# Advances on Quillen's conjecture

Kevin I. Piterman
(joint w. Stephen D. Smith)

Philipps-Universität Marburg

Groups 2023 - Northern Group Theory conference in honour of Bernd Fischer

September 29, 2023

# Setting

# Setting

Let $G$ be a finite group and $p$ a prime number.

# Setting

Let $G$ be a finite group and $p$ a prime number.

The Quillen poset is:

$$\mathcal{A}_p(G) = \{A \leq G \,:\, A \text{ is a non-trivial elementary abelian } p\text{-group}\},$$

# Setting

Let $G$ be a finite group and $p$ a prime number.

The Quillen poset is:

$$\mathcal{A}_p(G) = \{A \leq G \,:\, A \text{ is a non-trivial elementary abelian } p\text{-group}\},$$

- $\mathcal{A}_p(G)$ is a finite poset with the order induced by the inclusion.

# Setting

Let $G$ be a finite group and $p$ a prime number.

The Quillen poset is:

$$\mathcal{A}_p(G) = \{A \leq G \ : \ A \text{ is a non-trivial elementary abelian } p\text{-group}\},$$

- $\mathcal{A}_p(G)$ is a finite poset with the order induced by the inclusion.
- Regard $\mathcal{A}_p(G)$ as a topological space via its order-complex.

# Setting

Let $G$ be a finite group and $p$ a prime number.

The Quillen poset is:

$$\mathcal{A}_p(G) = \{A \leq G \,:\, A \text{ is a non-trivial elementary abelian } p\text{-group}\},$$

- $\mathcal{A}_p(G)$ is a finite poset with the order induced by the inclusion.
- Regard $\mathcal{A}_p(G)$ as a topological space via its order-complex.
- $G$ acts on $\mathcal{A}_p(G)$ by conjugation.

Let $G$ be a finite group and $p$ a prime number.

The Quillen poset is:

$$\mathcal{A}_p(G) = \{A \leq G \ : \ A \text{ is a non-trivial elementary abelian } p\text{-group}\},$$

- $\mathcal{A}_p(G)$ is a finite poset with the order induced by the inclusion.
- Regard $\mathcal{A}_p(G)$ as a topological space via its order-complex.
- $G$ acts on $\mathcal{A}_p(G)$ by conjugation.

**General goal.** Establish connections between properties of $G$ and combinatorial/topological properties of $\mathcal{A}_p(G)$.

# Why do we study $p$-group complexes?

# Why do we study $p$-group complexes?

1. (D. Quillen, '71) The Atiyah-Swan conjecture holds. Namely,

   Krull dimension of $H^*(G, k) = p$-rank of $G = 1 + \dim \mathcal{A}_p(G)$.

## Why do we study $p$-group complexes?

1. (D. Quillen, '71) The Atiyah-Swan conjecture holds. Namely,

   Krull dimension of $H^*(G, k) = p$-rank of $G = 1 + \dim \mathcal{A}_p(G)$.

2. (K. Brown, '94) $H^*_G(\mathcal{A}_p(G), p) \cong H^*(G, p)$.

# Why do we study $p$-group complexes?

1. (D. Quillen, '71) The Atiyah-Swan conjecture holds. Namely,

   Krull dimension of $H^*(G, k) = p$-rank of $G = 1 + \dim \mathcal{A}_p(G)$.

2. (K. Brown, '94) $H^*_G(\mathcal{A}_p(G), p) \cong H^*(G, p)$.

3. (Quillen, '78) $\mathcal{A}_p(G)$ is disconnected if and only if $G$ has a strongly $p$-embedded subgroup.

## Why do we study $p$-group complexes?

1. (D. Quillen, '71) The Atiyah-Swan conjecture holds. Namely,

   Krull dimension of $H^*(G,k) = p$-rank of $G = 1 + \dim \mathcal{A}_p(G)$.

2. (K. Brown, '94) $H^*_G(\mathcal{A}_p(G), p) \cong H^*(G,p)$.

3. (Quillen, '78) $\mathcal{A}_p(G)$ is disconnected if and only if $G$ has a strongly $p$-embedded subgroup. These groups are crucial in the **CFSG**!

# Why do we study $p$-group complexes?

1. (D. Quillen, '71) The Atiyah-Swan conjecture holds. Namely,

$$\text{Krull dimension of } H^*(G,k) = p\text{-rank of } G = 1 + \dim \mathcal{A}_p(G).$$

2. (K. Brown, '94) $H^*_G(\mathcal{A}_p(G), p) \cong H^*(G, p)$.
3. (Quillen, '78) $\mathcal{A}_p(G)$ is disconnected if and only if $G$ has a strongly $p$-embedded subgroup. These groups are crucial in the **CFSG**!
4. (Quillen, '78) $O_p(G) =$ largest normal $p$-subgroup of $G$.

# Why do we study $p$-group complexes?

1. (D. Quillen, '71) The Atiyah-Swan conjecture holds. Namely,

   Krull dimension of $H^*(G, k)$ = $p$-rank of $G = 1 + \dim \mathcal{A}_p(G)$.

2. (K. Brown, '94) $H^*_G(\mathcal{A}_p(G), p) \cong H^*(G, p)$.
3. (Quillen, '78) $\mathcal{A}_p(G)$ is disconnected if and only if $G$ has a strongly $p$-embedded subgroup. These groups are crucial in the **CFSG**!
4. (Quillen, '78) $O_p(G)$ = largest normal $p$-subgroup of $G$.
   If $O_p(G) \neq 1$ then $\mathcal{A}_p(G)$ is contractible.

# Why do we study $p$-group complexes?

1. (D. Quillen, '71) The Atiyah-Swan conjecture holds. Namely,

   Krull dimension of $H^*(G,k) = p$-rank of $G = 1 + \dim \mathcal{A}_p(G)$.

2. (K. Brown, '94) $H^*_G(\mathcal{A}_p(G), p) \cong H^*(G,p)$.
3. (Quillen, '78) $\mathcal{A}_p(G)$ is disconnected if and only if $G$ has a strongly $p$-embedded subgroup. These groups are crucial in the **CFSG**!
4. (Quillen, '78) $O_p(G) =$ largest normal $p$-subgroup of $G$.
   If $O_p(G) \neq 1$ then $\mathcal{A}_p(G)$ is contractible.

**Quillen's conjecture**

If $O_p(G) = 1$ then $\mathcal{A}_p(G)$ is not contractible.

1. (D. Quillen, '71) The Atiyah-Swan conjecture holds. Namely,

$$\text{Krull dimension of } H^*(G,k) = p\text{-rank of } G = 1 + \dim \mathcal{A}_p(G).$$

2. (K. Brown, '94) $H^*_G(\mathcal{A}_p(G), p) \cong H^*(G, p)$.
3. (Quillen, '78) $\mathcal{A}_p(G)$ is disconnected if and only if $G$ has a strongly $p$-embedded subgroup. These groups are crucial in the **CFSG**!
4. (Quillen, '78) $O_p(G)$ = largest normal $p$-subgroup of $G$.
   If $O_p(G) \neq 1$ then $\mathcal{A}_p(G)$ is contractible.

**Quillen's conjecture**

If $O_p(G) = 1$ then $\mathcal{A}_p(G)$ is not contractible.

**(Strong) Quillen's conjecture**

If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.

# On Quillen's conjecture

**(H-QC)** If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.

# On Quillen's conjecture

$$\textbf{(H-QC)} \quad \text{If } O_p(G) = 1 \text{ then } \tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0.$$

Quillen proved the following cases of (H-QC):

**(H-QC)** If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.

Quillen proved the following cases of (H-QC):

1. $G$ is a group of Lie type in characteristic $p$;

$$\textbf{(H-QC)} \quad \text{If } O_p(G) = 1 \text{ then } \tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0.$$

Quillen proved the following cases of (H-QC):

1. $G$ is a group of Lie type in characteristic $p$;
2. $p$-rank of $G = m_p(G) \leq 2$;

$$\textbf{(H-QC)} \quad \text{If } O_p(G) = 1 \text{ then } \tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0.$$

Quillen proved the following cases of (H-QC):

1. $G$ is a group of Lie type in characteristic $p$;
2. $p$-rank of $G = m_p(G) \leq 2$;
3. $G$ is solvable.

$$\textbf{(H-QC)} \quad \text{If } O_p(G) = 1 \text{ then } \tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0.$$

Quillen proved the following cases of (H-QC):

1. $G$ is a group of Lie type in characteristic $p$;
2. $p$-rank of $G = m_p(G) \leq 2$;
3. $G$ is solvable. Moreover, if $O_p(G) = 1$ then $G$ satisfies $(\mathcal{QD})_p$.

# On Quillen's conjecture

$$\textbf{(H-QC)} \quad \text{If } O_p(G) = 1 \text{ then } \tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0.$$

Quillen proved the following cases of (H-QC):

1. $G$ is a group of Lie type in characteristic $p$;
2. $p$-rank of $G = m_p(G) \leq 2$;
3. $G$ is solvable. Moreover, if $O_p(G) = 1$ then $G$ satisfies $(\mathcal{QD})_p$.

- $H$ satisfies $(\mathcal{QD})_p$ if $\mathcal{A}_p(H)$ has non-zero homology in top-degree:

$$\tilde{H}_{m_p(H)-1}(\mathcal{A}_p(H), \mathbb{Q}) \neq 0.$$

$$\textbf{(H-QC)} \quad \text{If } O_p(G) = 1 \text{ then } \tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0.$$

Quillen proved the following cases of (H-QC):

1. $G$ is a group of Lie type in characteristic $p$;
2. $p$-rank of $G = m_p(G) \leq 2$;
3. $G$ is solvable. Moreover, if $O_p(G) = 1$ then $G$ satisfies $(\mathcal{QD})_p$.

- $H$ satisfies $(\mathcal{QD})_p$ if $\mathcal{A}_p(H)$ has non-zero homology in top-degree:

$$\tilde{H}_{m_p(H)-1}(\mathcal{A}_p(H), \mathbb{Q}) \neq 0.$$

**Theorem.** If $G$ is $p$-solvable and $O_p(G) = 1$ then $G$ satisfies $(\mathcal{QD})_p$,

$$\textbf{(H-QC)} \quad \text{If } O_p(G) = 1 \text{ then } \tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0.$$

Quillen proved the following cases of (H-QC):

1. $G$ is a group of Lie type in characteristic $p$;
2. $p$-rank of $G = m_p(G) \leq 2$;
3. $G$ is solvable. Moreover, if $O_p(G) = 1$ then $G$ satisfies $(\mathcal{QD})_p$.

- $H$ satisfies $(\mathcal{QD})_p$ if $\mathcal{A}_p(H)$ has non-zero homology in top-degree:

$$\tilde{H}_{m_p(H)-1}(\mathcal{A}_p(H), \mathbb{Q}) \neq 0.$$

**Theorem.** If $G$ is $p$-solvable and $O_p(G) = 1$ then $G$ satisfies $(\mathcal{QD})_p$, and hence (H-QC).

$$\textbf{(H-QC)} \quad \text{If } O_p(G) = 1 \text{ then } \tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0.$$

Quillen proved the following cases of (H-QC):

1. $G$ is a group of Lie type in characteristic $p$;
2. $p$-rank of $G = m_p(G) \leq 2$;
3. $G$ is solvable. Moreover, if $O_p(G) = 1$ then $G$ satisfies $(\mathcal{QD})_p$.

- $H$ satisfies $(\mathcal{QD})_p$ if $\mathcal{A}_p(H)$ has non-zero homology in top-degree:

$$\tilde{H}_{m_p(H)-1}(\mathcal{A}_p(H), \mathbb{Q}) \neq 0.$$

**Theorem.** If $G$ is $p$-solvable and $O_p(G) = 1$ then $G$ satisfies $(\mathcal{QD})_p$, and hence (H-QC). (Use of **CFSG**).

## The Aschbacher-Smith result

# The Aschbacher-Smith result

## Aschbacher-Smith Theorem

(H-QC) holds for $G$ if $p > 5$ and:

## Aschbacher-Smith Theorem

(H-QC) holds for $G$ if $p > 5$ and:

**(HU)** If $L \cong \mathrm{PSU}_n(q)$, $p \mid q+1$, $q$ odd, is a component of $G$, then $p$-extensions of $\mathrm{PSU}_m(q^e)$ satisfy $(\mathcal{QD})_p$ $\forall m \leq n, e \in \mathbb{Z}$.

## Aschbacher-Smith Theorem

(H-QC) holds for $G$ if $p > 5$ and:

**(HU)** If $L \cong \mathrm{PSU}_n(q)$, $p \mid q+1, q$ odd, is a component of $G$, then $p$-extensions of $\mathrm{PSU}_m(q^e)$ satisfy $(\mathcal{QD})_p$ $\forall m \leq n, e \in \mathbb{Z}$.

A $p$-extension of $L$ is a split-extension of $L$ by some $B \in \mathcal{A}_p(\mathrm{Out}(L)) \cup \{1\}$.

**Aschbacher-Smith Theorem**

(H-QC) holds for $G$ if $p > 5$ and:

**(HU)** If $L \cong \mathrm{PSU}_n(q)$, $p \mid q+1, q$ odd, is a component of $G$, then $p$-extensions of $\mathrm{PSU}_m(q^e)$ satisfy $(\mathcal{QD})_p$ $\forall m \leq n, e \in \mathbb{Z}$.

A $p$-extension of $L$ is a split-extension of $L$ by some $B \in \mathcal{A}_p(\mathrm{Out}(L)) \cup \{1\}$.

**Why $(\mathcal{QD})_p$?**

## Aschbacher-Smith Theorem

(H-QC) holds for $G$ if $p > 5$ and:

**(HU)** If $L \cong \mathrm{PSU}_n(q)$, $p \mid q+1$, $q$ odd, is a component of $G$, then $p$-extensions of $\mathrm{PSU}_m(q^e)$ satisfy $(\mathcal{QD})_p$ $\forall m \leq n, e \in \mathbb{Z}$.

A $p$-extension of $L$ is a split-extension of $L$ by some $B \in \mathcal{A}_p(\mathrm{Out}(L)) \cup \{1\}$.

**Why $(\mathcal{QD})_p$?**

- Under a minimal counterexample to (H-QC) for $p > 5$, Aschbacher-Smith prove that *every* component has a $p$-extension that fails $(\mathcal{QD})_p$.

## Aschbacher-Smith Theorem

(H-QC) holds for $G$ if $p > 5$ and:

**(HU)** If $L \cong \mathrm{PSU}_n(q)$, $p \mid q+1$, $q$ odd, is a component of $G$, then $p$-extensions of $\mathrm{PSU}_m(q^e)$ satisfy $(\mathcal{QD})_p$ $\forall m \leq n, e \in \mathbb{Z}$.

A $p$-extension of $L$ is a split-extension of $L$ by some $B \in \mathcal{A}_p(\mathrm{Out}(L)) \cup \{1\}$.

**Why $(\mathcal{QD})_p$?**

- Under a minimal counterexample to (H-QC) for $p > 5$, Aschbacher-Smith prove that *every* component has a $p$-extension that fails $(\mathcal{QD})_p$.
- Then they list all simple groups with a potential $p$-extension failing $(\mathcal{QD})_p$ for $p$ odd.

# On Quillen's conjecture: new results

More general combinatorial arguments that do not depend on the prime $p$ and the **CFSG**, allow us to extend *some* reductions on a minimal counterexample to (H-QC) to **every prime** $p$.

More general combinatorial arguments that do not depend on the prime $p$ and the **CFSG**, allow us to extend *some* reductions on a minimal counterexample to (H-QC) to **every prime** $p$.

**Theorem (P.-Smith, '21)**

Aschbacher-Smith Theorem extends to $p = 3, 5$.

# On Quillen's conjecture: new results

More general combinatorial arguments that do not depend on the prime $p$ and the **CFSG**, allow us to extend *some* reductions on a minimal counterexample to (H-QC) to **every prime** $p$.

**Theorem (P.-Smith, '21)**

Aschbacher-Smith Theorem extends to $p = 3, 5$.

**Theorem (Antonio Díaz Ramos, '23)**

For $p$ odd, $p \mid q+1$, $q \neq 2$, $\mathrm{PSU}_n(q)$ and $\mathrm{PGU}_n(q)$ satisfy $(\mathcal{QD})_p$.

# On Quillen's conjecture: new results

More general combinatorial arguments that do not depend on the prime $p$ and the **CFSG**, allow us to extend *some* reductions on a minimal counterexample to (H-QC) to **every prime** $p$.

**Theorem (P.-Smith, '21)**

Aschbacher-Smith Theorem extends to $p = 3, 5$.

**Theorem (Antonio Díaz Ramos, '23)**

For $p$ odd, $p \mid q + 1$, $q \neq 2$, $\mathrm{PSU}_n(q)$ and $\mathrm{PGU}_n(q)$ satisfy $(\mathcal{QD})_p$.

**Corollary**

If $p$ is odd and for all $q \neq 2$ with $p \mid q + 1$, $\mathrm{PGU}_n(q)$ extended by a field automorphism of order $p$ satisfies $(\mathcal{QD})_p$, then (H-QC) holds for $p$.

# Results for $p = 2$

# Results for $p = 2$

**Theorem (P.-Smith, '21)**

If $p = 2$ and $G$ is a minimal counterexample to (H-QC), then:

**Theorem (P.-Smith, '21)**

If $p = 2$ and $G$ is a minimal counterexample to (H-QC), then:

1. $O_{2'}(G) = 1$,

**Theorem (P.-Smith, '21)**

If $p = 2$ and $G$ is a minimal counterexample to (H-QC), then:

1. $O_{2'}(G) = 1$,
2. every component $L$ of $G$ has non-trivial 2-extension $LB \leq G$,

**Theorem (P.-Smith, '21)**

If $p = 2$ and $G$ is a minimal counterexample to (H-QC), then:

1. $O_{2'}(G) = 1$,
2. every component $L$ of $G$ has non-trivial 2-extension $LB \leq G$,
3. every component $L$ of $G$ has a 2-extension in $G$ failing $(\mathcal{QD})_2$,

**Theorem (P.-Smith, '21)**

If $p = 2$ and $G$ is a minimal counterexample to (H-QC), then:

1. $O_{2'}(G) = 1$,
2. every component $L$ of $G$ has non-trivial 2-extension $LB \leq G$,
3. every component $L$ of $G$ has a 2-extension in $G$ failing $(\mathcal{QD})_2$,
4. $G$ has a component $L$ of Lie type such that $\mathrm{char}(L) \neq 2, 3$ or

$$L \cong \mathrm{PSL}_n(2^a)(n \geq 3), D_n(2^a)(n \geq 4), \text{ or } E_6(2^a).$$

# Results for $p = 2$

**Theorem (P.-Smith, '21)**

If $p = 2$ and $G$ is a minimal counterexample to (H-QC), then:

1. $O_{2'}(G) = 1$,
2. every component $L$ of $G$ has non-trivial 2-extension $LB \leq G$,
3. every component $L$ of $G$ has a 2-extension in $G$ failing $(\mathcal{QD})_2$,
4. $G$ has a component $L$ of Lie type such that $\mathrm{char}(L) \neq 2, 3$ or

$$L \cong \mathrm{PSL}_n(2^a)(n \geq 3), D_n(2^a)(n \geq 4), \text{ or } E_6(2^a).$$

Only reduction 4 depends on the CFSG to exclude the non-Lie type components.

# On Quillen's conjecture: more recent results for $p = 2$

# On Quillen's conjecture: more recent results for $p = 2$

By the previous theorem, if $G$ is a minimal counterexample for (H-QC) and $p = 2$, then every simple component of $G$ has a 2-extension failing $(\mathcal{QD})_2$:

By the previous theorem, if $G$ is a minimal counterexample for (H-QC) and $p = 2$, then every simple component of $G$ has a 2-extension failing $(\mathcal{QD})_2$:

$LB \leq G$, $L$ a simple component and $LB$ a 2-extension such that

$$\text{not-}(\mathcal{QD})_2 \quad H_{m_2(LB)-1}(\mathcal{A}_2(LB), \mathbb{Q}) = 0.$$

By the previous theorem, if $G$ is a minimal counterexample for (H-QC) and $p = 2$, then every simple component of $G$ has a 2-extension failing $(\mathcal{QD})_2$:

$LB \leq G$, $L$ a simple component and $LB$ a 2-extension such that

$$\text{not-}(\mathcal{QD})_2 \quad H_{m_2(LB)-1}(\mathcal{A}_2(LB), \mathbb{Q}) = 0.$$

**Problem.** Classify simple groups $L$ satisfying the following condition:

**(E-$(\mathcal{QD})$)** Every 2-extension of $L$ satisfies $(\mathcal{QD})_2$.

By the previous theorem, if $G$ is a minimal counterexample for (H-QC) and $p = 2$, then every simple component of $G$ has a 2-extension failing $(\mathcal{QD})_2$:

$LB \leq G$, $L$ a simple component and $LB$ a 2-extension such that

$$\text{not-}(\mathcal{QD})_2 \qquad H_{m_2(LB)-1}(\mathcal{A}_2(LB), \mathbb{Q}) = 0.$$

**Problem.** Classify simple groups $L$ satisfying the following condition:

**(E-$(\mathcal{QD})$)** Every 2-extension of $L$ satisfies $(\mathcal{QD})_2$.

**Theorem (P., '22)**

Let $L$ be a simple group of exceptional Lie type in odd characteristic.

# On Quillen's conjecture: more recent results for $p = 2$

By the previous theorem, if $G$ is a minimal counterexample for (H-QC) and $p = 2$, then every simple component of $G$ has a 2-extension failing $(\mathcal{QD})_2$:

$LB \leq G$, $L$ a simple component and $LB$ a 2-extension such that

$$\text{not-}(\mathcal{QD})_2 \quad H_{m_2(LB)-1}(\mathcal{A}_2(LB), \mathbb{Q}) = 0.$$

**Problem.** Classify simple groups $L$ satisfying the following condition:

**(E-$(\mathcal{QD})$)** Every 2-extension of $L$ satisfies $(\mathcal{QD})_2$.

**Theorem (P., '22)**

Let $L$ be a simple group of exceptional Lie type in odd characteristic. If $L$ fails (E-$(\mathcal{QD})$), then it is one of the following groups:

$${}^3D_4(9), F_4(3), F_4(9), G_2(3), G_2(9), {}^2G_2(3)', E_8(3), E_8(9).$$

# Idea of the proof

1. Establish (E-($\mathcal{QD}$)) for the low-rank groups $\mathrm{PSL}_2, \mathrm{PSL}_3, \mathrm{PSU}_3$.

# Idea of the proof

1. Establish (E-($\mathcal{QD}$)) for the low-rank groups $\mathrm{PSL}_2, \mathrm{PSL}_3, \mathrm{PSU}_3$. Use counting arguments (conjugacy classes of 2-subgroups, involutions, field and graph automorphisms).

# Idea of the proof

1. Establish (E-($\mathcal{QD}$)) for the low-rank groups $\mathrm{PSL}_2, \mathrm{PSL}_3, \mathrm{PSU}_3$. Use counting arguments (conjugacy classes of 2-subgroups, involutions, field and graph automorphisms).
2. For $L$ an exceptional group, we look for maximal subgroups $H$ of the 2-extensions $LB$ such that

# Idea of the proof

1. Establish (E-($\mathcal{QD}$)) for the low-rank groups $\mathrm{PSL}_2, \mathrm{PSL}_3, \mathrm{PSU}_3$. Use counting arguments (conjugacy classes of 2-subgroups, involutions, field and graph automorphisms).
2. For $L$ an exceptional group, we look for maximal subgroups $H$ of the 2-extensions $LB$ such that

$$H \leq LB, \quad m_2(H) = m_2(LB), \quad \text{and}$$

$$0 \neq H_{m_2(H)-1}(\mathcal{A}_2(H), \mathbb{Q}) \subseteq H_{m_2(LB)-1}(\mathcal{A}_2(LB), \mathbb{Q}).$$

## Idea of the proof

1. Establish (E-($\mathcal{QD}$)) for the low-rank groups $\mathrm{PSL}_2, \mathrm{PSL}_3, \mathrm{PSU}_3$. Use counting arguments (conjugacy classes of 2-subgroups, involutions, field and graph automorphisms).
2. For $L$ an exceptional group, we look for maximal subgroups $H$ of the 2-extensions $LB$ such that

$$H \leq LB, \quad m_2(H) = m_2(LB), \quad \text{and}$$

$$0 \neq H_{m_2(H)-1}(\mathcal{A}_2(H), \mathbb{Q}) \subseteq H_{m_2(LB)-1}(\mathcal{A}_2(LB), \mathbb{Q}).$$

3. If $H$ is parabolic, it has a solvable subgroup $K$ with $m_2(H) = m_2(K)$ and $O_2(K) = 1$, so we are done by Quillen's result.

## Idea of the proof

1. Establish (E-($\mathcal{QD}$)) for the low-rank groups $\mathrm{PSL}_2, \mathrm{PSL}_3, \mathrm{PSU}_3$. Use counting arguments (conjugacy classes of 2-subgroups, involutions, field and graph automorphisms).
2. For $L$ an exceptional group, we look for maximal subgroups $H$ of the 2-extensions $LB$ such that

$$H \leq LB, \quad m_2(H) = m_2(LB), \quad \text{and}$$

$$0 \neq H_{m_2(H)-1}(\mathcal{A}_2(H), \mathbb{Q}) \subseteq H_{m_2(LB)-1}(\mathcal{A}_2(LB), \mathbb{Q}).$$

3. If $H$ is parabolic, it has a solvable subgroup $K$ with $m_2(H) = m_2(K)$ and $O_2(K) = 1$, so we are done by Quillen's result.
4. Otherwise, look for $H = H_1 \times H_2$, where the $H_i$ satisfy $(\mathcal{QD})_2$ and

$$H_{m_2(H)-1}(\mathcal{A}_2(H), \mathbb{Q}) = H_{m_2(H_1)-1}(\mathcal{A}_2(H_1), \mathbb{Q}) \otimes H_{m_2(H_2)-1}(\mathcal{A}_2(H_2), \mathbb{Q}).$$

# Consequence on Quillen's conjecture for $p = 2$

**Corollary**

Let $G$ be a minimal counterexample to (H-QC) for $p = 2$.

# Consequence on Quillen's conjecture for $p = 2$

**Corollary**

Let $G$ be a minimal counterexample to (H-QC) for $p = 2$. Then $G$ contains a component of Lie type in characteristic $r \neq 3$.

# Consequence on Quillen's conjecture for $p = 2$

## Corollary

Let $G$ be a minimal counterexample to (H-QC) for $p = 2$. Then $G$ contains a component of Lie type in characteristic $r \neq 3$.
Moreover, every such component fails (E-($\mathcal{QD}$)) and belongs to one of the following families:

$$\mathrm{PSL}_n(2^a)(n \geq 3), D_n(2^a)(n \geq 4), E_6(2^a),$$

$$\mathrm{PSL}_n^{\pm}(q)(n \geq 4), \Omega_{2n+1}(q)(n \geq 2), \mathrm{PSp}_{2n}(q)(n \geq 3), D_n^{\pm}(q)(n \geq 4),$$

where $q = r^b$ and $r > 3$.

# Consequence on Quillen's conjecture for $p = 2$

## Corollary

Let $G$ be a minimal counterexample to (H-QC) for $p = 2$. Then $G$ contains a component of Lie type in characteristic $r \neq 3$.
Moreover, every such component fails (E-($\mathcal{QD}$)) and belongs to one of the following families:

$$\mathrm{PSL}_n(2^a)(n \geq 3), D_n(2^a)(n \geq 4), E_6(2^a),$$

$$\mathrm{PSL}_n^{\pm}(q)(n \geq 4), \Omega_{2n+1}(q)(n \geq 2), \mathrm{PSp}_{2n}(q)(n \geq 3), D_n^{\pm}(q)(n \geq 4),$$

where $q = r^b$ and $r > 3$.

**What could we do next?** Study (E-($\mathcal{QD}$)) for the classical groups.

# Consequence on Quillen's conjecture for $p = 2$

## Corollary

Let $G$ be a minimal counterexample to (H-QC) for $p = 2$. Then $G$ contains a component of Lie type in characteristic $r \neq 3$.
Moreover, every such component fails (E-($\mathcal{QD}$)) and belongs to one of the following families:

$$\mathrm{PSL}_n(2^a)(n \geq 3), D_n(2^a)(n \geq 4), E_6(2^a),$$

$$\mathrm{PSL}_n^{\pm}(q)(n \geq 4), \Omega_{2n+1}(q)(n \geq 2), \mathrm{PSp}_{2n}(q)(n \geq 3), D_n^{\pm}(q)(n \geq 4),$$

where $q = r^b$ and $r > 3$.

**What could we do next?** Study (E-($\mathcal{QD}$)) for the classical groups.
Partial results for $\Omega_{2n+1}(q), \mathrm{PSp}_{2n}(q)$ and some of the $D_n^{\pm}(q)$.

**Corollary**

Let $G$ be a minimal counterexample to (H-QC) for $p = 2$. Then $G$ contains a component of Lie type in characteristic $r \neq 3$.
Moreover, every such component fails (E-($\mathcal{QD}$)) and belongs to one of the following families:

$$\mathrm{PSL}_n(2^a)(n \geq 3), D_n(2^a)(n \geq 4), E_6(2^a),$$

$$\mathrm{PSL}_n^{\pm}(q)(n \geq 4), \Omega_{2n+1}(q)(n \geq 2), \mathrm{PSp}_{2n}(q)(n \geq 3), D_n^{\pm}(q)(n \geq 4),$$

where $q = r^b$ and $r > 3$.

**What could we do next?** Study (E-($\mathcal{QD}$)) for the classical groups. Partial results for $\Omega_{2n+1}(q), \mathrm{PSp}_{2n}(q)$ and some of the $D_n^{\pm}(q)$. But I'd try a different argument to eliminate them...

# Where are alternating and sporadic components?

# Where are alternating and sporadic components?

**Theorem (P.-Smith, '21)**

Let $p$ be a prime and $G$ a group such that for a fixed component $L$ the following hold:

**Theorem (P.-Smith, '21)**

Let $p$ be a prime and $G$ a group such that for a fixed component $L$ the following hold:

1. $p$ divides the order of $L$,

# Where are alternating and sporadic components?

**Theorem (P.-Smith, '21)**

Let $p$ be a prime and $G$ a group such that for a fixed component $L$ the following hold:

1. $p$ divides the order of $L$,
2. $\widehat{L} =$ central product of conjugates of $L$,

# Where are alternating and sporadic components?

**Theorem (P.-Smith, '21)**

Let $p$ be a prime and $G$ a group such that for a fixed component $L$ the following hold:

1. $p$ divides the order of $L$,
2. $\widehat{L} =$ central product of conjugates of $L$,
3. $C_G(\widehat{L})$ satisfies (H-QC),

# Where are alternating and sporadic components?

**Theorem (P.-Smith, ’21)**

Let $p$ be a prime and $G$ a group such that for a fixed component $L$ the following hold:

1. $p$ divides the order of $L$,
2. $\widehat{L} =$ central product of conjugates of $L$,
3. $C_G(\widehat{L})$ satisfies (H-QC),
4. $\mathcal{A}_p(L) \to \mathcal{A}_p(\mathrm{Aut}_G(L))$ is non-zero in homology.

## Where are alternating and sporadic components?

**Theorem (P.-Smith, '21)**

Let $p$ be a prime and $G$ a group such that for a fixed component $L$ the following hold:

1. $p$ divides the order of $L$,
2. $\widehat{L} =$ central product of conjugates of $L$,
3. $C_G(\widehat{L})$ satisfies (H-QC),
4. $\mathcal{A}_p(L) \to \mathcal{A}_p(\mathrm{Aut}_G(L))$ is non-zero in homology.

Then $G$ satisfies (H-QC).

# Where are alternating and sporadic components?

**Theorem (P.-Smith, '21)**

Let $p$ be a prime and $G$ a group such that for a fixed component $L$ the following hold:

1. $p$ divides the order of $L$,
2. $\widehat{L} =$ central product of conjugates of $L$,
3. $C_G(\widehat{L})$ satisfies (H-QC),
4. $\mathcal{A}_p(L) \to \mathcal{A}_p(\mathrm{Aut}_G(L))$ is non-zero in homology.

Then $G$ satisfies (H-QC).

In particular, if $G$ is a minimal counterexample to (H-QC) then:

# Where are alternating and sporadic components?

**Theorem (P.-Smith, '21)**

Let $p$ be a prime and $G$ a group such that for a fixed component $L$ the following hold:

1. $p$ divides the order of $L$,
2. $\widehat{L} =$ central product of conjugates of $L$,
3. $C_G(\widehat{L})$ satisfies (H-QC),
4. $\mathcal{A}_p(L) \to \mathcal{A}_p(\mathrm{Aut}_G(L))$ is non-zero in homology.

Then $G$ satisfies (H-QC).

In particular, if $G$ is a minimal counterexample to (H-QC) then:

- If $p$ is odd, $G$ does not contain alternating or sporadic components.

**Theorem (P.-Smith, '21)**

Let $p$ be a prime and $G$ a group such that for a fixed component $L$ the following hold:

1. $p$ divides the order of $L$,
2. $\widehat{L}$ = central product of conjugates of $L$,
3. $C_G(\widehat{L})$ satisfies (H-QC),
4. $\mathcal{A}_p(L) \to \mathcal{A}_p(\mathrm{Aut}_G(L))$ is non-zero in homology.

Then $G$ satisfies (H-QC).

In particular, if $G$ is a minimal counterexample to (H-QC) then:

- If $p$ is odd, $G$ does not contain alternating or sporadic components.
- If $p = 2$, $G$ does not contain components $L$ with $\mathrm{Out}(L) = 1$, or components $L = \mathrm{Alt}_6, \mathrm{Alt}_8, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{HS}$.

# Some examples

# Some examples

**Fact.** If $A$ is obtained from $L$ by an extension of order $p$,

**Fact.** If $A$ is obtained from $L$ by an extension of order $p$, then $\mathcal{A}_p(A)$ is homotopy equivalent to $\mathcal{A}_p(L)$ after attaching a cone over $\mathcal{A}_p(C_L(B))$ for each $B \in \mathcal{A}_p(A)$ of order $p$ with $B \cap L = 1$.

# Some examples

**Fact.** If $A$ is obtained from $L$ by an extension of order $p$, then $\mathcal{A}_p(A)$ is homotopy equivalent to $\mathcal{A}_p(L)$ after attaching a cone over $\mathcal{A}_p(C_L(B))$ for each $B \in \mathcal{A}_p(A)$ of order $p$ with $B \cap L = 1$.

$$\ldots \widetilde{H}_{n+1}(\mathcal{A}_p(A)) \to \bigoplus_B \widetilde{H}_n(\mathcal{A}_p(C_L(B))) \to \widetilde{H}_n(\mathcal{A}_p(L)) \to \widetilde{H}_n(\mathcal{A}_p(A)) \to \ldots$$

## Some examples

**Fact.** If $A$ is obtained from $L$ by an extension of order $p$, then $\mathcal{A}_p(A)$ is homotopy equivalent to $\mathcal{A}_p(L)$ after attaching a cone over $\mathcal{A}_p(C_L(B))$ for each $B \in \mathcal{A}_p(A)$ of order $p$ with $B \cap L = 1$.

$$\ldots \widetilde{H}_{n+1}(\mathcal{A}_p(A)) \to \bigoplus_B \widetilde{H}_n(\mathcal{A}_p(C_L(B))) \to \widetilde{H}_n(\mathcal{A}_p(L)) \to \widetilde{H}_n(\mathcal{A}_p(A)) \to \ldots$$

### Eliminating $\mathrm{Alt}_8$

We show that $\mathcal{A}_2(\mathrm{Alt}_8) \to \mathcal{A}_2(\mathrm{Sym}_8)$ is non-zero in homology.

**Fact.** If $A$ is obtained from $L$ by an extension of order $p$, then $\mathcal{A}_p(A)$ is homotopy equivalent to $\mathcal{A}_p(L)$ after attaching a cone over $\mathcal{A}_p(C_L(B))$ for each $B \in \mathcal{A}_p(A)$ of order $p$ with $B \cap L = 1$.

$$\ldots \widetilde{H}_{n+1}(\mathcal{A}_p(A)) \to \bigoplus_B \widetilde{H}_n(\mathcal{A}_p(C_L(B))) \to \widetilde{H}_n(\mathcal{A}_p(L)) \to \widetilde{H}_n(\mathcal{A}_p(A)) \to \ldots$$

**Eliminating** $\mathrm{Alt}_8$

We show that $\mathcal{A}_2(\mathrm{Alt}_8) \to \mathcal{A}_2(\mathrm{Sym}_8)$ is non-zero in homology.

1. The "outer" subgroups $B$ of order $p$ have $C_{\mathrm{Alt}_8}(B) = \mathrm{Sym}_6 = \mathrm{Sp}_4(2)$.

**Fact.** If $A$ is obtained from $L$ by an extension of order $p$, then $\mathcal{A}_p(A)$ is homotopy equivalent to $\mathcal{A}_p(L)$ after attaching a cone over $\mathcal{A}_p(C_L(B))$ for each $B \in \mathcal{A}_p(A)$ of order $p$ with $B \cap L = 1$.

$$\ldots \widetilde{H}_{n+1}(\mathcal{A}_p(A)) \to \bigoplus_B \widetilde{H}_n(\mathcal{A}_p(C_L(B))) \to \widetilde{H}_n(\mathcal{A}_p(L)) \to \widetilde{H}_n(\mathcal{A}_p(A)) \to \ldots$$

**Eliminating** $\mathrm{Alt}_8$

We show that $\mathcal{A}_2(\mathrm{Alt}_8) \to \mathcal{A}_2(\mathrm{Sym}_8)$ is non-zero in homology.

1. The "outer" subgroups $B$ of order $p$ have $C_{\mathrm{Alt}_8}(B) = \mathrm{Sym}_6 = \mathrm{Sp}_4(2)$.
2. Then $\mathcal{A}_2(C_{\mathrm{Alt}_8}(B))$ is a wedge of 1-spheres.

**Fact.** If $A$ is obtained from $L$ by an extension of order $p$, then $\mathcal{A}_p(A)$ is homotopy equivalent to $\mathcal{A}_p(L)$ after attaching a cone over $\mathcal{A}_p(C_L(B))$ for each $B \in \mathcal{A}_p(A)$ of order $p$ with $B \cap L = 1$.

$$\ldots \widetilde{H}_{n+1}(\mathcal{A}_p(A)) \to \bigoplus_B \widetilde{H}_n(\mathcal{A}_p(C_L(B))) \to \widetilde{H}_n(\mathcal{A}_p(L)) \to \widetilde{H}_n(\mathcal{A}_p(A)) \to \ldots$$

**Eliminating** $\mathrm{Alt}_8$

We show that $\mathcal{A}_2(\mathrm{Alt}_8) \to \mathcal{A}_2(\mathrm{Sym}_8)$ is non-zero in homology.

1. The "outer" subgroups $B$ of order $p$ have $C_{\mathrm{Alt}_8}(B) = \mathrm{Sym}_6 = \mathrm{Sp}_4(2)$.
2. Then $\mathcal{A}_2(C_{\mathrm{Alt}_8}(B))$ is a wedge of 1-spheres.
3. $\mathrm{Alt}_8 \cong \mathrm{PSL}_4(2)$, so $\mathcal{A}_2(\mathrm{Alt}_8)$ is a wedge of 2-spheres.

## Some examples

**Fact.** If $A$ is obtained from $L$ by an extension of order $p$, then $\mathcal{A}_p(A)$ is homotopy equivalent to $\mathcal{A}_p(L)$ after attaching a cone over $\mathcal{A}_p(C_L(B))$ for each $B \in \mathcal{A}_p(A)$ of order $p$ with $B \cap L = 1$.

$$\ldots \widetilde{H}_{n+1}(\mathcal{A}_p(A)) \to \bigoplus_B \widetilde{H}_n(\mathcal{A}_p(C_L(B))) \to \widetilde{H}_n(\mathcal{A}_p(L)) \to \widetilde{H}_n(\mathcal{A}_p(A)) \to \ldots$$

**Eliminating** $\mathrm{Alt}_8$

We show that $\mathcal{A}_2(\mathrm{Alt}_8) \to \mathcal{A}_2(\mathrm{Sym}_8)$ is non-zero in homology.

1. The "outer" subgroups $B$ of order $p$ have $C_{\mathrm{Alt}_8}(B) = \mathrm{Sym}_6 = \mathrm{Sp}_4(2)$.
2. Then $\mathcal{A}_2(C_{\mathrm{Alt}_8}(B))$ is a wedge of 1-spheres.
3. $\mathrm{Alt}_8 \cong \mathrm{PSL}_4(2)$, so $\mathcal{A}_2(\mathrm{Alt}_8)$ is a wedge of 2-spheres.

$$n = 2 \,:\, 0 = \bigoplus_B \widetilde{H}_2(\mathcal{A}_2(C_{\mathrm{Alt}_8}(B))) \to \widetilde{H}_2(\mathcal{A}_2(\mathrm{Alt}_8)) \to \widetilde{H}_2(\mathcal{A}_2(\mathrm{Sym}_8))$$

# Example: eliminating HS

## Example: eliminating HS

**Goal.** Show that $\mathcal{A}_2(\mathrm{HS}) \to \mathcal{A}_2(\mathrm{Aut}(\mathrm{HS}))$ is non-zero in homology.

## Example: eliminating HS

**Goal.** Show that $\mathcal{A}_2(\mathrm{HS}) \to \mathcal{A}_2(\mathrm{Aut}(\mathrm{HS}))$ is non-zero in homology.

Let $L := \mathrm{HS}$ and $A := \mathrm{Aut}(\mathrm{HS})$.

# Example: eliminating HS

**Goal.** Show that $\mathcal{A}_2(\mathrm{HS}) \to \mathcal{A}_2(\mathrm{Aut}(\mathrm{HS}))$ is non-zero in homology.

Let $L := \mathrm{HS}$ and $A := \mathrm{Aut}(\mathrm{HS})$.

- $m_2(L) = 4$, $m_2(A) = 5$ and $A = \mathrm{HS} : 2C = \mathrm{HS} : 2D$.

# Example: eliminating HS

**Goal.** Show that $\mathcal{A}_2(\mathrm{HS}) \to \mathcal{A}_2(\mathrm{Aut}(\mathrm{HS}))$ is non-zero in homology.

Let $L := \mathrm{HS}$ and $A := \mathrm{Aut}(\mathrm{HS})$.

- $m_2(L) = 4$, $m_2(A) = 5$ and $A = \mathrm{HS} : 2C = \mathrm{HS} : 2D$.
- $\mathcal{A}_2(A)$ is obtained from $\mathcal{A}_2(L)$ by gluing cones over $\mathcal{A}_2(C_L(t))$, where $t$ runs over the conjugates of $2C$ and $2D$:

## Example: eliminating HS

**Goal.** Show that $\mathcal{A}_2(\mathrm{HS}) \to \mathcal{A}_2(\mathrm{Aut}(\mathrm{HS}))$ is non-zero in homology.

Let $L := \mathrm{HS}$ and $A := \mathrm{Aut}(\mathrm{HS})$.

- $m_2(L) = 4$, $m_2(A) = 5$ and $A = \mathrm{HS} : 2C = \mathrm{HS} : 2D$.
- $\mathcal{A}_2(A)$ is obtained from $\mathcal{A}_2(L)$ by gluing cones over $\mathcal{A}_2(C_L(t))$, where $t$ runs over the conjugates of $2C$ and $2D$:

$$\ldots \widetilde{H}_{n+1}(\mathcal{A}_2(A)) \to \bigoplus_t \widetilde{H}_n(\mathcal{A}_2(C_L(t))) \to \widetilde{H}_n(\mathcal{A}_2(L)) \to \widetilde{H}_n(\mathcal{A}_2(A)) \to \ldots$$

# Example: eliminating HS

**Goal.** Show that $\mathcal{A}_2(\mathrm{HS}) \to \mathcal{A}_2(\mathrm{Aut}(\mathrm{HS}))$ is non-zero in homology.

Let $L := \mathrm{HS}$ and $A := \mathrm{Aut}(\mathrm{HS})$.

- $m_2(L) = 4$, $m_2(A) = 5$ and $A = \mathrm{HS} : 2C = \mathrm{HS} : 2D$.
- $\mathcal{A}_2(A)$ is obtained from $\mathcal{A}_2(L)$ by gluing cones over $\mathcal{A}_2(C_L(t))$, where $t$ runs over the conjugates of $2C$ and $2D$:

$$\dots \widetilde{H}_{n+1}(\mathcal{A}_2(A)) \to \bigoplus_t \widetilde{H}_n(\mathcal{A}_2(C_L(t))) \to \widetilde{H}_n(\mathcal{A}_2(L)) \to \widetilde{H}_n(\mathcal{A}_2(A)) \to \dots$$

**(2C)** If $t = 2C$, $O_2(C_L(2C)) \neq 1$ and $\mathcal{A}_2(C_L(t)) \simeq *$.

# Example: eliminating HS

**Goal.** Show that $\mathcal{A}_2(\mathrm{HS}) \to \mathcal{A}_2(\mathrm{Aut}(\mathrm{HS}))$ is non-zero in homology.

Let $L := \mathrm{HS}$ and $A := \mathrm{Aut}(\mathrm{HS})$.

- $m_2(L) = 4$, $m_2(A) = 5$ and $A = \mathrm{HS} : 2C = \mathrm{HS} : 2D$.
- $\mathcal{A}_2(A)$ is obtained from $\mathcal{A}_2(L)$ by gluing cones over $\mathcal{A}_2(C_L(t))$, where $t$ runs over the conjugates of $2C$ and $2D$:

$$\ldots \widetilde{H}_{n+1}(\mathcal{A}_2(A)) \to \bigoplus_t \widetilde{H}_n(\mathcal{A}_2(C_L(t))) \to \widetilde{H}_n(\mathcal{A}_2(L)) \to \widetilde{H}_n(\mathcal{A}_2(A)) \to \ldots$$

**(2C)** If $t = 2C$, $O_2(C_L(2C)) \neq 1$ and $\mathcal{A}_2(C_L(t)) \simeq *$.

**(2D)** If $t = 2D$, $C_L(2D) = \mathrm{Sym}_8$ and $\mathcal{A}_2(\mathrm{Sym}_8) \simeq \bigvee S^2$ (dimension 2).

# Example: eliminating HS

**Goal.** Show that $\mathcal{A}_2(\mathrm{HS}) \to \mathcal{A}_2(\mathrm{Aut}(\mathrm{HS}))$ is non-zero in homology.

Let $L := \mathrm{HS}$ and $A := \mathrm{Aut}(\mathrm{HS})$.

- $m_2(L) = 4$, $m_2(A) = 5$ and $A = \mathrm{HS} : 2C = \mathrm{HS} : 2D$.
- $\mathcal{A}_2(A)$ is obtained from $\mathcal{A}_2(L)$ by gluing cones over $\mathcal{A}_2(C_L(t))$, where $t$ runs over the conjugates of $2C$ and $2D$:

$$\ldots \widetilde{H}_{n+1}(\mathcal{A}_2(A)) \to \bigoplus_t \widetilde{H}_n(\mathcal{A}_2(C_L(t))) \to \widetilde{H}_n(\mathcal{A}_2(L)) \to \widetilde{H}_n(\mathcal{A}_2(A)) \to \ldots$$

**(2C)** If $t = 2C$, $O_2(C_L(2C)) \neq 1$ and $\mathcal{A}_2(C_L(t)) \simeq *$.

**(2D)** If $t = 2D$, $C_L(2D) = \mathrm{Sym}_8$ and $\mathcal{A}_2(\mathrm{Sym}_8) \simeq \bigvee S^2$ (dimension 2).

1. Hence $H_2(\mathcal{A}_2(L)) \to H_2(\mathcal{A}_2(A))$ is surjective.

**Goal.** Show that $\mathcal{A}_2(\mathrm{HS}) \to \mathcal{A}_2(\mathrm{Aut}(\mathrm{HS}))$ is non-zero in homology.

Let $L := \mathrm{HS}$ and $A := \mathrm{Aut}(\mathrm{HS})$.

- $m_2(L) = 4$, $m_2(A) = 5$ and $A = \mathrm{HS} : 2C = \mathrm{HS} : 2D$.
- $\mathcal{A}_2(A)$ is obtained from $\mathcal{A}_2(L)$ by gluing cones over $\mathcal{A}_2(C_L(t))$, where $t$ runs over the conjugates of $2C$ and $2D$:

$$\ldots \widetilde{H}_{n+1}(\mathcal{A}_2(A)) \to \bigoplus_t \widetilde{H}_n(\mathcal{A}_2(C_L(t))) \to \widetilde{H}_n(\mathcal{A}_2(L)) \to \widetilde{H}_n(\mathcal{A}_2(A)) \to \ldots$$

**(2C)** If $t = 2C$, $O_2(C_L(2C)) \neq 1$ and $\mathcal{A}_2(C_L(t)) \simeq *$.

**(2D)** If $t = 2D$, $C_L(2D) = \mathrm{Sym}_8$ and $\mathcal{A}_2(\mathrm{Sym}_8) \simeq \bigvee S^2$ (dimension 2).

1. Hence $H_2(\mathcal{A}_2(L)) \to H_2(\mathcal{A}_2(A))$ is surjective.
2. $H_4(\mathcal{A}_2(A)) = 0$ and $\tilde{\chi}(\mathcal{A}_2(A)) = 1204224$, so $H_2(\mathcal{A}_2(A)) \neq 0$.

## Example: eliminating HS

**Goal.** Show that $\mathcal{A}_2(\mathrm{HS}) \to \mathcal{A}_2(\mathrm{Aut}(\mathrm{HS}))$ is non-zero in homology.

Let $L := \mathrm{HS}$ and $A := \mathrm{Aut}(\mathrm{HS})$.

- $m_2(L) = 4$, $m_2(A) = 5$ and $A = \mathrm{HS} : 2C = \mathrm{HS} : 2D$.
- $\mathcal{A}_2(A)$ is obtained from $\mathcal{A}_2(L)$ by gluing cones over $\mathcal{A}_2(C_L(t))$, where $t$ runs over the conjugates of $2C$ and $2D$:

$$\dots \widetilde{H}_{n+1}(\mathcal{A}_2(A)) \to \bigoplus_t \widetilde{H}_n(\mathcal{A}_2(C_L(t))) \to \widetilde{H}_n(\mathcal{A}_2(L)) \to \widetilde{H}_n(\mathcal{A}_2(A)) \to \dots$$

**(2C)** If $t = 2C$, $O_2(C_L(2C)) \neq 1$ and $\mathcal{A}_2(C_L(t)) \simeq *$.

**(2D)** If $t = 2D$, $C_L(2D) = \mathrm{Sym}_8$ and $\mathcal{A}_2(\mathrm{Sym}_8) \simeq \bigvee S^2$ (dimension 2).

1. Hence $H_2(\mathcal{A}_2(L)) \to H_2(\mathcal{A}_2(A))$ is surjective.
2. $H_4(\mathcal{A}_2(A)) = 0$ and $\tilde{\chi}(\mathcal{A}_2(A)) = 1204224$, so $H_2(\mathcal{A}_2(A)) \neq 0$.

(Similarly, for $L = \mathrm{M}_{12}$ and $L = \mathrm{M}_{22}$, $O_2(C_L(B)) \neq 1$ if $|B| = 2$).

Vielen Dank!