# ASYMPTOTIC BEHAVIOR OF THE BEST SOBOLEV TRACE CONSTANT IN EXPANDING AND CONTRACTING DOMAINS

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ABSTRACT. We study the asymptotic behavior for the best constant and extremals of the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ on expanding and contracting domains. We find that the behavior strongly depends on $p$ and $q$. For contracting domains we prove that the behavior of the best Sobolev trace constant depends on the sign of $qN - pN + p$ while for expanding domains it depends on the sign of $q - p$. We also give some results regarding the behavior of the extremals, for contracting domains we prove that they converge to a constant when rescaled is a suitable way and for expanding domains we observe when a concentration phenomena takes place.

## 1. INTRODUCTION.

Let $\Omega$ be a smooth bounded domain in ${}^N$, $N \geq 2$. Of importance in the study of boundary value problems for differential operators in $\Omega$ are the Sobolev trace inequalities. For any $1 < p < N$, and $1 < q \leq p^* = p(N-1)/(N-p)$ we have that $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ and hence the following inequality holds:

$$S_q \|u\|^p_{L^q(\partial\Omega)} \leq \|u\|^p_{W^{1,p}(\Omega)},$$

for all $u \in W^{1,p}(\Omega)$. This is known as the Sobolev trace embedding Theorem. The best constant for this embedding is the largest $S_q$ such that the above inequality holds, that is,

$$S_q(\Omega) = \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p + |u|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |u|^q \, d\sigma\right)^{p/q}}. \tag{1.1}$$

Moreover, if $1 < q < p^*$ the embedding is compact and as a consequence we have the existence of extremals, i.e. functions where the infimum is attained, see [8]. These extremals are weak solutions of the following problem

$$\begin{cases} \Delta_p u = |u|^{p-2} u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian and $\frac{\partial}{\partial \nu}$ is the outer unit normal derivative.

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Standard regularity theory and the strong maximum principle, [16], show that any extremal $u$ belongs to the class $C^{1,\alpha}_{\rm loc}(\Omega)\cap C^{\alpha}(\overline{\Omega})$ and that is strictly one signed in $\Omega$, so we can assume that $u>0$ in $\Omega$.

Let us fix $p$, $q$ with $1<q<p^*$ and $\Omega$ a bounded smooth domain in $\mathbb{R}^N$, $C^1$ is enough for our calculations. For $\mu>0$ we consider the family of domains

$$\Omega_\mu=\mu\Omega=\{\mu x\ ;\ x\in\Omega\}.$$

The purpose of this work is to describe the asymptotic behavior of the best Sobolev trace constants $S_q(\Omega_\mu)$ as $\mu\to 0+$ and $\mu\to+\infty$.

As a precedent, see [4] for a detailed analysis of the behavior of extremals and best Sobolev constants in expanding domains for $p=2$ and $q>2$. In that paper it is proved that the extremals develops a peak near the point where the curvature of the boundary attains a maximum. In [5] and [13] a related problem in the half-space $\mathbb{R}^N_+$ for the critical exponent is studied. See also [6], [7] for other geometric problems that leads to nonlinear boundary conditions.

Let us call $u_\mu$ an extremal corresponding to $\Omega_\mu$. Making a change of variables, we go back to the original domain $\Omega$. If we define $v_\mu(x)=u_\mu(\mu x)$, we have that $v_\mu\in W^{1,p}(\Omega)$ and

$$S_q(\Omega_\mu)=\mu^{(Nq-Np+p)/q}\,\frac{\displaystyle\int_\Omega \mu^{-p}|\nabla v_\mu|^p+|v_\mu|^p\,dx}{\left(\displaystyle\int_{\partial\Omega}|v_\mu|^q\,d\sigma\right)^{p/q}}. \tag{1.3}$$

We can assume, and we do so, that the functions $u_\mu$ are chosen so that

$$\int_{\partial\Omega}|v_\mu|^q\,d\sigma=1.$$

We remark that the quantity (1.1) is not homogeneous under dilations or contractions of the domain. This is a remarkable difference with the study of the Sobolev embedding $W^{1,p}_0(\Omega)\hookrightarrow L^q(\Omega)$.

First, we deal with the case $\mu\to 0+$. As we will see the behavior of the Sobolev constant and extremals is very different when the domain is contracted than when it is expanded.

Our first result is the following:

**Theorem 1.1.** *Let $1<q<p^*$, then*

$$\lim_{\mu\to 0+}\frac{S_q(\Omega_\mu)}{\mu^{(Nq-Np+p)/q}}=\frac{|\Omega|}{|\partial\Omega|^{p/q}} \tag{1.4}$$

*and if we scale the extremals $u_\mu$ to the original domain $\Omega$ as $v_\mu(x)=u_\mu(\mu x)$, $x\in\Omega$, with $\|v_\mu\|_{L^q(\partial\Omega)}=1$, then $v_\mu$ is nearly constant in the sense that*

$$v_\mu\to\frac{1}{|\partial\Omega|^{1/q}}\qquad \text{in } W^{1,p}(\Omega).$$

Observe that the behavior of the Sobolev trace constant, strongly depends on $p$ and $q$. If we call $\beta_{pq}=(Nq-Np+p)/q$ then we have that, as $\mu\to 0+$,

$$\begin{array}{ll} S_q\to 0 & \text{if } \beta_{pq}>0,\\ S_q\to+\infty & \text{if } \beta_{pq}<0,\\ S_q\to C\neq 0 & \text{if } \beta_{pq}=0.\end{array}$$

Also let us remark that the influence of the geometry of the domain appears in the limit (1.4).

In the special case $p = q$, problem (1.2) becomes a nonlinear eigenvalue problem. For $p = 2$, this eigenvalue problem is known as the *Steklov* problem, [2].

In [8] it is proved, applying the Ljusternik-Schnirelman critical point Theory on $C^1$ manifolds, that there exists a sequence of variational eigenvalues $\lambda_k \nearrow +\infty$ and it is easy to see that the first eigenvalue $\lambda_1(\Omega)$ verifies $\lambda_1(\Omega) = S_p(\Omega)$. So Theorem 1.1 shows a difference in the behavior of the first eigenvalue of (1.2) with respect to the domain with the behavior of the first eigenvalue of the following Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where it is a well known fact that $\lambda_1$ increases as the domain decreases, see [1], [10].

The variational eigenvalues $\lambda_k$ of (1.2) are characterized by

$$\frac{1}{\lambda_k} = \sup_{C \in C_k} \min_{u \in C} \frac{\|u\|^p_{L^p(\partial\Omega)}}{\|u\|^p_{W^{1,p}(\Omega)}}, \tag{1.5}$$

where $C_k = \{C \subset W^{1,p}(\Omega);\ C$ is compact, symmetric and $\gamma(C) \geq k\}$ and $\gamma$ is the Krasnoselski genus (see [11]). It is shown in [9] that there exists a second eigenvalue for (1.2) and that it coincides with the second variational eigenvalue $\lambda_2$. Moreover, the following characterization of the second eigenvalue $\lambda_2$ holds

$$\lambda_2 = \inf_{u \in A} \left\{ \int_\Omega |\nabla u|^p + |u|^p \, dx \right\}, \tag{1.6}$$

where $A = \{u \in W^{1,p}(\Omega);\ \|u\|_{L^p(\partial\Omega)} = 1$ and $|\partial\Omega^\pm| \geq c\}$, $\partial\Omega^+ = \{x \in \partial\Omega;\ u(x) > 0\}$ and $\partial\Omega^-$ is defined analogously.

Concerning this second eigenvalue, we have the following result.

**Theorem 1.2.** *There exists a constant $\widetilde{\lambda}_2$ such that*

$$\lim_{\mu \to 0+} \mu^{p-1} \lambda_2(\Omega_\mu) = \widetilde{\lambda}_2.$$

*This constant $\widetilde{\lambda}_2$ is the first nonzero eigenvalue of the following problem*

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \tilde{\lambda}|u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

*Moreover, if we take an eigenfunction $u_{2,\mu}$ associated to $\lambda_2(\Omega_\mu)$ and scale it to $\Omega$ as in Theorem 1.1, we obtain that*

$$v_{2,\mu} \to \widetilde{v}_2 \qquad \text{in } W^{1,p}(\Omega),$$

*where $\widetilde{v}_2$ is an eigenfunction of (1.7) associated to $\widetilde{\lambda}_2$.*

Observe that the first eigenvalue of (1.7) is zero with associated eigenfunction a constant. Hence Theorem 1.1 says that the first eigenvalue and the first eigenfunction of our problem (1.2) converges to the ones of (1.7).

As a consequence of Theorem 1.2 we obtain the following Theorems.

**Theorem 1.3.** *Every eigenvalue $\lambda_2(\Omega_\mu) \le \lambda(\Omega_\mu) \le \lambda_k(\Omega_\mu)$ of (1.2) (variational or not) behaves as*

$$\lambda(\Omega_\mu) \sim \mu^{1-p} \qquad \text{as } \mu \to 0+ .$$

**Theorem 1.4.** *Let $\mu_j \to 0$ and $\lambda_j = \lambda(\Omega_{\mu_j})$ be a sequence of eigenvalues such that there exists $\lambda$ with*

$$\lim_{j\to\infty} \frac{\lambda_j}{\mu_j^{1-p}} = \lambda.$$

*Let $(v_j)$ be the sequence of associated eigenfunctions rescaled as in Theorem 1.1. Then $(v_j)$ has a convergent subsequence $(v_{j_k})$ and a limit $v$, that is an eigenfunction of (1.7) with eigenvalue $\lambda$.*

These theorems say that $\lambda(\Omega_\mu) \to +\infty$ as $\mu \to 0+$ and that problem (1.7) is a limit problem for (1.2) when $\mu \to 0+$.

Now, we deal with the case $\mu \to +\infty$. In this case we find, as before, that the behavior strongly depends on $p$ and $q$. We prove,

**Theorem 1.5.** *Let $\beta_{pq} = (qN - pN + p)/q$. It holds*

1. *If $1 < q < p$,*
$$0 < c_1 \mu^{\beta_{pq}-1} \le S_q(\Omega_\mu) \le c_2 \mu^{\beta_{pq}-1}.$$
2. *If $p \le q < p^*$,*
$$0 < c_1 \le S_q(\Omega_\mu) \le c_2 < \infty.$$

*For the lower bound in (2) for the case $p < q < p^*$ we have to assume that the corresponding extremals $v_\mu$ rescaled such that $\max_{\overline{\Omega}} v_\mu = 1$ verify $|\nabla v_\mu| \le C/\mu$.*

*Moreover, for all cases, we have that the corresponding extremals $u_\mu$ rescaled as in Theorem 1.1 concentrates at the boundary, in the sense that*

$$\int_\Omega |v_\mu|^p \, dx \le C \mu^{-\beta_{pq}} \to 0 \quad \text{as } \mu \to +\infty, \qquad \text{if } q \ge p,$$

$$\int_\Omega |v_\mu|^p \, dx \le C \mu^{-1} \to 0 \quad \text{as } \mu \to +\infty, \qquad \text{if } q < p,$$

*with*

$$\int_{\partial\Omega} |v_\mu|^q \, d\sigma = 1.$$

As before the behavior of the Sobolev trace constant depends on $p$ and $q$. We have that, as $\mu \to +\infty$,

$$\begin{array}{ll} S_q \to 0 & \text{if } \beta_{pq} - 1 < 0, \text{ i.e. } q < p, \\ 0 < c_1 \le S_q \le c_2 < \infty & \text{if } \beta_{pq} - 1 \ge 0, \text{ i.e. } q \ge p. \end{array}$$

The hypothesis $|\nabla v_\mu| \le C/\mu$ is a regularity assumption, see [15] for $C^{1,\alpha}_{\text{loc}}$ regularity results.

As a consequence of our arguments we have that the extremals do not develop a peak if $1 < q < p$ as in this case we have that

$$c_1 \le \int_{\partial\Omega} |v_\mu|^p \, d\sigma \le c_2,$$

and

$$\int_{\partial\Omega} |v_\mu|^q \, d\sigma = 1.$$

For $p = q$ it is proved in [12] that the first eigenvalue $\lambda_1(\Omega_\mu) = S_p(\Omega_\mu)$ is isolated and simple. As a consequence of this if $\Omega$ is a ball the extremal $v_\mu$ is radial and hence it does not develop a peak. Finally, for $q > p$ the extremals develop peaking concentration phenomena in the sense that, for every $a > 0$,

$$a^p |\partial\Omega \cap \{v_\mu > a\}| \to 0, \qquad \text{as } \mu \to +\infty,$$

with

$$\max_{\overline{\Omega}} v_\mu = 1.$$

This is in concordance with the results of [4] where for $p = 2$, $q > 2$ they find that the extremals concentrates, with the formation of a peak, near a point of the boundary where the curvature maximizes.

We believe that for $q > p$, extremals develop a single peak as in the case $p = 2$. However that kind of analysis needs some fine knowledge of the limit problem in $^N_+$ that is not yet available for the $p-$Laplacian.

Let us give an idea of the proof of the lower bounds. In the case $p = q$ we can obtain the lower bound by an approximation procedure. We replace $W^{1,p}(\Omega)$ by an increasing sequence of subspaces in the minimization problem. Then we prove a convergence result and find a uniform bound from below for the approximating problems. We believe that this idea can be used in other contexts. For the case $q > p$ we use our assumption $|\nabla v_\mu| \le C/\mu$ to prove a reverse Hölder inequality for the extremals on the boundary that allows us to reduce to the case $p = q$.

Finally, for large $\mu$, in the case $p = q$ we can prove that every eigenvalue is bounded.

**Theorem 1.6.** *Let $\lambda_1(\Omega_\mu) \le \lambda(\Omega_\mu) \le \lambda_k(\Omega_\mu)$ be an eigenvalue of (1.2) in $\Omega_\mu$ (variational or not). Then there exists two constants, $C_1, C_2 > 0$, independent of $\mu$ such that*

$$0 < C_1 \le \lambda(\Omega_\mu) \le C_2 < +\infty,$$

*for every $\mu$ large.*

Throughout the paper, by $C$ we mean a constant that may vary from line to line but remains independent of the relevant quantities.

The rest of the paper is organized as follows. In §2, we deal with the case $\mu \to 0$ and in §3, we study the case $\mu \to +\infty$.

## 2. Behavior as $\mu \to 0+$

In this section we focus on the case $\mu \to 0+$. First we prove Theorem 1.1 and then study the case where $q = p$ (the eigenvalue problem).

Let us begin with the following Lemma.

**Lemma 2.1.** *Under the assumptions of Theorem 1.1, it follows that*

$$S_q(\Omega_\mu) \le \mu^{(Nq-Np+p)/q} \frac{|\Omega|}{|\partial\Omega|^{p/q}}.$$

*Proof.* Let us recall that

$$S_q(\Omega_\mu) = \inf_{u\in W^{1,p}(\Omega_\mu)\setminus\{0\}} \frac{\displaystyle\int_{\Omega_\mu} |\nabla u|^p + |u|^p\,dx}{\left(\displaystyle\int_{\partial\Omega_\mu} |u|^q\,d\sigma\right)^{p/q}}.$$

Then, taking $u\equiv 1$ it follows that

$$S_q(\Omega_\mu) \le \mu^{(Nq-Np+p)/q}\frac{|\Omega|}{|\partial\Omega|^{p/q}},$$

as we wanted to see. ∎

This Lemma shows that the ratio $S_q(\Omega_\mu)/\mu^{(Nq-Np+p)/q}$ is bounded. So a natural question will be to determine if it converges to some value. This is answered in Theorem 1.1 that we prove next.

*Proof of Theorem 1.1.* Let $u_\mu \in W^{1,p}(\Omega_\mu)$ be a extremal for $S_q(\Omega_\mu)$ and define $v_\mu(x) = u_\mu(\mu x)$, we have that $v_\mu \in W^{1,p}(\Omega)$.

We can assume that the functions $u_\mu$ are chosen so that

$$\int_{\partial\Omega} |v_\mu|^q\,d\sigma = 1.$$

Equation (1.3) and Lemma 2.1 give, for $\mu<1$,

$$\|v_\mu\|^p_{W^{1,p}(\Omega)} \le \int_\Omega \mu^{-p}|\nabla v_\mu|^p + |v_\mu|^p\,dx \le \frac{|\Omega|}{|\partial\Omega|^{p/q}},$$

so there exists a function $v\in W^{1,p}(\Omega)$ and a sequence $\mu_j\to 0+$ such that

$$\begin{aligned} v_{\mu_j} &\rightharpoonup v \quad \text{weakly in } W^{1,p}(\Omega),\\ v_{\mu_j} &\to v \quad \text{in } L^p(\Omega),\\ v_{\mu_j} &\to v \quad \text{in } L^q(\partial\Omega). \end{aligned}$$

Moreover,

$$\int_\Omega |\nabla v_\mu|^p\,dx \le \frac{|\Omega|}{|\partial\Omega|^{p/q}}\mu^p.$$

Hence $\nabla v_\mu \to 0$ in $L^p(\Omega)$. It follows that the limit $v$ is a constant and must verify $\int_{\partial\Omega}|v|^q = 1$, hence

$$v = constant = \frac{1}{|\partial\Omega|^{1/q}}$$

and so the full sequence $v_\mu$ converges weakly in $W^{1,p}(\Omega)$ to $v$. From our previous bounds we have

$$v_\mu \to \frac{1}{|\partial\Omega|^{1/q}} \text{ in } L^p(\Omega) \qquad \text{and} \qquad \int_\Omega |\nabla v_\mu|^p\,dx \to 0.$$

Therefore, we have strong convergence,

$$v_\mu \to \frac{1}{|\partial\Omega|^{1/q}} \qquad \text{in } W^{1,p}(\Omega).$$

The proof is finished. ∎

Now we turn our attention to the case $p = q$ which is a nonlinear eigenvalue problem. We recall that Theorem 1.1 says that $\lambda_1(\Omega_\mu) = S_p(\Omega_\mu) \sim \mu \to 0$.

First we focus on the behavior of the second eigenvalue $\lambda_2$ this is the content of Theorem 1.2.

For the proof of Theorem 1.2 we need the following Lemmas. We believe that these results have independent interest.

**Lemma 2.2.** *Let $h \in L^{p'}(\partial\Omega)$. Then, problem*

$$\begin{cases} \Delta_p w = 0 & \text{in } \Omega, \\ \\ |\nabla w|^{p-2}\frac{\partial w}{\partial \nu} = h(x) & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

*has a weak solution if and only if $\int_{\partial\Omega} h(x)\, d\sigma = 0$. Moreover, the solution is unique up to an additive constant.*

*Proof.* It is straightforward to check that if there exists a weak solution to (2.1) then $\int_{\partial\Omega} h(x)\, d\sigma = 0$.

Now, let $X = \{w \in W^{1,p}(\Omega);\ \int_\Omega w\, dx = 0\}$. By a standard compactness argument, one can verify that the following Poincare inequality holds,

$$\|w\|_{L^p(\Omega)} \le C\|\nabla w\|_{L^p(\Omega)}, \tag{2.2}$$

for every $w \in X$ and some constant $C$.

Let us now define

$$\Phi(w) = \int_\Omega |\nabla w|^p\, dx - \int_{\partial\Omega} h(x) w\, d\sigma. \tag{2.3}$$

Critical points of $\Phi$ in $W^{1,p}(\Omega)$ are weak solutions of (2.1).

By (2.2), $\Phi$ is a strictly convex, bounded below functional on $X$, and so there exists a unique function $w \in X$ such that $\Phi'(w)(v) = 0$ for every $v \in X$.

Now, using the fact that $\int_{\partial\Omega} h(x)\, d\sigma = 0$, it is easy to see that $\Phi'(w)(v) = 0$ for every $v \in W^{1,p}(\Omega)$ and the proof is now complete. □

Now we find a variational characterization of the first non-zero eigenvalue of the limit problem (1.7).

**Lemma 2.3.** *Let $\tilde\lambda_2$ be defined by*

$$\tilde\lambda_2 = \inf_{u \in Y - \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p\, dx}{\displaystyle\int_{\partial\Omega} |u|^p\, d\sigma}, \tag{2.4}$$

*where $Y = \{u \in W^{1,p}(\Omega);\ \int_{\partial\Omega} |u|^{p-2}u\, d\sigma = 0\}$. Then the infimum is attained.*

*Proof.* Let $u_n$ be a minimizing sequence with $\|u_n\|_{L^p(\partial\Omega)} = 1$. By a compactness argument we can extract a subsequence, that we still call $u_n$, such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega), \\ u_n &\to u \quad \text{in } L^p(\Omega), \\ u_n &\to u \quad \text{in } L^p(\partial\Omega). \end{aligned}$$

Hence $u \in Y - \{0\}$, $\|u\|_{L^p(\partial\Omega)} = 1$. Moreover, we have that

$$\int_\Omega |\nabla u|^p\, dx \le \liminf \int_\Omega |\nabla u_n|^p\, dx = \tilde\lambda_2.$$

Therefore $u$ is a minimizer. $\square$

Now we are ready to deal with the proof of Theorem 1.2.

*Proof of Theorem 1.2.* We can assume that $0 \in \Omega$ and then we can take $u(x) = x_1$ in the characterization of $\lambda_2$ given by (1.6) to obtain

$$\lambda_2(\Omega_\mu) \le \frac{|\Omega_\mu| + \displaystyle\int_{\Omega_\mu} |x_1|^p\,dx}{\displaystyle\int_{\partial\Omega_\mu} |x_1|^p\,d\sigma} = \mu^{1-p} \frac{|\Omega| + \mu^p \displaystyle\int_{\Omega} |y_1|^p\,dy}{\displaystyle\int_{\partial\Omega} |y_1|^p\,d\sigma} \le C\mu^{1-p}.$$

Hence if we consider $v_{2,\mu}$ any eigenfunction associated to $\lambda_2(\Omega_\mu)$ with

$$\|v_{2,\mu}\|_{L^p(\partial\Omega)} = 1$$

we get

$$C\mu^{1-p} \ge \lambda_2(\Omega_\mu) = \mu^{1-p}\left(\int_\Omega |\nabla v_{2,\mu}|^p\,dx + \mu^p \int_\Omega |v_{2,\mu}|^p\,dx\right).$$

Therefore

$$\|\nabla v_{2,\mu}\|_{L^p(\Omega)} \le C.$$

As we have that $\|v_{2,\mu}\|_{L^p(\partial\Omega)} = 1$, it follows that

$$\|v_{2,\mu}\|_{W^{1,p}(\Omega)} \le C,$$

hence we can extract a subsequence $\mu_j \to 0+$ such that

$$\begin{aligned} v_{2,\mu_j} &\rightharpoonup \tilde{v}_2 \quad \text{weakly in } W^{1,p}(\Omega),\\ v_{2,\mu_j} &\to \tilde{v}_2 \quad \text{in } L^p(\Omega),\\ v_{2,\mu_j} &\to \tilde{v}_2 \quad \text{in } L^p(\partial\Omega). \end{aligned}$$

Therefore we have that

$$\int_{\partial\Omega} |\tilde{v}_2|^p\,d\sigma = 1.$$

As it is proved in [9], $|\{v_{2,\mu_j} > 0\} \cap \partial\Omega|, |\{v_{2,\mu_j} < 0\} \cap \partial\Omega| > c$ independent of $\mu_j$, then $\tilde{v}_2$ changes sign. Hence, we get

$$\int_\Omega |\nabla \tilde{v}_2|^p\,dx \ne 0.$$

Taking a subsequence, if necessary, we can assume that

$$\frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} \to \bar{\lambda} \qquad \text{as } \mu \to 0+$$

and, as

$$\frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} = \int_\Omega |\nabla v_{2,\mu}|^p\,dx + \mu^p \int_\Omega |v_{2,\mu}|^p\,dx,$$

passing to the limit

$$0 \ne \int_\Omega |\nabla \tilde{v}_2|^p\,dx \le \liminf \int_\Omega |\nabla v_{2,\mu}|^p\,dx = \bar{\lambda},$$

hence we obtain that $\bar{\lambda} \ne 0$.

Taking $\varphi \equiv 1$ in the weak form of the equation satisfied by $v_{2,\mu}$ we get that

$$\mu^p \int_\Omega |v_{2,\mu}|^{p-2} v_{2,\mu}\,dx = \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} \int_{\partial\Omega} |v_{2,\mu}|^{p-2} v_{2,\mu}\,d\sigma.$$

Passing again to the limit we have that

$$\tilde{v}_2 \in Y = \{u \in W^{1,p}(\Omega);\ \int_{\partial\Omega} |u|^{p-2}u\, d\sigma = 0\}.$$

Let $w$ be a function where the infimum (2.4) is attained with $\|w\|_{L^p(\partial\Omega)} = 1$. As $w \in A$ (see (1.6)), we have

$$\int_\Omega |\nabla w|^p + \mu^p |w|^p\, dx \geq \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} = \int_\Omega |\nabla v_{2,\mu}|^p + \mu^p |v_{2,\mu}|^p\, dx.$$

Taking the limit as $\mu \to 0+$ we get

$$\tilde{\lambda}_2 = \int_\Omega |\nabla w|^p\, dx \geq \lim_{\mu\to 0} \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} \geq \int_\Omega |\nabla \tilde{v}_2|^p\, dx \geq \inf_{\|z\|_{L^p(\partial\Omega)}=1, z\in Y} \int_\Omega |\nabla z|^p = \tilde{\lambda}_2.$$

Therefore

$$\lim_{\mu\to 0} \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} = \tilde{\lambda}_2$$

and

$$\int_\Omega |\nabla v_{2,\mu}|^p\, dx \to \int_\Omega |\nabla \tilde{v}_2|^p\, dx,$$

from where it follows that

$$v_{2,\mu} \to \tilde{v}_2 \qquad \text{strongly in } W^{1,p}(\Omega).$$

One again, we pass to the limit as $\mu \to 0+$ in the weak formulation satisfied by $v_{2,\mu}$ to get that $\tilde{v}_2$ is an eigenfunction associated to $\tilde{\lambda}_2$. By the characterization of $\tilde{\lambda}_2$ given in Lemma 2.4 we get that this is the first non-zero eigenvalue for problem (1.7). This completes the proof. $\square$

Now we find the behavior of the remaining eigenvalues.

*Proof of Theorem 1.3.* Let $\lambda(\Omega_\mu)$ be an eigenvalue (variational or not). Then, as the variational eigenvalues $\lambda_k(\Omega_\mu)$ form an unbounded sequence, there exists $k$ such that $\lambda_2(\Omega_\mu) \leq \lambda(\Omega_\mu) \leq \lambda_k(\Omega_\mu)$. Now, let $x_1, \dots, x_k \in \partial\Omega$ and $r = r(k)$ be such that $dist(x_i, x_j) > 2r$. Let $\phi \in C^\infty(\Omega)$ be a nonnegative function with support $B(0,r)$ and let $\phi_j(x) = \phi(x - x_j)$.

Now, let us define $S_k = span\{\phi_1, \dots, \phi_k\} \cap \{v \in W^{1,p}(\Omega);\ \|v\|_{W^{1,p}(\Omega)} = 1\}$ and $S_{k,\mu} = \{v(x/\mu);\ v \in S_k\}$, then $\gamma(S_k) = \gamma(S_{k,\mu}) = k$. Hence

$$\frac{1}{\lambda_k(\Omega_\mu)} = \sup_{\gamma(S)\geq k} \inf_{u\in S} \frac{\displaystyle\int_{\partial\Omega_\mu} |u|^p\, d\sigma}{\displaystyle\int_{\Omega_\mu} |\nabla u|^p + |u|^p\, dx} \geq \inf_{u\in S_{k,\mu}} \frac{\displaystyle\int_{\partial\Omega_\mu} |u|^p\, d\sigma}{\displaystyle\int_{\Omega_\mu} |\nabla u|^p + |u|^p\, dx}.$$

Changing variables we get,

$$\frac{1}{\lambda_k(\Omega_\mu)} \geq \mu^{p-1} \inf_{v\in S_k} \frac{\displaystyle\int_{\partial\Omega} |v|^p\, d\sigma}{\displaystyle\int_\Omega |\nabla v|^p + \mu^p |v|^p\, dx}. \tag{2.5}$$

As $\phi_i$ have disjoint support,

$$\|v\|^p_{L^p(\Omega)} = \left\| \sum_{i=1}^k a_i \phi_i \right\|^p_{L^p(\Omega)} = \sum_{i=1}^k |a_i|^p \|\phi_i\|^p_{L^p(\Omega)} \leq \sum_{i=1}^k |a_i|^p \|\phi\|^p_{L^p(B(0,r))}$$

and

$$\|\nabla v\|_{L^p(\Omega)}^p = \left\| \sum_{i=1}^k a_i \nabla \phi_i \right\|_{L^p(\Omega)}^p = \sum_{i=1}^k |a_i|^p \|\nabla \phi_i\|_{L^p(\Omega)}^p \leq \sum_{i=1}^k |a_i|^p \|\nabla \phi\|_{L^p(B(0,r))}^p.$$

As the boundary of $\Omega$ is regular we have that there exists a constant $C_k$ such that

$$\|v\|_{L^p(\partial\Omega)}^p = \left\| \sum_{i=1}^k a_i \phi_i \right\|_{L^p(\partial\Omega)}^p = \sum_{i=1}^k |a_i|^p \|\phi_i\|_{L^p(\partial\Omega)}^p \geq C_k \sum_{i=1}^k |a_i|^p.$$

Using these estimates in (2.5) we obtain

$$0 < c \leq \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} \leq \frac{\lambda(\Omega_\mu)}{\mu^{1-p}} \leq \frac{\lambda_k(\Omega_\mu)}{\mu^{1-p}} \leq C_k < +\infty$$

and the result follows. □

Finally we study the convergence of the eigenvalues and eigenfunctions corresponding to the rest of the spectrum.

*Proof of Theorem 1.4.* By our hypotheses we have that

$$\lim_{j\to\infty} \frac{\lambda_j}{\mu_j^{1-p}} = \lambda.$$

As $v_j$ is bounded in $W^{1,p}(\Omega)$ we can extract a subsequence (that we still call $v_j$) such that

$$\begin{aligned} v_j &\rightharpoonup v \quad \text{weakly in } W^{1,p}(\Omega), \\ v_j &\to v \quad \text{in } L^p(\Omega), \\ v_j &\to v \quad \text{in } L^p(\partial\Omega). \end{aligned}$$

Using that $v_j$ are solutions of (1.2), we obtain

$$\int_\Omega |\nabla v_j|^{p-2} \nabla v_j \nabla \phi + \mu_j^p |v_j|^{p-2} v_j \phi \, dx = \frac{\lambda_j}{\mu_j^{1-p}} \int_{\partial\Omega} |v_j|^{p-2} v_j \phi \, d\sigma. \tag{2.6}$$

Taking $\phi \equiv 1$ we get

$$\int_\Omega \mu_j^p |v_j|^{p-2} v_j \, dx = \frac{\lambda_j}{\mu_j^{1-p}} \int_{\partial\Omega} |v_j|^{p-2} v_j \, d\sigma.$$

The limit as $j \to \infty$ gives us

$$0 = \lambda \int_{\partial\Omega} |v|^{p-2} v \, d\sigma$$

and, as $\lambda \neq 0$, we obtain that

$$0 = \int_{\partial\Omega} |v|^{p-2} v \, d\sigma. \tag{2.7}$$

By Lemma 2.2 and (2.7), there exists a unique $w \in W^{1,p}(\Omega)$ with

$$\int_{\partial\Omega} |w|^{p-2} w \, d\sigma = 0$$

that satisfies

$$(2.8)\qquad \begin{cases} \Delta_p w = 0 & \text{in } \Omega, \\ |\nabla w|^{p-2}\frac{\partial w}{\partial \nu} = \lambda |v|^{p-2} v & \text{on } \partial\Omega. \end{cases}$$

Combining (2.6), the variational formulation of (2.8) with $\phi = v_j - w$ and the fact that we are dealing with a strongly monotone operator (see [3]), we get

$$\begin{aligned} \alpha \quad & \|\nabla v_j - \nabla w\|^p_{L^p(\Omega)} \le \int_\Omega (|\nabla v_j|^{p-2}\nabla v_j - |\nabla w|^{p-2}\nabla w)(\nabla v_j - \nabla w)\, dx \\ = \quad & -\mu_j^p \int_\Omega |v_j|^{p-2} v_j (v_j - w)\, dx + \frac{\lambda_j}{\mu_j^{1-p}} \int_{\partial\Omega} |v_j|^{p-2} v_j (v_j - w)\, d\sigma \\ & -\lambda \int_{\partial\Omega} |v|^{p-2} v (v_j - w)\, d\sigma \\ \le \quad & C\mu_j^p + \left( \frac{\lambda_j}{\mu_j^{1-p}} - \lambda \right) \int_{\partial\Omega} |v_j|^{p-2} v_j (v_j - w)\, d\sigma \\ & +\lambda \int_{\partial\Omega} (|v_j|^{p-2} v_j - |v|^{p-2} v)(v_j - w)\, d\sigma. \end{aligned}$$

The first two terms go to zero as $j \to \infty$. Concerning the last one, we have that it is bounded by

$$\begin{aligned} & (\|v_j\|_{L^p(\partial\Omega)} + \|v\|_{L^p(\partial\Omega)})^{p-2} \|v_j - v\|_{L^p(\partial\Omega)} \|v_j - w\|_{L^p(\partial\Omega)} && \text{if } p \ge 2, \\ & M \|v_j - v\|^{p-1}_{L^p(\partial\Omega)} \|v_j - w\|_{L^p(\partial\Omega)} && \text{if } p < 2. \end{aligned}$$

Therefore, taking the limit $j \to \infty$, we get

$$\nabla v_j \to \nabla w \quad \text{in } L^p(\Omega)$$

and as $\nabla v_j \rightharpoonup \nabla v$ weakly in $L^p(\Omega)$ we conclude that $\nabla v = \nabla w$ and so $v = w$ and $v_j \to v$ strongly in $W^{1,p}(\Omega)$.

Finally, taking limits in (2.6) we obtain that $v$ is a weak solution of (1.7) as we wanted to prove. □

## 3. Behavior as $\mu \to +\infty$

In this section we study the behavior of the Sobolev constant in expanding domains, that is when $\mu \to +\infty$. To clarify the exposition we divide the proof of Theorem 1.5 in several Lemmas. Let us begin by the upper bounds.

**Lemma 3.1.** *Let $p = q$, then there exists a constant $C > 0$ such that*

$$S_p(\Omega_\mu) = \lambda_1(\Omega_\mu) \le C,$$

*for every $\mu$ large.*

*Proof.* We have $p = q$ and look for a bound on the first eigenvalue $\lambda_1(\Omega_\mu)$. Changing variables as before we have that

$$\lambda_1(\Omega_\mu) = \inf_{v\in W^{1,p}(\Omega)} \frac{\mu \left( \displaystyle\int_\Omega \mu^{-p} |\nabla v|^p + |v|^p \, dx \right)}{\displaystyle\int_{\partial\Omega} |v|^p d\sigma}.$$

We choose $v(x)$ such that $v = a = constant$ on $\partial\Omega$ and $v = 0$ in $\Omega_r = \{x \in \Omega \; ; \; dist(x, \partial\Omega) \geq r\}$ with $|\nabla v| \leq C/r$. We fix $a$ such that

$$\int_{\partial\Omega} |v|^p \, d\sigma = 1,$$

that is

$$a = \frac{1}{|\partial\Omega|^{1/p}}.$$

As for $r$ small we have that $|\Omega \setminus \Omega_r| \sim r|\partial\Omega|$ we get

$$\int_\Omega |v|^p \, d\sigma \leq Cr.$$

Using that $|\nabla v| \leq C/r$ we obtain

$$\int_\Omega |\nabla v|^p \, d\sigma \leq \frac{C}{r^{p-1}},$$

therefore

$$\lambda_1(\Omega_\mu) \leq C\mu \left( C\frac{\mu^{-p}}{r^{p-1}} + Cr \right).$$

Finally, choose $r = \mu^{-1}$ to obtain the desired result. $\square$

**Lemma 3.2.** *Let $p < q < p^*$, then there exists a constant $C > 0$ such that*

$$S_q(\Omega_\mu) \leq C,$$

*for every $\mu$ large.*

*Proof.* As we mentioned in the introduction, we have that

$$S_q(\Omega_\mu) = \mu^{(Nq-Np+p)/q} \inf_{v\in W^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega \mu^{-p}|\nabla v|^p + |v|^p \, dx}{\left( \displaystyle\int_{\partial\Omega} |v|^q \, d\sigma \right)^{p/q}}. \tag{3.9}$$

Now, let us choose a point $x_0 \in \partial\Omega$ and let $\phi \in C^\infty(\Omega)$ with support $B(x_0, \mu^{-1})$, and $\|\phi\|^q_{L^q(\partial\Omega)} = 1$.

Arguing as in Section 2, we have that

$$\mu^{(Nq-Np+p)/q} \int_\Omega |\phi|^p \, dx \leq C,$$

and

$$\mu^{(Nq-Np+p)/q} \mu^{-p} \int_\Omega |\nabla\phi|^p \, dx \leq C.$$

Therefore, taking $\phi = v$ in (3.9), we get

$$S_q(\Omega_\mu) \leq C,$$

and this ends the proof. $\square$

**Lemma 3.3.** *Let $1 < q < p$, then we have*

$$S_q(\Omega_\mu) \le C\mu^{(N-1)(q-p)/q},$$

*for some constant $C > 0$. Remark that this says that*

$$\lim_{\mu\to\infty} S_q(\Omega_\mu) = 0.$$

*Proof.* We observe that the same calculations of Lemma 3.2 show that $S_q$ is bounded independently of $\mu$ for $1 < q < p$. Now, as in the case $p = q$ (Lemma 3.1), let us take $v(x)$ such that $v = a = constant$ on $\partial\Omega$ and $v = 0$ in $\Omega_r = \{x \in \Omega\ ;\ dist(x, \partial\Omega) \ge r\}$. We fix $a$ such that

$$\int_{\partial\Omega} |v|^q \, d\sigma = 1.$$

Using the same arguments as in Lemma 3.1 we get

$$S_q(\Omega_\mu) \le C\mu^{(Nq-Np+p)/q} \left( C\frac{\mu^{-p}}{r^{p-1}} + Cr \right)$$

and choosing $r = \mu^{-1}$ we obtain

$$S_q(\Omega_\mu) \le C\mu^{(Nq-Np+p-q)/q},$$

as we wanted to prove. □

Now let us prove that the extremals concentrates at the boundary.

**Lemma 3.4.** *Let $1 < q < p^*$. The extremals concentrate at the boundary in the sense that*

$$\int_\Omega |v_\mu|^p \, dx \to 0 \qquad \textit{as } \mu \to +\infty,$$

*while*

$$\int_{\partial\Omega} |v_\mu|^q \, d\sigma = 1.$$

*Proof.* Let $v_\mu$ be an extremal such that $\|v_\mu\|_{L^q(\partial\Omega)} = 1$. From our previous bound we get, for $p = q$,

$$\mu^{1-p} \int_\Omega |\nabla v_\mu|^p \, dx + \mu \int_\Omega |v_\mu|^p \, dx \le C$$

Hence

$$\int_\Omega |v_\mu|^p \, dx \le \frac{C}{\mu} \to 0 \qquad \text{as } \mu \to +\infty.$$

Now we turn back to the case $1 < q < p$. We have, from our previous calculations,

$$S_q(\Omega_\mu) \le C\mu^{(Nq-Np+p-q)/q}.$$

Hence

$$\int_\Omega |v_\mu|^p \, dx \le C\mu^{(N-1)(q-p)/q} \to 0 \qquad \mu \to +\infty.$$

Finally, for $p < q < p^*$ we get that

$$\mu^{(Nq-Np+p)/q} \int_\Omega |v_\mu|^p \, dx \le C$$

and therefore, as we are in the case $q > p$ and so $Nq > p(N-1)$, we get

$$\int_\Omega |v_\mu|^p \, dx \le \frac{C}{\mu^{(Nq-Np+p)/q}} \to 0 \qquad \text{as } \mu \to +\infty.$$

The proof is now complete. □

To get the bound from below for $\lambda_1$ in the case $p = q$ we use the following idea, first we replace the minimization problem in $W^{1,p}(\Omega)$ with a minimization problem in a sequence of increasing subspaces and next we find that for an adequate choice of the subspaces we get a uniform lower bound for the approximate problems. This idea combined with a convergence result for the approximations gives the desired result. So, let us first state and prove the convergence result. Since this procedure works for every $1 < q < p^*$ we prove it in full generality.

Now we want to describe a general approximation procedure for $S_q$. These results are essentially contained in [14] but we reproduce the main arguments here in order to make the paper self-contained.

The Sobolev trace constant $S_q$ can be characterized as

$$S_q = \inf_{v \in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla v|^p + |v|^p \, dx; \qquad \int_{\partial\Omega} |v|^q \, d\sigma = 1. \right\}. \tag{3.10}$$

As we have already mentioned, the idea is to replace the space $W^{1,p}(\Omega)$ with a subspace $V_h$ in the minimization problem (3.10). To this end, let $V_h$ be an increasing sequence of closed subspaces of $W^{1,p}(\Omega)$, such that

$$\begin{aligned} &\left\{ u_h \in V_h; \int_{\partial\Omega} |u_h|^q \, d\sigma = 1 \right\} \neq \emptyset \\ \text{and} \quad & \\ &\lim_{h \to 0} \inf_{u_h \in V_h} \|v - u_h\|_{W^{1,p}(\Omega)} = 0, \quad \forall \|v\|_{W^{1,p}(\Omega)} = 1. \end{aligned} \tag{3.11}$$

We observe that the only requirement on the subspaces $V_h$ is (3.11). This allows us to choose $V_h$ as the usual finite elements spaces, for example.

With this sequence of subspaces $V_h$ we define our approximation of $S_q$ by

$$S_{q,h} = \inf_{u_h \in V_h} \left\{ \int_\Omega |\nabla u_h|^p + |u_h|^p \, dx; \qquad \int_{\partial\Omega} |u_h|^q \, d\sigma = 1 \right\}. \tag{3.12}$$

We have that, under hypothesis (3.11), $S_{q,h}$ approximates $S_q$ when $h \to 0$.

**Theorem 3.1.** *Let $v$ be an extremal for* (3.10)*. Then, there exists a constant $C$ independent of $h$ such that,*

$$|S_q - S_{q,h}| \le C \inf_{u_h \in V_h} \|u_h - v\|_{W^{1,p}(\Omega)},$$

*for every $h$ small enough.*

*Proof.* As $V_h \subset W^{1,p}(\Omega)$ we have that

$$S_q \le S_{q,h}. \tag{3.13}$$

Let us choose $w \in V_h$ such that $\|w - v\|_{W^{1,p}(\Omega)} \le \inf_{V_h} \|v - u_h\|_{W^{1,p}(\Omega)} + \varepsilon$. We have

$$\begin{aligned} S_{q,h}^{1/p} = \|u_h\|_{W^{1,p}(\Omega)} \le \frac{\|w\|_{W^{1,p}(\Omega)}}{\|w\|_{L^q(\partial\Omega)}} &\le \frac{\|w - v\|_{W^{1,p}(\Omega)} + \|v\|_{W^{1,p}(\Omega)}}{\|w\|_{L^q(\partial\Omega)}} \\ &= \left( \frac{\|w - v\|_{W^{1,p}(\Omega)} + S_q^{1/p}}{\|w\|_{L^q(\partial\Omega)}} \right). \end{aligned}$$

Now we use that

$$|\|w\|_{L^q(\partial\Omega)} - 1| \le |\|w\|_{L^q(\partial\Omega)} - \|v\|_{L^q(\partial\Omega)}| \le \|w - v\|_{L^q(\partial\Omega)} \le C\|w - v\|_{W^{1,p}(\Omega)}$$

and hypothesis (3.11) to obtain that for every $h$ small enough,

$$S_{q,h} \leq \left( \frac{\|w - v\|_{W^{1,p}(\Omega)} + S_q^{1/p}}{1 - C\|w - v\|_{W^{1,p}(\Omega)}} \right)^p \leq S_q + C\|w - v\|_{W^{1,p}(\Omega)}. \tag{3.14}$$

The result follows from (3.13) and (3.14). □

Now we prove a result regarding the convergence of the approximate extremals. We will not use it but it completes the analysis of the approximations.

**Theorem 3.2.** *Let $u_h$ be a function in $V_h$ where the infimum* (3.12) *is achived. Then from any sequence $h \to 0$ we can extract a subsequence $h_j \to 0$ such that $u_{h_j}$ converges strongly to an extremal in $W^{1,p}(\Omega)$. That is, there exists an extremal of* (3.10)*, $v$, with*

$$\lim_{h_j \to 0} \|u_{h_j} - v\|_{W^{1,p}(\Omega)} = 0.$$

*Proof.* Theorem 3.1 and hypothesis (3.11) gives that

$$\lim_{h \to 0} \|u_h\|^p_{W^{1,p}(\Omega)} = \lim_{h \to 0} S_{q,h} = S_q.$$

Hence there exists a constant $C$ such that for every $h$ small enough,

$$\|u_h\|_{W^{1,p}(\Omega)} \leq C.$$

Therefore we can extract a subsequence, that we denote by $u_{h_j}$, such that

$$\begin{aligned} u_{h_j} &\rightharpoonup w \qquad \text{weakly in } W^{1,p}(\Omega), \\ u_{h_j} &\to w \qquad \text{strongly in } L^p(\Omega), \\ u_{h_j} &\to w \qquad \text{strongly in } L^q(\partial\Omega). \end{aligned} \tag{3.15}$$

Hence, from the $L^q(\partial\Omega)$ convergence we have,

$$1 = \lim_{h_j \to 0} \int_{\partial\Omega} |u_{h_j}|^q \, d\sigma = \int_{\partial\Omega} |w|^q \, d\sigma.$$

Therefore $w$ is an admissible function in the minimization problem (3.10). Now we observe that, if $v$ is an extremal,

$$\begin{aligned} \|v\|^p_{W^{1,p}(\Omega)} &\leq \|w\|^p_{W^{1,p}(\Omega)} \leq \liminf_{h_j \to 0} \|u_{h_j}\|^p_{W^{1,p}(\Omega)} \\ &\leq \lim_{h_j \to 0} \|u_{h_j}\|^p_{W^{1,p}(\Omega)} = \lim_{h_j \to 0} S_{q,h} = S_q = \|v\|^p_{W^{1,p}(\Omega)}, \end{aligned}$$

and therefore,

$$\lim_{h_j \to 0} \|u_{h_j}\|_{W^{1,p}(\Omega)} = \|w\|_{W^{1,p}(\Omega)} = S_q^{1/p}. \tag{3.16}$$

The space $W^{1,p}(\Omega)$ being uniformly convex, the weak convergence, (3.15), and the convergence of the norms, (3.16), imply the convergence in norm. Therefore $u_{h_j} \to w$ in $W^{1,p}(\Omega)$. This limit $w$ verifies $\|w\|^p_{W^{1,p}(\Omega)} = S_q$ and $\|w\|_{L^q(\partial\Omega)} = 1$. Hence it is an extremal and we have that

$$\lim_{h_j \to 0} \|u_{h_j} - w\|_{W^{1,p}(\Omega)} = 0,$$

as we wanted to prove. □

With these convergence results we can prove the lower bound in the case $p = q$.

**Lemma 3.5.** *Let $p = q$, then*

$$S_p(\Omega_\mu) = \lambda_1(\Omega_\mu) \geq C,$$

*for every $\mu$ large.*

*Proof.* Let us choose a particular subspace $V_h$ of $W^{1,p}(\Omega)$. As the boundary of $\Omega$ is smooth, we can define new coordinates near the boundary as follows. As before we denote by $\Omega_r = \{x \in \Omega; dist(x, \partial\Omega) \geq r\}$ and by $\partial\Omega_r = \{x \in \Omega; dist(x, \partial\Omega) = r\}$ and we use the following construction. We define

$$\Phi(\xi, r) = \xi - r\nu(\xi),$$

where $\nu(\xi)$ is the exterior normal vector at $\xi \in \partial\Omega$.

$$\Phi : \partial\Omega \times (0, R) \mapsto \Omega \setminus \overline{\Omega}_R.$$

We recall that $\Phi$ is a difeomorphism if $R$ is small enough.

With this application $\Phi$ we can define a triangulation as follows. First, choose a uniform regular triangulation of size $h$ of the set $\partial\Omega \times (0, R)$. Now, by the application $\Phi$ we can get a triangulation of the strip $\Omega \setminus \overline{\Omega}_R$. In fact, we can select as nodes $x_{ij}$ the points $\Phi(\xi_i, r_j)$, where $(\xi_i, r_j)$ is a node of the uniform mesh of $\partial\Omega \times (0, R)$.

Our space $V_h$ is defined by all the continuous functions in $W^{1,p}(\Omega)$ that are linear over each triangle of the strip $\Omega \setminus \overline{\Omega}_R$. This space is the usual space of linear finite elements in special triangulations defined using the mapping $\Phi$, see [3] for detailed information on the finite elements method.

Let us call $u_h$ the functions in $V_h$. We have indexed the nodes $x_{ij}$ in a way such that $x_{i1} \in \partial\Omega$ and $x_{ij}$ is at distance $j - 1$ (in nodes) from the boundary, $\partial\Omega$. We denote by $u_{ij}$ the value of $u_h$ at the node $x_{ij}$ and by $a_{ij}$ the value of the gradient of $u_h$ on the triangle $T_{ij}$. We assume that the index $i$ runs from 1 to $l$ and $j$ from 1 to $k_0$. Remark that $k_0 \sim R/h$ and $l \sim |\partial\Omega|/h^{N-1}$.

We want to find a lower bound (independent of $h$ and $\mu$) on the approximation of the first eigenvalue,

$$\lambda_{1,h}(\Omega_\mu) = \inf_{u_h \in V_h} \left\{ \mu^{1-p} \int_\Omega |\nabla u_h|^p \, dx + \mu \int_\Omega |u_h|^p \, dx; \qquad \int_{\partial\Omega} |u_h|^p \, d\sigma = 1 \right\}.$$

To this end we consider a function $u_h \in V_h$ such that

$$\int_{\partial\Omega} |u_h|^p \, d\sigma = 1,$$

that is

$$\sum_{i=1}^{l} |u_{i1}|^p h^{N-1} \geq C_1$$

Let $k$ be the first integer in $[1, k_0]$ such that

$$\sum_{i=1}^{l} |u_{ik}|^p h^{N-1} \leq \frac{C_1}{2}$$

First, let us observe that if $k = k_0$ (there are $k_0$ triangles between the two boundaries of $\Omega \setminus \Omega_r$), then we have

$$\begin{aligned}\mu\int_\Omega |u_h|^p\,dx \geq & \quad \mu\sum_{j=2}^{k_0}\sum_{i=1}^{l}\int_{T_{ij}}|u_h|^p\,dx \geq C\mu\sum_{j=2}^{k_0}\sum_{i=1}^{l}|u_{ij}|^p h^N\\ = & \quad Ch\mu\sum_{j=2}^{k_0}\sum_{i=1}^{l}|u_{ij}|^p h^{N-1} \geq Ch\mu k_0\frac{C_1}{2}.\end{aligned}$$

As $k_0 \sim R/h$ we get that

$$\lambda_{1,h}(\Omega_\mu) = \inf_{u_h\in V_h}\left\{\mu^{1-p}\int_\Omega |\nabla u_h|^p\,dx + \mu\int_\Omega |u_h|^p\,dx; \qquad \int_{\partial\Omega}|u_h|^p\,d\sigma = 1\right\} \geq$$
$$\inf_{u_h\in V_h}\left\{\mu\int_\Omega |u_h|^p\,dx; \qquad \int_{\partial\Omega}|u_h|^p\,d\sigma = 1\right\} \geq C\mu > 1$$

and we are done. Hence let us assume that $k < k_0$. As before we can bound the term $\mu\int_\Omega |u_h|^p$ by

$$\mu\int_\Omega |u_h|^p\,dx \geq C\mu\sum_{j=2}^{k}\sum_{i=1}^{l}|u_{ij}|^p h^N = Ch\mu\sum_{j=2}^{k}\sum_{i=1}^{l}|u_{ij}|^p h^{N-1} \geq Ch\mu k\frac{C_1}{2}. \tag{3.17}$$

Now we observe that

$$u_{i1} - u_{ik} = \sum_{j=1}^{k} a_{ij}h.$$

Using this fact we get,

$$\begin{aligned}C &\leq \left|\left(\frac{1}{l}\sum_{i=1}^{l}|u_{i1}|^p\right)^{1/p} - \left(\frac{1}{l}\sum_{i=1}^{l}|u_{ik}|^p\right)^{1/p}\right|\\ &\leq \left(\frac{1}{l}\sum_{i=1}^{l}|u_{i1}-u_{ik}|^p\right)^{1/p} = \left(\frac{k^p}{l}\sum_{i=1}^{l}\left|\frac{1}{k}\sum_{j=1}^{k}a_{ij}h\right|^p\right)^{1/p}.\end{aligned}$$

Hence we get

$$\frac{Cl}{k^{p-1}h^p} \leq \sum_{i=1}^{l}\frac{1}{k}\sum_{j=1}^{k}|a_{ij}|^p$$

and finally,

$$\mu^{1-p}\int_\Omega |\nabla u_h|^p\,dx \geq \frac{C\mu^{1-p}lh^{N-1}}{k^{p-1}h^{p-1}} \geq \frac{C\mu^{1-p}}{k^{p-1}h^{p-1}}. \tag{3.18}$$

Using (3.17) and (3.18) we obtain

$$\begin{aligned}\lambda_{1,h}(\Omega_\mu) &= \inf_{u_h\in V_h}\left\{\mu^{1-p}\int_\Omega |\nabla u_h|^p\,dx + \mu\int_\Omega |u_h|^p\,dx; \quad \int_{\partial\Omega}|u_h|^p\,d\sigma = 1\right\}\\ &\geq C(\mu hk) + \frac{C}{(\mu hk)^{p-1}}.\end{aligned}$$

Hence, if we call $\tau = \mu hk$ we get that

$$\lambda_{1,h}(\Omega_\mu) \geq F(\tau) \equiv C\tau + \frac{C}{\tau^{p-1}} \geq C.$$

Since the subspaces that we have chosen verify hypotheses (3.11), we can use the convergence result, Theorem 3.1, to get that

$$\lambda_1(\Omega_\mu) = \lim_{h\to 0} \lambda_{1,h}(\Omega_\mu) \geq C,$$

as we wanted to prove. □

Let us look at the case $1 < q < p$ more carefully, and obtain a bound from below using the lower bound obtained for $\lambda_1(\Omega_\mu)$.

**Lemma 3.6.** *Let $1 < q < p$. Then, for every $\mu$ large,*

$$S_q(\Omega_\mu) \geq C\mu^{\beta_{pq}-1}.$$

*Moreover this shows that, if $v$ is an extremal,*

$$c_1 \left( \int_{\partial\Omega} |v|^q \, d\sigma \right)^{1/q} \geq \left( \int_{\partial\Omega} |v|^p \, d\sigma \right)^{1/p} \geq c_2 \left( \int_{\partial\Omega} |v|^q \, d\sigma \right)^{1/q}.$$

*Hence there is no peaking formation in this case.*

*Proof.* As we mentioned in the introduction, we have that

$$\begin{aligned} S_q(\Omega_\mu) = \; & \mu^{(Nq-Np+p)/q} \inf_{v\in W^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega \mu^{-p}|\nabla v|^p + |v|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q \, d\sigma\right)^{p/q}} \\ = \; & \mu^{\beta_{pq}-1} \inf_{v\in W^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega \mu^{1-p}|\nabla v|^p + \mu|v|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q \, d\sigma\right)^{p/q}} \\ = \; & \mu^{\beta_{pq}-1} \inf_{v\in W^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega \mu^{1-p}|\nabla v|^p + \mu|v|^p \, dx}{\displaystyle\int_{\partial\Omega} |v|^p \, d\sigma} \frac{\displaystyle\int_{\partial\Omega} |v|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q \, d\sigma\right)^{p/q}}. \end{aligned}$$

Using that $1 < q < p$ we get that, by Holder's inequality

$$\frac{\displaystyle\int_{\partial\Omega} |v|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q \, d\sigma\right)^{p/q}} \geq C.$$

Hence, using our previous lower bound for $\lambda_1(\Omega_\mu)$ we get that there exists a constant $C$ such that

$$S_q(\Omega_\mu) \geq C\mu^{\beta_{pq}-1}.$$

The upper bound proved in Lemma 3.3, $S_q(\Omega_\mu) \le C\mu^{\beta_{pq}-1}$, gives that

$$C\mu^{\beta_{pq}-1} \ge S_q(\Omega_\mu) = \mu^{\beta_{pq}-1} \frac{\displaystyle\int_\Omega \mu^{1-p}|\nabla v_\mu|^p + \mu |v_\mu|^p \, dx}{\displaystyle\int_{\partial\Omega} |v_\mu|^p \, d\sigma} \frac{\displaystyle\int_{\partial\Omega} |v_\mu|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |v_\mu|^q \, d\sigma\right)^{p/q}}$$

$$\ge C\mu^{\beta_{pq}-1} \frac{\displaystyle\int_{\partial\Omega} |v_\mu|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |v_\mu|^q \, d\sigma\right)^{p/q}}.$$

Hence

$$\int_{\partial\Omega} |v_\mu|^p \, dx \le C \left(\int_{\partial\Omega} |v_\mu|^q \, d\sigma\right)^{p/q}.$$

This ends the proof. □

To finish the proof of Theorem 1.5 we need the following Lemma.

**Lemma 3.7.** *Let $p < q < p^*$. Then, for large $\mu$,*

$$S_q(\Omega_\mu) \ge C.$$

*Moreover, the extremals concentrates in the sense that*

$$a^p|\partial\Omega \cap \{v_\mu > a\}| \to 0, \qquad \text{as } \mu \to +\infty,$$

*with*

$$\max_{\overline{\Omega}} v_\mu = 1.$$

*Proof.* First we prove that there exists a constant $C$ such that $S_q(\Omega_\mu) \ge C$. Let $v_\mu$ be an extremal in $\Omega$. By rescaling $v_\mu$ we can obtain an extremal $\tilde{v}_\mu$ such that $\max_{\overline{\Omega}} \tilde{v}_\mu = 1$. That is, $0 < \tilde{v}_\mu \le 1$ and there exits a point $x_0 \in \partial\Omega$ with $\tilde{v}_\mu(x_0) = 1$. Arguing as in Lemma 3.6 we have

$$S_q(\Omega_\mu) = \mu^{\beta_{pq}-1} \frac{\displaystyle\int_\Omega \mu^{1-p}|\nabla \tilde{v}_\mu|^p + \mu |\tilde{v}_\mu|^p \, dx}{\displaystyle\int_{\partial\Omega} |\tilde{v}_\mu|^p \, d\sigma} \frac{\displaystyle\int_{\partial\Omega} |\tilde{v}_\mu|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |\tilde{v}_\mu|^q \, d\sigma\right)^{p/q}}. \tag{3.19}$$

As $\tilde{v}_\mu$ satisfies (1.2), by our hypothesis, we have that

$$|\nabla \tilde{v}_\mu| \le \frac{C}{\mu}.$$

Hence

$$\{x \in \partial\Omega;\, \tilde{v}_\mu(x) \ge 1/2\} \supseteq B(x_0, c/\mu) \cap \partial\Omega.$$

As $q > p$ and $0 < \tilde{v}_\mu \le 1$ we have that

$$\int_{\partial\Omega} |\tilde{v}_\mu|^p \, d\sigma \ge \int_{\partial\Omega} |\tilde{v}_\mu|^q \, d\sigma.$$

Therefore

$$\mu^{\beta_{pq}-1}\frac{\displaystyle\int_{\partial\Omega}|\tilde v_\mu|^p\,dx}{\left(\displaystyle\int_{\partial\Omega}|\tilde v_\mu|^q\,d\sigma\right)^{p/q}} \geq \mu^{\beta_{pq}-1}\left(\int_{\partial\Omega}|\tilde v_\mu|^p\,dx\right)^{(q-p)/q}$$

$$\geq C\mu^{\beta_{pq}-1}\left(\int_{\partial\Omega\cap B(x_0,c/\mu)}\frac{1}{2^p}\,dx\right)^{(q-p)/q} \geq C.$$

Using this bound and the lower bound for $S_p(\Omega_\mu)$ in (3.19) we get the desired lower bound.

Next, we prove the concentration property for the extremals. Using the same arguments as before, we get

$$a^p|\partial\Omega\cap\{\tilde v_\mu > a\}| \leq \int_{\partial\Omega}|\tilde v_\mu|^p\,d\sigma \leq \frac{C}{\mu^{N-1}} \to 0, \qquad \text{as } \mu\to+\infty,$$

with

$$\max_{\Omega}\tilde v_\mu = 1.$$

This proves the concentration phenomena for the extremals. $\square$

We end the article proving that every eigenvalue is bounded as $\mu\to+\infty$.

*Proof of Theorem 1.6.* The idea is similar as the one used in the proof of Theorem 1.3, see Section 2.

Let $x_1,\dots,x_k\in\partial\Omega$ such that $dist(x_i,x_j)>2\mu$ and let $\phi_j\in C^\infty(\Omega)$ with support $B(x_j,\mu)$ and $\max\phi_j=1$.

Now, let us define $S_k = span\{\phi_1,\dots,\phi_k\}\cap\{u\in W^{1,p}(\Omega);\ \|u\|_{W^{1,p}(\Omega)}=1\}$ and $S_{k,\mu}=\{v(x/\mu);\ v\in S_k\}$. Then, $\gamma(S_k)=\gamma(S_{k,\mu})=k$. Hence

$$\frac{1}{\lambda_k(\Omega_\mu)} = \sup_{\gamma(S)\geq k}\inf_{u\in S}\frac{\displaystyle\int_{\partial\Omega_\mu}|u|^p\,d\sigma}{\displaystyle\int_{\Omega_\mu}|\nabla u|^p+|u|^p\,dx} \geq \inf_{u\in S_{k,\mu}}\frac{\displaystyle\int_{\partial\Omega_\mu}|u|^p\,d\sigma}{\displaystyle\int_{\Omega_\mu}|\nabla u|^p+|u|^p\,dx}.$$

Changing variables we get,

$$\frac{1}{\lambda_k(\Omega_\mu)} \geq \mu^{p-1}\inf_{v\in S_k}\frac{\displaystyle\int_{\partial\Omega}|v|^p\,d\sigma}{\displaystyle\int_{\Omega}|\nabla v|^p+\mu^p|v|^p\,dx}. \tag{3.20}$$

As $\phi_i$ have disjoint support,

$$\|v\|^p_{L^p(\Omega)} = \left\|\sum_{i=1}^k a_i\phi_i\right\|^p_{L^p(\Omega)} = \sum_{i=1}^k|a_i|^p\|\phi_i\|^p_{L^p(\Omega)} \leq C\sum_{i=1}^k|a_i|^p\mu^{-N}$$

and

$$\|\nabla v\|^p_{L^p(\Omega)} = \left\|\sum_{i=1}^k a_i\nabla\phi_i\right\|^p_{L^p(\Omega)} = \sum_{i=1}^k|a_i|^p\|\nabla\phi_i\|^p_{L^p(\Omega)} \leq C\sum_{i=1}^k|a_i|^p\mu^{-N+p}.$$

As the boundary of $\Omega$ is regular we have that there exists a constant $C$ such that

$$\|v\|^p_{L^p(\partial\Omega)} = \left\| \sum_{i=1}^k a_i \phi_i \right\|^p_{L^p(\partial\Omega)} = \sum_{i=1}^k |a_i|^p \|\phi_i\|^p_{L^p(\partial\Omega)} \geq C \sum_{i=1}^k |a_i|^p \mu^{1-N}.$$

Using these estimates in (3.20) we obtain

$$0 < c \leq \lambda_1(\Omega_\mu) \leq \lambda(\Omega_\mu) \leq \lambda_k(\Omega_\mu) \leq C_k < +\infty.$$

The result follows. □

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