(K)) \cup \mathcal{F}_G(HK)$.

Ideas of the proofs

Under certain reductions, we take $H = LB$, where

- L is a (simple) component of G, and
- LB is a p-extension of L in G.

We take then $K = C_G(LB)$ and:

1. Induction on $\mathcal{A}_p(K)$ to get non-zero cycles, (B is chosen s.t. $O_p(K) = 1$).
2. Look for more specific cycles in $\mathcal{A}_p(LB)$ (e.g. $(\mathcal{QD})_p$, cycles in top dimension).
3. Take advantage of $W_G(H, K)$ to show that $i : \mathcal{A}_p(H) \underline{*} \mathcal{A}_p(K) \hookrightarrow W_G(H, K)$ is non-zero in homology.

Variant! $\mathcal{A}_p(G) \ \simeq \ W_G^{\mathcal{B}}(H, K) = (\mathcal{B}_p(H) \underline{*} \mathcal{A}_p(K)) \cup \mathcal{F}_G(HK)$.

- $H = L$ simple of Lie type in characteristic p (homology in top dimension in $\mathcal{B}_p(L)$).

Thank you very much!