TRACES FOR FRACTIONAL SOBOLEV SPACES WITH VARIABLE EXPONENTS

LEANDRO M. DEL PEZZO AND JULIO D. ROSSI

ABSTRACT. In this note we prove a trace theorem in fractional spaces with variable exponents. To be more precise, we show that if $p\colon \overline{\Omega} \times \overline{\Omega} \to (1,\infty)$ and $q\colon \partial\Omega \to (1,\infty)$ are continuous functions such that

$$\frac{(n-1)p(x,x)}{n-sp(x,x)} > q(x) \qquad \text{in } \partial\Omega \cap \{x \in \overline{\Omega}\colon n - sp(x,x) > 0\},$$

then the inequality

$$\|f\|_{L^{q(\cdot)}(\partial\Omega)} \le C \left\{ \|f\|_{L^{\bar{p}(\cdot)}(\Omega)} + [f]_{s,p(\cdot,\cdot)} \right\}$$

holds. Here $\bar{p}(x) = p(x,x)$ and $[f]_{s,p(\cdot,\cdot)}$ denotes the fractional seminorm with variable exponent, that is given by

$$[f]_{s,p(\cdot,\cdot)} := \inf \left\{ \lambda > 0 \colon \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{n+sp(x,y)}} \, dx\, dy < 1 \right\}$$

and $\|f\|_{L^{q(\cdot)}(\partial\Omega)}$ and $\|f\|_{L^{\bar{p}(\cdot)}(\Omega)}$ are the usual Lebesgue norms with variable exponent.

1. INTRODUCTION

We begin this article remembering the definition of the variable exponents Lebesgue space, to this end we follow [4]. Let (A, Σ, μ) be a $\sigma-$finite complete measure space. Then by $\mathcal{M}(A, \mu)$ we denote the space of all $f\colon A \to [-\infty, +\infty]$ $\mu-$measurable functions. We say that $p \in \mathcal{M}(A, \mu)$ is a bounded variable exponent if

$$1 \le p_- := \operatorname*{ess\,inf}_{x \in A} p(x) \le p_+ := \operatorname*{ess\,sup}_{x \in A} p(x) < +\infty.$$

Then the variable exponent Lebesgue space $L^{p(\cdot)}(A, \mu)$ is defined as

$$L^{p(\cdot)}(A, \mu) := \left\{ f \in \mathcal{M}(A, \mu) \colon \exists \lambda > 0 \text{ such that } \int_A \left| \frac{f(x)}{\lambda} \right|^{r(x)} d\mu < +\infty \right\}$$

equipped with the norm

$$\|f\|_{L^{p(\cdot)}(A,\mu)} := \inf \left\{ \lambda > 0 \colon \int_A \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} d\mu(x) < 1 \right\}.$$

In the special case that μ_1 is the $n-$Lebsgue measure, μ_2 is the $(n-1)-$Huassdorff measure, Ω is a smooth bounded domain of $\mathbb{R}^n$, Σ_1 is the $\sigma-$algebra of μ_1-measurable set of Ω, Σ_2 is the $\sigma-$algebra of μ_2-measurable set of $\partial\Omega$, $p \in \mathcal{M}(\Omega, \mu_1)$ and $q \in \mathcal{M}(\Omega, \mu_2)$ are bounded variable exponents, we note $L^{p(\cdot)}(\Omega) := L^{p(\cdot)}(\Omega, \mu_1)$ and $L^{q(\cdot)}(\partial\Omega) := L^{q(\cdot)}(\partial\Omega, \mu_2)$.

From now on let Ω be a fixed smooth bounded domain in $\mathbb{R}^n$. Let p be a bounded variable exponent in $\overline{\Omega} \times \overline{\Omega}$, $\bar{p}(x) := p(x,x)$ and $0 < s < 1$. We now introduce the

variable exponent Sobolev fractional space as follows:

$$W^{s,p(\cdot,\cdot)}(\Omega) :=$$

$$\left\{ f \in L^{\bar{p}(\cdot)}(\Omega) \colon \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{n+sp(x,y)}} dxdy < +\infty \text{ for some } \lambda > 0 \right\},$$

and we set

$$[f]_{s,p(\cdot,\cdot)}(\Omega) := \inf \left\{ \lambda > 0 \colon \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{n+sp(x,y)}} dxdy < 1 \right\}$$

as the variable exponent seminorm. When there is no confusion we omit the set Ω from the notation.

It is easy to see that $W^{s,p(\cdot,\cdot)}(\Omega)$ is a Banach space with the norm

$$\|f\|_{s,p(\cdot,\cdot)} := \|f\|_{L^{\bar{p}(\cdot)}(\Omega)} + [f]_{s,p(\cdot,\cdot)}.$$

To show this fact, one just has to follow the arguments in [5] for the constant exponent case. This space $W^{s,p(\cdot,\cdot)}(\Omega)$ was recently introduced in [8]. For general theory of classical Sobolev spaces we refer the reader to [1, 3] and for the variable exponent case to [4]. From an applied point of view we recall that non-local energies with constant exponents (we quote here [9, 10]) and also local equations with variable exponents (see [2]) where used in image processing. The space $W^{s,p(\cdot,\cdot)}(\Omega)$ defined above combines the two features, it is given by a fractional seminorm with a variable exponent.

Now we consider two continuous variable exponents, one defined in $\overline{\Omega} \times \overline{\Omega}$ (that was used to define the previous space $W^{s,p(\cdot,\cdot)}(\Omega)$) and the other on $\partial\Omega$ (that is used for the usual Lebesgue space $L^{q(\cdot)}(\partial\Omega)$). We assume that both p and q are bounded away from 1 and ∞, that is,

$$1 < p_- \leq p_+ < +\infty \quad \text{and} \quad 1 < q_- \leq q_+ < +\infty.$$

Our main result in this note is the following compact embedding trace theorem into variable exponent Lebesgue spaces.

Theorem 1.1. *If $1 < sp_-$ and*

$$(1) \qquad p^\star(x) := \frac{(n-1)\bar{p}(x)}{n - s\bar{p}(x)} > q(x) \quad \text{in } \partial\Omega \cap \{x \in \overline{\Omega} \colon n - s\bar{p}(x) > 0\},$$

then there is a constant $C = C(n, s, p, q, \Omega)$ such that

$$\|f\|_{L^{q(\cdot)}(\partial\Omega)} \leq C\|f\|_{s,p(\cdot,\cdot)} \quad \forall f \in W^{s,p(\cdot,\cdot)}(\Omega).$$

That is, the space $W^{s,p(\cdot,\cdot)}(\Omega)$ is continuously embedded in $L^{q(\cdot)}(\partial\Omega)$. Moreover, this embedding is compact.

Remark 1.1. Observe that if p is a continuous bounded variable exponent in $\overline{\Omega}$ and we extend p to $\overline{\Omega} \times \overline{\Omega}$ as $p(x,y) := \frac{p(x)+p(y)}{2}$, then $p^*(x)$ coincides with the classical Sobolev trace exponent associated with $p(x)$.

Remark 1.2. We also want to observe that Theorem 1.1 is still holds if we replace the continuity hypotheses with the assumption that there is $\varepsilon > 0$ such that

$$p^\star(x) - \varepsilon > q(x) \quad \text{in } \partial\Omega \cap \{x \in \overline{\Omega} \colon n - s\bar{p}(x) > 0\}.$$

As a simple application of our trace theorem we can mention the following: For the local case we have that the Neumann problem

$$\begin{cases} -\Delta_{p(x)}u(x) + |u|^{p(x)-2}u(x) = 0 & \text{in } \Omega, \\ |\nabla u(x)|^{p(x)-2}\dfrac{\partial u}{\partial \eta}(x) = g(x) & \text{on } \partial\Omega, \end{cases}$$

can be solved minimizing the functional

$$F(u) := q \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)} \, dx + \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)} \, dx - \int_{\partial\Omega} g(x)u(x) \, d\sigma.$$

Here $\Delta_{p(x)}u = \mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is the $p(x)$−Laplacian and $\dfrac{\partial}{\partial\eta}$ is the outer normal derivative.

Here we show the following result that is analogous to the one that holds for the local case.

Theorem 1.2. *Let* $r\colon \partial\Omega \to (1,\infty)$ *be a continuous function such that* $1 < r_- \leq r_+ < +\infty$. *If* p *is symmetric (i.e.* $p(x,y) = p(y,x)$*) and*

$$p^\star(x) > \frac{r(x)}{r(x)-1} \quad \text{in } \partial\Omega \cap \{x \in \overline{\Omega} \colon n - sp(x,x) > 0\},$$

then, for any $g \in L^{r(\cdot)}(\partial\Omega)$, *there exists a unique minimizer of the functional*

$$G(u) := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{n+sp(x,y)}} \, dx \, dy + \int_\Omega \frac{|u(x)|^{p(x,x)}}{p(x,x)} \, dx - \int_{\partial\Omega} g(x)u(x) \, d\sigma$$

in $W^{s,p(\cdot,\cdot)}(\Omega)$ *that verifies*

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp(x,y)}} \, dx \, dy$$
$$+ \int_\Omega |u(x)|^{p(x,x)-2}u(x)\varphi(x) \, dx - \int_{\partial\Omega} g(x)\varphi(x) \, d\sigma = 0$$

for every $\varphi \in C^1$.

The rest of paper is organized as follows: in the next section, Section 2, we include as preliminaries the statements of known results that will be used in the proof of our main result; while in Section 3 we include the proof of Theorem 1.1; Finally, in Section 4 we prove Theorem 1.2.

2. Preliminaries

In this section we collect some well known results.

We begin by observing that if (A, Σ, μ) is a σ−finite complete space, and p is bounded variable exponent then $f \in L^{p(\cdot)}(A, \mu)$ if only if

$$\int_A |f(x)|^{p(x)} d\mu(x) < \infty.$$

Our first result in this section is the well known Holder's inequality for variable exponents, see [4, Lemma 3.2.20].

Theorem 2.1 (Holder's inequality)**.** *Let* (A, Σ, μ) *be a* σ−*finite complete space, and* p, q *and* r *be bounded variable exponent such that*

$$\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$$

for μ−*a.e.* $x \in A$. *If* $f \in L^{p(\cdot)}(A, \mu)$ *and* $g \in L^{q(\cdot)}(A, \mu)$, *then* $fg \in L^{r(\cdot)}(A, \mu)$ *and there is a positive constant* C *such that*

$$\|fg\|_{L^{r(\cdot)}(A)} \leq C\|f\|_{L^{p(\cdot)}(A)}\|g\|_{L^{q(\cdot)}(A)}.$$

Our second result is an embedding result.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $s \in (0,1)$ and p be a bounded variable exponent such that $p_- > 1$. If $t \in (0,s)$ and $r \in (1,p_-)$ then the space $W^{s,p(\cdot,\cdot)}(\Omega)$ is continuously embedded in $W^{t,r}(\Omega)$. In addition, there is a positive constant $C = C(p_-, p_+, r, s, t, N, \Omega)$ such that*

$$\|f\|_{L^r(\Omega)} \leq C\|f\|_{L^{\bar{p}(\cdot)}(\Omega)} \quad and \quad [f]_{t,r} \leq C[f]_{s,p(\cdot,\cdot)}$$

Proof. Given $f \in W^{s,p(\cdot,\cdot)}(\Omega)$ we want to show that $f \in W^{t,r}(\Omega)$. Since $r \in (1,p_-)$, $f \in L^{\bar{p}(\cdot)}(\Omega)$ and Ω has finite measure, by Holder's inequality, we have that $f \in L^r(\Omega)$. Then, we only need to show that

$$F_r(x,y) = \frac{|f(x) - f(y)|}{|x-y|^{N/r+t}} \in L^r(\Omega \times \Omega).$$

Observe that

$$F_r(x,y) = \frac{|f(x) - f(y)|}{|x-y|^{N/p(x,y)+s}} \frac{|x-y|^{s-t}}{|x-y|^{N(p(x,y)-r)/rp(x,y)}} := H(x,y)G(x,y).$$

Since $f \in W^{s,p(\cdot,\cdot)}(\Omega)$, we have that $H \in L^{p(\cdot,\cdot)}(\Omega \times \Omega)$. Then, by the Holder's inequality we only need to show that $G \in L^{q(\cdot,\cdot)}(\Omega \times \Omega)$, where

$$q(x,y) = \frac{p(x,y)r}{(p(x,y) - r)}.$$

That is, it is enough to show that

$$(2) \qquad \int_\Omega \int_\Omega \frac{|x-y|^{(s-t)rp(x,y)/(p(x,y)-r)}}{|x-y|^N} dx\,dy < +\infty.$$

To prove this, we set $d = \sup\{|x-y| : (x,y) \in \Omega \times \Omega\}$. Observe that

$$d^{(s-t)rp(x,y)/(p(x,y)-r)} \leq \max\{d^{(s-t)rp_-/(p_--r)}, d^{(s-t)rp_+/(p_+-r)}\},$$

$$\left(\frac{|x-y|}{d}\right)^{(s-t)rp(x,y)/(p(x,y)-r)} \leq \left(\frac{|x-y|}{d}\right)^{(s-t)rp_+/(p_+-r)}.$$

Then, there is a positive constant $C = C(p_+, p_-, r, s, t, d)$ such that

$$\int_\Omega \int_\Omega \frac{|x-y|^{(s-t)rp(x,y)/(p(x,y)-r)}}{|x-y|^N} dx\,dy \leq C \int_\Omega \int_\Omega \frac{|x-y|^{(s-t)rp_+/(p_+-r)}}{|x-y|^N} .dx\,dy$$

Therefore, since $(s-t)rp_+/(p_+-r) > 0$ and Ω is bounded, we have that (2) holds. $\quad\square$

Finally, we recall that in the constant exponent case we have the following fractional Sobolev trace embedding theorem. For the proof we refer to [6].

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be an smooth bounded domain, $0 < s < 1$ and $p \in [1, +\infty)$ such that $1 < sp < n$. Then there exists a positive constant $C = C(n, p, q, s, \Omega)$ such that, for any $f \in W^{s,p}(\Omega)$, we have*

$$\|f\|_{L^q(\partial\Omega)} \leq C\|f\|_{W^{s,p}(\Omega)}$$

for any q such that

$$1 \leq q \leq \frac{(n-1)p}{n-sp};$$

i.e., the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\partial\Omega)$. Moreover, this embedding is compact for $q \in [1, \frac{(n-1)p}{n-sp})$.

Remark 2.1. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $0 < s < 1$ and p be a bounded variable exponent such that $n > sp_- > 1$. Then there exist $t \in (0,s)$ and $r \in (1, p_-)$ such that $tr \in (1,n)$. Therefore, by Theorems 2.2 and 2.3, we have that $W^{s,p(\cdot,\cdot)}(\Omega)$ is continuously embedded in $L^q(\partial\Omega)$ for all $q \in [1, \frac{(n-1)r}{n-tr}]$. That is, for any $u \in W^{s,p(\cdot,\cdot)}(\Omega)$, $u|_{\partial\Omega}$ is well defined.

3. The trace theorem

Let us proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. Being p and q continuous, and $\partial\Omega$ compact, from our assumption (1) we get that there exists a positive constant k such that

$$(3) \qquad p^\star(x) - q(x) = \frac{(n-1)p(x,x)}{n - sp(x,x)} - q(x) \geq k > 0,$$

for every $x \in \partial\Omega$ (here $p^\star$ is understood as $+\infty$ when $n - sp(x,x) \leq 0$).

Since p and q are continuous, using (3) we can find a constant $\epsilon = \epsilon(p,q,k,s)$ and a finite family of open sets $B_i \subset \overline{\Omega}$ such that

$$\partial\Omega = \bigcup_{i=1}^{N} \overline{B_i} \cap \partial\Omega, \qquad diam(B_i) := \sup\{|x-y|\colon (x,y) \in B_i \times B_i\} < \epsilon,$$

and

$$\frac{(n-1)p(z,y)}{n - sp(z,y)} - q(x) \geq \frac{k}{2}$$

for every $x \in \partial\Omega \cap \overline{B_i}$ and $(z,y) \in \overline{B_i} \times \overline{B_i}$ (here we set again $\frac{(n-1)p(z,y)}{n-sp(z,y)}$ as $+\infty$ when $n - sp(z,y) \leq 0$).

Given $\delta > 0$ small we can select

$$p_i < \inf\{p(z,y) - \delta\colon (z,y) \in B_i \times B_i\} \qquad 1 < s_i p_i < n,$$

such that

$$\frac{(n-1)p_i}{n - s_i p_i} \geq \frac{k}{3} + q(x)$$

for each $x \in \overline{B_i} \cap \partial\Omega$. We can choose δ smaller is necessary in order to have

$$p_i - 1 > \delta > 0.$$

Hence, by Theorem 2.2 and the trace theorem for constant exponents (see Theorem 2.3), we obtain the existence of a constant $C = C(n, p_i, s_i, \epsilon, B_i)$ such that

$$(4) \qquad \|f\|_{L^{p_i^*}(\overline{B_i} \cap \partial\Omega)} \leq C\Big(\|f\|_{L^{p_i}(B_i)} + [f]_{t,p_i}(B_i)\Big).$$

Now we want to show that the following three statements hold.

(A) There exists a constant c_1 such that

$$\sum_{i=0}^{N} \|f\|_{L^{p_i^*}(\overline{B_i} \cap \partial\Omega)} \geq c_1 \|f\|_{L^{q(\cdot)}(\partial\Omega)}.$$

(B) There exists a constant c_2 such that

$$\sum_{i=0}^{N} \|f\|_{L^{p_i}(B_i)} \leq c_2 \|f\|_{L^{\bar{p}(\cdot)}(\Omega)}.$$

(C) There exists a constant c_3 such that

$$\sum_{i=0}^{N} [f]_{s_i, p_i}(B_i) \leq c_3 [f]_{s, p(\cdot, \cdot)}(\Omega).$$

These three inequalities and (4) give

$$\|f\|_{L^{q(\cdot)}(\partial\Omega)} \leq C \sum_{i=0}^{N} \|f\|_{L^{p_i^*}(\overline{B_i}\cap\partial\Omega)}$$

$$\leq C \sum_{i=0}^{N} \left(\|f\|_{L^{p_i}(B_i)} + [f]_{s_i,p_i}(B_i) \right)$$

$$\leq C \left(\|f\|_{L^{\bar{p}(\cdot)}(\Omega)} + [f]_{s,p(\cdot,\cdot)}(\Omega) \right)$$

$$= C\|f\|_{s,p(\cdot,\cdot)},$$

as we wanted to show.

Therefore, we have to show (A), (B) and (C). Let us start with (A). For $x \in \partial\Omega$ we have

$$|f(x)| \leq \sum_{i=0}^{N} |f(x)| \chi_{\overline{B_i}}(x).$$

Hence

$$\|f\|_{L^{q(\cdot)}(\partial\Omega)} \leq \sum_{i=0}^{N} \|f\|_{L^{q(\cdot)}(\overline{B_i}\cap\partial\Omega)},$$

Since we have

$$\frac{(n-1)p_i}{n-tp_i} \geq \frac{k_1}{3} + q(x) > q(x)$$

we can take $a_i(x)$ such that

$$\frac{1}{q(x)} = \frac{n-tp_i}{(n-1)p_i} + \frac{1}{a_i(x)}.$$

Using Theorem 2.1 we obtain

$$\|f\|_{L^{q(x)}(\overline{B_i}\cap\partial\Omega)} \leq c\|f\|_{L^{\frac{(n-1)p_i}{n-tp_i}}(\overline{B_i}\cap\partial\Omega)} \|1\|_{L^{a_i(x)}(\overline{B_i}\cap\partial\Omega)}$$

$$\leq C\|f\|_{L^{\frac{(n-1)p_i}{n-tp_i}}(\overline{B_i}\cap\partial\Omega)}.$$

Thus, we get (A).

To show (B) we argue in a similar way using that $p(x,x) > p_i$ for $x \in B_i$.

In order to prove (C) let us set

$$F(x,y) := \frac{|f(x) - f(y)|}{|x-y|^s},$$

and observe that

$$[f]_{t,p_i}(B_i) = \left(\int_{B_i} \int_{B_i} \frac{|f(x)-f(y)|^{p_i}}{|x-y|^{n+tp_i+sp_i-sp_i}} \, dxdy \right)^{\frac{1}{p_i}}$$

$$= \left(\int_{B_i} \int_{B_i} \left(\frac{|f(x)-f(y)|}{|x-y|^s} \right)^{p_i} \frac{dxdy}{|x-y|^{n+(t-s)p_i}} \right)^{\frac{1}{p_i}}$$

$$= \|F\|_{L^{p_i}(\mu,B_i\times B_i)}$$

$$\leq C\|F\|_{L^{p(\cdot,\cdot)}(\mu,B_i\times B_i)} \|1\|_{L^{b_i(\cdot,\cdot)}(\mu,B_i\times B_i)}$$

$$\leq C\|F\|_{L^{p(\cdot,\cdot)}(B_i\times B_i,\mu)},$$

where we have used Theorem 2.1 with

$$\frac{1}{p_i} = \frac{1}{p(x,y)} + \frac{1}{b_i(x,y)},$$

but considering the measure in $B_i \times B_i$ given by

$$d\mu(x,y) = \frac{dxdy}{|x-y|^{n+(t-s)p_i}}.$$

Now our aim is to show that

$$\|F\|_{L^{p(\cdot,\cdot)}(B_i \times B_i, \mu)} \leq C[f]_{s,p(\cdot,\cdot)}(B_i)$$

for every i. If this holds, then we immediately get (C) using that for every i it holds that

$$[f]_{s,p(\cdot,\cdot)}(B_i) \leq [f]_{s,p(\cdot,\cdot)}(\Omega).$$

Set $\lambda = [f]_{s,p(\cdot,\cdot)}(B_i)$. Then, using that $diam(B_i) < \epsilon < 1$, we get $|x-y| < 1$ for every $(x,y) \in B_i \times B_i$ and hence

$$\int_{B_i}\int_{B_i}\left(\frac{|f(x)-f(y)|}{\lambda|x-y|^s}\right)^{p(x,y)}\frac{dxdy}{|x-y|^{n+(t-s)p_i}}$$

$$= \int_{B_i}\int_{B_i}|x-y|^{(s-t)p_i}\frac{|f(x)-f(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dxdy$$

$$< \int_{B_i}\int_{B_i}\frac{|f(x)-f(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{n+sp(x,y)}}\,dxdy \leq 1.$$

Therefore

$$\|F\|_{L^{p(\cdot,\cdot)}(B_i \times B_i, \mu)} \leq \lambda = [f]_{s,p(\cdot,\cdot)}(B_i),$$

which implies the desired inequality.

Finally, we recall that the previous embedding is compact since in the constant exponent case we have that for subcritical exponents the embedding is compact. Hence, for a bounded sequence in $W^{s,p(\cdot,\cdot)}(\Omega)$, f_i, we can mimic the previous proof obtaining that for each B_i we can extract a convergent subsequence in $L^{q(\cdot)}(B_i \cap \partial\Omega)$. $\square$

Remark 3.1. Our result is sharp in the following sense: if

$$\frac{(n-1)p(x_0,x_0)}{n-sp(x_0,x_0)} < q(x_0)$$

for some $x_0 \in \partial\Omega$, then the embedding of $W^{s,p(\cdot,\cdot)}(\Omega)$ in $L^{q(\cdot)}(\partial\Omega)$ cannot hold. In fact, from our continuity conditions on p and q there is a small ball $B_\delta(x_0)$ such that

$$\max_{\overline{B}_\delta(x_0)\times\overline{B}_\delta(x_0)}\frac{(n-1)p(x,y)}{n-sp(x,y)} < \min_{\overline{B}_\delta(x_0)\cap\partial\Omega}q(x).$$

In this situation, with the same arguments that hold for the constant exponent case, one can find a sequence f_k supported inside $B_\delta(x_0)$ such that $\|f_k\|_{s,p(\cdot,\cdot)} \leq C$ and $\|f_k\|_{L^{q(\cdot)}(\overline{B_\delta(x_0)}\cap\partial\Omega)} \to +\infty$. In fact, we just consider a smooth, compactly supported function g and take

$$f_k(x) = k^a g(k(x-x_0))$$

with a such that $ap(y,z) - n + sp(y,z) \leq 0$ and $aq(x) - (n-1) > 0$ for $x \in \overline{B}_\delta(x_0)\cap\partial\Omega$ and $y, z \in \overline{B}_\delta(x_0)$.

Finally, we mention that the critical case

$$\frac{(n-1)p(x,x)}{n-sp(x,x)} \geq q(x)$$

with equality for some $x_0 \in \partial\Omega$ is left open.

Remark 3.2. We observe that with the same arguments we can also deal with variable s in the fractional Sobolev space. That is, given

$$s : \overline{\Omega} \times \overline{\Omega} \to (0,1)$$

a symmetric and continuous function and p as before, we can consider the seminorm

$$[f]_{s(\cdot,\cdot),p(\cdot,\cdot)}(\Omega) := \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{n + s(x,y)p(x,y)}} < 1 \right\}$$

and, as before, the norm

$$\|f\|_{s(\cdot,\cdot),p(\cdot,\cdot)} := \|f\|_{L^{\bar{p}(\cdot)}(\Omega)} + [f]_{s(\cdot,\cdot),p(\cdot,\cdot)}(\Omega).$$

In this case, we have that

$$\frac{(n-1)p(x,x)}{n - s(x,x)p(x,x)} > q(x),$$

for $x \in \partial\Omega \cap \{n - s(x,x)p(x,x) > 0\}$ implies the existence of a constant C such that

$$\|f\|_{L^{q(\cdot)}(\partial\Omega)} \leq C\|f\|_{W^{s(\cdot,\cdot),p(\cdot,\cdot)}(\Omega)}.$$

Remark 3.3. We also have a Sobolev-Sobolev trace embedding. Using that for constant p one has the embedding

$$W^{s,p}(B) \to W^{s - \frac{1}{p}, p}(\partial B)$$

(see [6]) one can show (arguing exactly as before) the following result: let

$$s \colon \overline{\Omega} \times \overline{\Omega} \to (0,1), \qquad p \colon \overline{\Omega} \times \overline{\Omega} \to (1,\infty)$$

and

$$t \colon \partial\Omega \times \partial\Omega \to (0,1), \qquad q \colon \partial\Omega \times \partial\Omega \to (1,\infty)$$

be continuous functions with

$$\bar{p}(x)\bar{s}(x) := p(x,x)s(x,x) > 1$$

and

$$\bar{t}(x) := t(x,x) < \bar{s}(x) - \frac{1}{\bar{p}(x)} \qquad \text{and} \qquad \bar{q}(x) := q(x,x) < \bar{p}(x)$$

for every $x \in \partial\Omega$. Then it holds that

$$W^{s(\cdot,\cdot),p(\cdot,\cdot)}(\Omega) \to W^{t(\cdot,\cdot),q(\cdot,\cdot)}(\partial\Omega).$$

Notice that here we let (as the notation suggests)

$$[f]_{t(\cdot,\cdot),q(\cdot,\cdot)}(\partial\Omega) := \inf \left\{ \lambda > 0 : \int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^{q(x,y)}}{\lambda^{q(x,y)} |x - y|^{n + t(x,y)q(x,y)}} d\sigma d\sigma < 1 \right\}$$

and the norm

$$\|f\|_{W^{t(\cdot,\cdot),q(\cdot,\cdot)}(\partial\Omega)} := \|f\|_{L^{\bar{q}(\cdot)}(\partial\Omega)} + [f]_{t(\cdot,\cdot),q(\cdot,\cdot)}(\partial\Omega).$$

In fact, to prove this result, one first observe that the trace theorem in its Sobolev-Lebesgue version gives that

$$\|f\|_{L^{\bar{q}(\cdot)}(\partial\Omega)} \leq C\|f\|_{s(\cdot,\cdot),p(\cdot,\cdot)},$$

see Remark 3.2 (notice that we have $\bar{q}(x) < \bar{p}(x) < \frac{(n-1)\bar{p}(x)}{n - \bar{s}(x)\bar{p}(x)}$ for $x \in \partial\Omega$ due to the fact that we assumed $\bar{p}(x)\bar{s}(x) > 1$). Hence we are left with the proof of an inequality of the form

$$[f]_{t(\cdot,\cdot),q(\cdot,\cdot)}(\partial\Omega) \leq C\|f\|_{s(\cdot,\cdot),p(\cdot,\cdot)}.$$

Here one can mimic the same proof as in the Sobolev-Lebesgue trace theorem using that there exist a finite number of sets B_i such that $\cup_i^N \overline{B_i}$ cover $\partial\Omega$ and constant exponents s_i, t_i, p_i, q_i such that

$$p(x,y) > p_i, \qquad s_i > s(x,y), \qquad \forall x,y \in \overline{B_i} \times \overline{B_i}$$

$$q(x,y) < q_i \qquad t_i < t(x,y), \qquad \forall x,y \in \overline{B_i} \cap \partial\Omega \times \overline{B_i} \cap \partial\Omega$$

with

$$p_i s_i > 1, \qquad t_i < s_i - \frac{1}{p_i} \qquad \text{and} \qquad q_i < p_i$$

and then use the Sobolev-Sobolev trace theorem with constant exponents

$$W^{s_i, p_i}(B) \to W^{t_i, q_i}(\partial B),$$

add over i and conclude as before.

4. An application

Now we turn our attention to the proof of Theorem 1.2.

Proof of Theorem 1.2. We just observe that we can apply the direct method of calculus of variations. Note that the functional G is strictly convex (this holds since for any x and y the function $t \mapsto t^{p(x,y)}$ is strictly convex) and weakly lower semicontinuous.

From our previous results, $W^{s,p(\cdot,\cdot)}(\Omega)$ is compactly embedded in $L^{q(\cdot)}(\partial\Omega)$ for $q(x) < p^*(x)$, see Theorem 1.1. In particular, we have that $W^{s,p(\cdot,\cdot)}(\Omega)$ is compactly embedded in $L^{\frac{r(\cdot)}{r(\cdot)-1}}(\partial\Omega)$.

Let us see that G is coercive. We have

$$G(u) = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}p(x,y)} dx dy + \int_\Omega \frac{|u(x)|^{p(x,x)}}{p(x,x)} dx - \int_{\partial\Omega} g(x)u(x)\, d\sigma$$

$$\geq \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}p(x,y)} dx dy + \int_\Omega \frac{|u(x)|^{p(x,x)}}{p(x,x)} dx$$
$$\quad - \|g\|_{L^{r(x)}(\partial\Omega)} \|u\|_{L^{\frac{r(x)}{r(x)-1}}(\partial\Omega)}$$

$$\geq \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}p(x,y)} dx dy + \int_\Omega \frac{|u(x)|^{p(x,x)}}{p(x,x)} dx - C\|u\|_{s,p(\cdot,\cdot)}.$$

Now, we choose a sequence u_j such that $\|u_j\|_{s,p(\cdot,\cdot)} \to \infty$ as $j \to \infty$. Let us assume that $\|u_j\|_{s,p(\cdot,\cdot)} > 1$. Then we have

$$\frac{G(u_j)}{\|u_j\|_{s,p(\cdot,\cdot)}} \geq \frac{1}{\|u_j\|_{s,p(\cdot,\cdot)}} \left(\int_\Omega \int_\Omega \frac{|u_j(x)-u_j(y)|^{p(x,y)}}{|x-y|^{n+sp(x,y)}p(x,y)} dx dy + \int_\Omega \frac{|u_j(x)|^{p(x,x)}}{p(x,x)} dx \right) - C$$
$$\geq \|u_j\|_{s,p(\cdot,\cdot)}^{p_- - 1} - C.$$

Then we obtain

$$G(u_j) \geq \|u_j\|_{s,p(\cdot,\cdot)}^{p_- - 1} - C\|u_j\|_{s,p(\cdot,\cdot)} \to \infty,$$

and we conclude that G is coercive. Therefore, there is a unique minimizer of G in $W^{s,p(\cdot,\cdot)}(\Omega)$.

To show that it holds that the minimizer verifies

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp(x,y)}} dx\, dy$$

$$+ \int_\Omega |u(x)|^{p(x,x)-2}u(x)\varphi(x)\, dx - \int_{\partial\Omega} g(x)\varphi(x)\, d\sigma = 0$$

for every $\varphi \in C^1$ one just have to differentiate

$$t \mapsto G(u + tv)$$

and use that this derivative vanishes at $t = 0$ since u is a minimum of G. $\qquad\square$

References

[1] R. Adams, J. Fournier, *Sobolev spaces*. Second edition. Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003. xiv+305 pp.

[2] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), no. 4, 1383–1406-

[3] F. Demengel, G. Demengel, *Functional spaces for the theory of elliptic partial differential equations*, Universitext, Springer, London, 2012. Translated from the 2007 French original by Reinie Erné.

[4] L. Diening, P. Harjulehto, P. Hasto, M. Ruzicka, Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics, Vol. 2017, Springer-Verlag, Heidelberg, 2011.

[5] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.

[6] P- Grisvard, Elliptic problems in nonsmooth domains. Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA, 1985. xiv+410 pp.

[7] P. Harjulehto, P. Hasto, U. Le, M. Nuortio, *Overview of differential equations with non-standard growth*, Nonlinear Anal. **72** (2010), 4551–4574.

[8] U. Kaufmann, J. D. Rossi and R. Vidal. *Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians.* Preprint.

[9] Y. Lou, X. Zhang, S. Osher, Stanley, A. Bertozzi, *Image recovery via nonlocal operators*. J. Sci. Comput. **42** (2010), no. 2, 185–197.

[10] G. Gilboa, S. Osher, *Nonlocal operators with applications to image processing*. Multiscale Model. Simul. **7** (2008), no. 3, 1005–1028.

Leandro M. Del Pezzo
CONICET and
Departamento de Matemáticas y Estadística
Universidad Torcuato Di Tella
Av. Figueroa Alcorta 7350 (C1428BCW)
Buenos Aires, Argentina.

E-mail address: `ldelpezzo@utdt.edu`

URL: `http://cms.dm.uba.ar/Members/ldpezzo/`

Julio D. Rossi
CONICET and
Departamento de Matemática, FCEyN,
Universidad de Buenos Aires,
Pabellon I, Ciudad Universitaria (1428),
Buenos Aires, Argentina.

E-mail address: `jrossi@dm.uba.ar`

URL: `http://mate.dm.uba.ar/~jrossi/`