NEUMANN BOUNDARY CONDITIONS FOR THE INFINITY LAPLACIAN AND THE MONGE-KANTOROVICH MASS TRANSPORT PROBLEM

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ABSTRACT. In this note we review some recent results concerning the natural Neumann boundary condition for the ∞-Laplacian and its relation with the Monge-Kantorovich mass transport problem.

(1) We study the limit as $p \to \infty$ of solutions of $-\Delta_p u_p = 0$ in a domain Ω with $|Du_p|^{p-2}\partial u_p/\partial\nu = g$ on $\partial\Omega$. We obtain a natural minimization problem that is verified by a limit point of $\{u_p\}$ and a limit problem that is satisfied in the viscosity sense. It turns out that the limit variational problem is related to the Monge-Kantorovich mass transport problems when the measures are supported on $\partial\Omega$.
(2) Next, we study the limit of Monge-Kantorovich mass transport problems when the involved measures are supported in a small strip near the boundary of a bounded smooth domain, Ω. Given an absolutely continuous measure (with respect to the surface measure) supported on the boundary $\partial\Omega$ with zero mean value, by performing a suitable extension of the measures to a strip of width ε near the boundary of the domain Ω we consider the mass transfer problem for the extensions. Then we study the limit as ε goes to zero of the Kantorovich potentials for the extensions and obtain that it coincides with a solution of the original mass transfer problem.
(3) Also we present a Steklov like eigenvalue problem that appears as the limit of the usual Steklov eigenvalue problem for the $p-$Laplacian as $p \to \infty$.

1. INTRODUCTION. THE MONGE-KANTOROVICH MASS TRANSPORTATION PROBLEM AND THE $\infty-$LAPLACIAN

In this note we review some recent results obtained by the authors in [8], [13], [14] and [15]. We study the Monge-Kantorovich mass transport problem when the involved measures are supported on the boundary of the domain. This problem is related to the natural Neumann boundary conditions that appear when one considers the ∞-Laplacian in a smooth bounded domain as limit of the Neumann problem for the p-Laplacian as $p \to \infty$.

To formalize the mass transportation problem, let g be a measure with zero total mass and let Ω be a domain with $supp(g) \subset \overline{\Omega}$. We want to determine the most

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efficient way of transport the measures g_+ to g_-, that is, we want to find a function $T : supp(g_+) \to supp(g_-)$ in such a way that T minimizes the total transport cost

$$L(T) = \int_\Omega |x - T(x)| dg_+(x).$$

We refer to [24] and [10] for references and details.

On the other hand, let $\Delta_p u = \mathrm{div}\left(|Du|^{p-2} Du\right)$ be the p-Laplacian. The ∞-Laplacian is the limit operator $\Delta_\infty = \lim_{p\to\infty} \Delta_p$ given by

$$\Delta_\infty u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial x_i} \frac{\partial u}{\partial x_i}$$

in the viscosity sense (to be more precise, in the sense that a uniform limit of $p-$harmonic functions (solutions to $\Delta_p u = 0$) is $\infty-$harmonic (a solution to $\Delta_\infty u = 0$)). This operator appears naturally when one considers absolutely minimizing Lipschitz extensions of a boundary function f; see [2], [3], and [17]. A fundamental result of Jensen [17] establishes that the Dirichlet problem for Δ_∞ is well posed in the viscosity sense.

The close relation between the Monge-Kantorovich problem and the limit as $p \to \infty$ for solutions to $\Delta_p u = f$ was first noticed by Evans and Gangbo in [11]. They considered mass transfer optimization problems between absolutely continuous measures (with respect to the Lebesgue measure) that appear as limits of p-Laplacian problems. A very general approach is discussed in [7], where a problem related to but different from ours is discussed (see Remark 4.3 in [7]).

Here we study the Neumann problem for the ∞-Laplacian obtained as the limit as $p \to \infty$ of the problems

$$\left\{ \begin{array}{ll} -\Delta_p u = 0 & \text{in } \Omega, \\ |Du|^{p-2} \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{array} \right. \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. The boundary data g is a continuous function that necessarily verifies the compatibility condition $\int_{\partial\Omega} g = 0$, otherwise there is no solution to (1). Imposing the normalization $\int_\Omega u = 0$ there exists a unique solution to problem (1) that we denote by u_p.

We will find a variational problem that is verified by a limit point of $\{u_p\}$ and a limit partial differential equation that is satisfied in the viscosity sense. Next, we will study the limit of Monge-Kantorovich mass transfer problems when the involved measures are supported in a small strip near the boundary. Given an absolutely continuos measure (with respect to the surface measure) supported on the boundary $\partial\Omega$ with zero mean value, by performing a suitable extension of the measures to a strip of width ε near the boundary of the domain Ω we consider the mass transfer problem for the extensions. Then we study the limit as ε goes to zero of the Kantorovich potentials for the extensions and obtain that it coincides with a solution of the original mass transfer problem. Moreover, recent results from game theory allow to give a probabilistic interpretation of the infinity Laplacian (see Section 5 for details). Here, we use these results and show that the PDE that is solved by the continuous value of the game is actually a mixed boundary value problem for the infinity Laplacian. In addition, this game theory interpretation provides a proof of the uniqueness of viscosity solutions to this mixed problem. Also, at the end of this paper, we will indicate a Steklov like eigenvalue problem that appears as the limit of the usual Steklov eigenvalue problem for the $p-$Laplacian as $p \to \infty$.

When considering the Neumann problem, boundary conditions that involve the outer normal derivative, $\partial u/\partial \nu$ have been addressed from the point of view of viscosity solutions for fully nonlinear equations in [4] and [16]. In these references it is proved that there exist viscosity solutions and comparison principles between them when appropriate hypothesis are satisfied. In particular a suitable strict monotonicity is needed and such property does not hold in our case of interest.

The rest of the paper is organized as follows: in Section 2 we deal with a variational setting, in Section 3 we perform a viscosity analysis of the limit as $p \to \infty$ in (1), in Section 4 we approximate these problems (both variationally and in the viscosity sense) by problems with measures supported in small strips near the boundary, in Section 5 we present some results using a game theory approach and finally in Section 6 we present a related eigenvalue problem.

2. A variational approach

A solution to (1) can be obtained by a variational principle. In fact, up to a Lagrange multiplier, $\lambda_p \to 1$ as $p \to \infty$, we can write

$$(2) \qquad \int_{\partial\Omega} u_p\, g = \max\left\{ \int_{\partial\Omega} wg \colon w \in W^{1,p}(\Omega),\ \int_\Omega w = 0\,,\ \int_\Omega |Dw|^p \le 1 \right\}.$$

Our first result states that there exist accumulation points of the family $\{u_p\}_{p>1}$ as $p \to \infty$ which are maximizers of a variational problem that is the natural limit of variational problems (2). Observe that for $q > N$ the set $\{u_p\}_{p>q}$ is bounded in $C^{1-p/q}(\overline{\Omega})$.

Theorem 1. *Let v_∞ be a uniform limit of a subsequence $\{u_{p_i}\}$, $p_i \to \infty$, then v_∞ is a solution to the maximization problem*

$$(3) \qquad \int_{\partial\Omega} v_\infty g = \max\left\{ \int_{\partial\Omega} wg \colon w \in W^{1,\infty}(\Omega),\ \int_\Omega w = 0,\ \|Dw\|_\infty \le 1 \right\}.$$

An equivalent dual statement is the minimization problem

$$(4) \qquad \|Dv_\infty\|_\infty = \min\left\{ \|Dw\|_\infty \colon w \in W^{1,\infty}(\Omega),\ \int_\Omega w = 0,\ \int_{\partial\Omega} wg \ \ge 1 \right\}.$$

The maximization problem (3) is also obtained by applying the Kantorovich optimality principle to a mass transfer problem for the measures $\mu^+ = g^+\mathcal{H}^{N-1} \llcorner\, \partial\Omega$ and $\mu^- = g^-\mathcal{H}^{N-1} \llcorner\, \partial\Omega$ that are concentrated on $\partial\Omega$. The mass transfer compatibility condition $\mu^+(\partial\Omega) = \mu^-(\partial\Omega)$ holds since g has zero average on $\partial\Omega$. The maximizers of (3) are called maximal Kantorovich potentials [1].

To prove Theorem 1 we follow [13]. We review some previous estimates.

Suppose that we have a sequence $\{u_p\}$ of solutions to (1). Since we are interested in large values of p we may assume that $p > N$ and hence $u_p \in C^\alpha(\overline{\Omega})$. Multiplying the equation by u_p and integrating we obtain,

$$\int_\Omega |Du_p|^p = \int_{\partial\Omega} u_p\, g \le \left(\int_{\partial\Omega} |u_p|^p \right)^{1/p} \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p'} \tag{5}$$

where p' is the exponent conjugate to p, that is $1/p' + 1/p = 1$. Recall the following trace inequality, see for example [9],

$$\int_{\partial\Omega} |\phi|^p d\sigma \le Cp \left(\int_\Omega |\phi|^p + |D\phi|^p dx \right),$$

where C is a constant that does not depend on p. Going back to (5), we get,

$$\int_\Omega |Du_p|^p \le \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p'} C^{1/p} p^{1/p} \left(\int_\Omega |u_p|^p + |Du_p|^p dx \right)^{1/p}.$$

On the other hand, for large p we have

$$|u_p(x) - u_p(y)| \le C_p |x-y|^{1-\frac{N}{p}} \left(\int_\Omega |Du_p|^p dx \right)^{1/p}.$$

Since we are assuming that $\int_\Omega u_p = 0$, we may choose a point y such that $u_p(y) = 0$, and hence

$$|u_p(x)| \le C(p,\Omega) \left(\int_\Omega |Du_p|^p dx \right)^{1/p}.$$

The arguments in [9], pag. 266-267, show that the constant $C(p,\Omega)$ can be chosen uniformly in p. Hence, we obtain

$$\int_\Omega |Du_p|^p \le \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p'} C^{1/p} p^{1/p} (C_2^p + 1)^{1/p} \left(\int_\Omega |Du_p|^p dx \right)^{1/p}.$$

Taking into account that $p' = p/(p-1)$, for large values of p we get

$$\left(\int_\Omega |Du_p|^p \right)^{1/p} \le \alpha_p \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p}$$

where $\alpha_p \to 1$ as $p \to \infty$. Next, fix m, and take $p > m$. We have,

$$\left(\int_\Omega |Du_p|^m \right)^{1/m} \le |\Omega|^{\frac{1}{m}-\frac{1}{p}} \left(\int_\Omega |Du_p|^p \right)^{1/p} \le |\Omega|^{\frac{1}{m}-\frac{1}{p}} \alpha_p \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p},$$

where $|\Omega|^{\frac{1}{m}-\frac{1}{p}} \to |\Omega|^{\frac{1}{m}}$ as $p \to \infty$. Hence, there exists a weak limit in $W^{1,m}(\Omega)$ that we will denote by v_∞. This weak limit has to verify

$$\left(\int_\Omega |Dv_\infty|^m \right)^{1/m} \le |\Omega|^{\frac{1}{m}}.$$

As the above inequality holds for every m, we get that $v_\infty \in W^{1,\infty}(\Omega)$ and moreover, taking the limit $m \to \infty$,

$$|Dv_\infty| \le 1, \qquad \text{a.e. } x \in \Omega.$$

Lemma 1. *The subsequence u_{p_i} converges to v_∞ uniformly in $\overline{\Omega}$.*

Proof. From our previous estimates we know that

$$\left(\int_\Omega |Du_p|^p dx \right)^{1/p} \le C,$$

uniformly in p. Therefore we conclude that u_p is bounded (independently of p) and has a uniform modulus of continuity. Hence u_p converges uniformly to v_∞. □

Proof of Theorem 1. Multiplying by u_p, passing to the limit, and using Lemma 1, we obtain,

$$\lim_{p\to\infty} \int_\Omega |Du_p|^p = \lim_{p\to\infty} \int_{\partial\Omega} u_p g = \int_{\partial\Omega} v_\infty g.$$

If we multiply (1) by a test function w, we have, for large enough p,

$$\begin{aligned} \int_{\partial\Omega} wg \quad &\le \left(\int_\Omega |Du_p|^p \right)^{(p-1)/p} \left(\int_\Omega |Dw|^p \right)^{1/p} \\ &\le \left(\int_{\partial\Omega} v_\infty g d\sigma + \delta \right)^{(p-1)/p} \left(\int_\Omega |Dw|^p \right)^{1/p}. \end{aligned}$$

As the previous inequality holds for every $\delta > 0$, passing to the limit as $p \to \infty$ we conclude,

$$\int_{\partial\Omega} wg \le \left(\int_{\partial\Omega} v_\infty g \right) \|Dw\|_\infty.$$

Hence, the function v_∞ verifies,

$$\int_{\partial\Omega} v_\infty g = \max \left\{ \int_{\partial\Omega} wg \, : \, w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_\infty \le 1 \right\},$$

or equivalently,

$$\|Dv_\infty\|_\infty = \min \left\{ \|Dw\|_\infty \, : \, w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \int_{\partial\Omega} wg \le 1 \right\}.$$

This ends the proof. □

On the other hand, taking as a test function in the maximization problem v_∞ itself we obtain the following corollary.

Corollary 1. *If $g \not\equiv 0$, then $\|Dv_\infty\|_{L^\infty(\Omega)} = 1$.*

3. Viscosity setting

In this section we discuss the equation that v_∞, a uniform limit of u_p as $p \to \infty$, satisfies in the viscosity sense.

Following [4] let us recall the definition of viscosity solution taking into account general boundary conditions. Assume

$$F : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{S}^{N\times N} \to \mathbb{R}$$

a continuous function. The associated equation

$$F(x, Du, D^2u) = 0$$

is called (degenerate) elliptic if

$$F(x, \xi, X) \le F(x, \xi, Y) \quad \text{if } X \ge Y.$$

Definition 1. *Consider the boundary value problem*

$$\begin{cases} F(x,\nabla u, D^2 u)=0 & \text{in } \Omega, \\ B(x,u,\nabla u)=0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

(1) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi\in C^2(\overline{\Omega})$ such that $u-\phi$ has a strict minimum at the point $x_0\in\overline{\Omega}$ with $u(x_0)=\phi(x_0)$ we have: If $x_0\in\partial\Omega$ the inequality*

$$\max\{B(x_0,\phi(x_0),\nabla\phi(x_0)),\ F(x_0,\nabla\phi(x_0),D^2\phi(x_0))\}\geq 0$$

holds and if $x_0\in\Omega$ then we require

$$F(x_0,\nabla\phi(x_0),D^2\phi(x_0))\geq 0.$$

(2) *An upper semi-continuous function u is a subsolution if for every $\psi\in C^2(\overline{\Omega})$ such that $u-\psi$ has a strict maximum at the point $x_0\in\overline{\Omega}$ with $u(x_0)=\psi(x_0)$ we have: If $x_0\in\partial\Omega$ the inequality*

$$\min\{B(x_0,\psi(x_0),\nabla\psi(x_0)),\ F(x_0,\nabla\psi(x_0),D^2\psi(x_0))\}\leq 0$$

holds, and if $x_0\in\Omega$ then we require

$$F(x_0,\nabla\psi(x_0),D^2\psi(x_0))\leq 0.$$

(3) *Finally, u is a viscosity solution if it is a super and a subsolution.*

The main result in this section is the following theorem.

Theorem 2. *A limit v_∞ is a solution of*

$$\begin{cases} \Delta_\infty u=0 & \text{in } \Omega, \\ B(x,u,Du)=0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

in the viscosity sense. Here

$$B(x,u,Du)\equiv\begin{cases} \min\left\{|Du|-1\ ,\ \frac{\partial u}{\partial\nu}\right\} & \text{if } g(x)>0, \\ \max\{1-|Du|\ ,\ \frac{\partial u}{\partial\nu}\} & \text{if } g(x)<0, \\ H(|Du|)\frac{\partial u}{\partial\nu} & \text{if } g(x)=0, \\ \frac{\partial u}{\partial\nu}=0 & \text{if } x\in\{g(x)=0\}^o, \end{cases}$$

where $\{g(x)=0\}^o$ is the interior of the zero level set of g and $H(a)$ is given by

$$H(a)=\begin{cases} 1 & \text{if } a\geq 1, \\ 0 & \text{if } 0\leq a<1. \end{cases}$$

Proof of Theorem 2. Again we follow [13]. First, let us check that $-\Delta_\infty v_\infty=0$ in the viscosity sense in Ω. Let us recall the standard proof. Let ϕ be a smooth test function such that $v_\infty-\phi$ has a strict maximum at $x_0\in\Omega$. Since u_{p_i} converges uniformly to v_∞ we get that $u_{p_i}-\phi$ has a maximum at some point $x_i\in\Omega$ with $x_i\to x_0$. Next we use the fact that u_{p_i} is a viscosity solution (see [13] for a proof of this fact) of

$$-\Delta_{p_i}u_{p_i}=0$$

and we obtain

$$-(p_i-2)|D\phi|^{p_i-4}\Delta_\infty\phi(x_i)-|D\phi|^{p_i-2}\Delta\phi(x_i)\leq 0. \tag{8}$$

If $D\phi(x_0) = 0$ we get $-\Delta_\infty \phi(x_0) \le 0$. If this is not the case, we have that $D\phi(x_i) \neq 0$ for large i and then

$$-\Delta_\infty \phi(x_i) \le \frac{1}{p_i - 2} |D\phi|^2 \Delta\phi(x_i) \to 0, \quad \text{as } i \to \infty.$$

We conclude that

$$-\Delta_\infty \phi(x_0) \le 0.$$

That is v_∞ is a viscosity subsolution of $-\Delta_\infty u = 0$.

A similar argument shows that v_∞ is also a supersolution and therefore a solution of $-\Delta_\infty v_\infty = 0$ in Ω.

Let us check the boundary condition. There are six cases to be considered. Here we deal only with one and refer to [13] for the rest of the cases.

Assume that $v_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$. Using the uniform convergence of u_{p_i} to v_∞ we obtain that $u_{p_i} - \phi$ has a minimum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \phi(x_0) \ge 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|D\phi|^{p_i - 2}(x_i) \frac{\partial \phi}{\partial \nu}(x_i) \ge g(x_i).$$

Since $g(x_0) > 0$, we have $D\phi(x_0) \neq 0$, and we obtain

$$|D\phi|(x_0) \ge 1.$$

Moreover, we also have

$$\frac{\partial \phi}{\partial \nu}(x_0) \ge 0.$$

Hence, if $v_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$, we have

$$\max\left\{ \min\{-1 + |D\phi|(x_0), \frac{\partial \phi}{\partial \nu}(x_0)\} \ , -\Delta_\infty \phi(x_0) \right\} \ge 0. \tag{9}$$

This ends the proof. □

Remark 1. *The function v_∞ is a viscosity solution of $\Delta_\infty v_\infty = 0$ in Ω and therefore it is an absolutely minimizing function,* [3]. *It is a minimizer of the Lipschitz constant of u among functions that coincide with v_∞ on $\partial\Omega'$ in every subdomain Ω' of Ω. Therefore we can rewrite the maximization problem* (3) *as a maximization problem on $\partial\Omega$: $v_\infty|_{\partial\Omega}$ is a function that has Lipschitz constant less or equal than one on $\partial\Omega$ and maximizes $\int_{\partial\Omega} ug$.*

Concerning the limit PDE, note that there is no uniqueness of viscosity solutions of (7), see [13]. Nevertheless we can say something about uniqueness under some favorable geometric assumptions on g and Ω. The proof of uniqueness is based on some tools from [11]. To state our uniqueness result let us describe the required geometrical hypothesis on the boundary data. Let $\partial\Omega_+ = \operatorname{supp} g^+$ and $\partial\Omega_- = \operatorname{supp} g^-$. For a given v_∞ a maximizer in (3) following [11] we define the transport set as

$$T(v_\infty) = \left\{ \begin{array}{rl} z \in \overline{\Omega} : \exists x \in \partial\Omega_+, y \in \partial\Omega_-, & v_\infty(z) = v_\infty(x) - |x - z| \\ \text{and} & v_\infty(z) = v_\infty(y) + |y - z| \end{array} \right\}.$$

Observe that this set T is closed. We have the following property (see [11])

Proposition 1. *Suppose that Ω is a convex domain. Let v_∞ be a maximizer of (3) with $\Delta_\infty v_\infty = 0$, then $|Dv_\infty(x)| = 1$, for a.e. $x \in T(v_\infty)$.*

Define a transport ray by $R_x = \{z \,|\, |v_\infty(x) - v_\infty(z)| = |x - z|\}$. Notice that two transport rays cannot intersect in Ω unless they are identical. Indeed, assume $z \in T$ then there exist $x, y \in \overline{\Omega}$ such that $v_\infty(x) - v_\infty(z) = |x - z|$ and $v_\infty(z) - v_\infty(y) = |z - y|$, then $|x - y| \le |x - z| + |z - y| = v_\infty(x) - v_\infty(y)$. If x, y and z are not colinear we contradict the Lipschitz condition verified by v_∞.

Our first geometric hypothesis for uniqueness is then

$$\partial\Omega \subset T(v_\infty).$$

Note that with similar ideas but using the uniqueness of viscosity solutions to a mixed problem for the infinity Laplacian, this hypothesis can be relaxed (see the last remark of Section 3).

We have:

Theorem 3. *Assume that we have a convex domain Ω and a boundary datum g on $\partial\Omega$ such that every maximizer v_∞ with $\Delta_\infty v_\infty = 0$ verifies $\partial\Omega \subset T(v_\infty)$, then there exists a unique infinite harmonic solution, u_∞ to (3). Hence, the limit $\lim_{p\to\infty} u_p = u_\infty$, uniformly in Ω exists.*

Remark 2. *Observe that if $\{g = 0\}$ has empty interior on the boundary then the uniqueness of the limit holds since for every v_∞ we get $\partial\Omega \subset T(v_\infty)$.*

Examples. To illustrate our results we present some examples. In an interval $\Omega = (-L, L)$ with $g(L) = -g(-L) > 0$ the limit of the solutions of (1), u_p, turns out to be $u_\infty(x) = x$. It is easy to check that this function is indeed the unique solution of the maximization problem (3) and of the problem (7).

This example can be easily generalized to the case where Ω is an annulus, $\Omega = \{r_1 < |x| < r_2\}$, and the function g is a positive constant g_1 on $|x| = r_1$ and a negative constant g_2 on $|x| = r_2$ with the constraint $\int_{\partial\Omega} g = \int_{|x|=r_1} g + \int_{|x|=r_2} g = 0$. The solutions u_p of (1) in the annulus converge uniformly as $p \to \infty$ to a cone $u_\infty(x) = C - |x|$. However one can modify the function g on $|x| = r_2$ in such a way it does not change its sign and that the cone does not maximize (3). Hence, there is no uniqueness for (7) even for non-vanishing boundary data.

An example of a domain and boundary data such that uniqueness of the limit holds is a disk in $\mathbb{R}^2$, $D = \{|(x, y)| < 1\}$ with $g(x, y) > 0$ for $x > 0$ and $g(x, y) < 0$ for $x < 0$ with $\int_{\partial D} g = 0$.

4. Approximations by measures supported in small strips near the boundary

In this section we will show that these variational problems can be achieved as a singular limit of mass transport problems where the measures are supported in small strips near the boundary. In this sense we get a natural Neumann problem for the $p-$Laplacian while in the paper [11] it appears a Dirichlet condition in a large ball.

Precisely, let us consider the subset of Ω,

$$\omega_\delta = \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) < \delta\}\,.$$

Note that this set has measure $|\omega_\delta| \sim \delta|\partial\Omega|$ for small values of δ. Then for sufficiently small $s > 0$ we can define the *parallel* interior boundary $\Gamma_s = \{z - s\nu(z),\ z \in \partial\Omega\}$ where $\nu(z)$ denotes the outwards normal unit at $z \in \partial\Omega$. Note that $\Gamma_0 = \partial\Omega$. Then we can also look at the set ω_δ as the neighborhood of Γ_0 defined by

$$\omega_\delta = \{y = z - s\nu(z),\ z \in \partial\Omega,\ s \in (0,\delta)\} = \bigcup_{0<s<\delta} \Gamma_s$$

for sufficiently small δ, say $0 < \delta < \delta_0$. We also denote $\Omega_s = \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) > s\}$ and for s small we have that $\partial\Omega_s = \Gamma_s$.

Let us consider the transport problem for a suitable extension of g. To define this extension, as we have mentioned, let us denote by $d\sigma$ and $d\sigma_s$ the surface measures on the sets $\partial\Omega$ and Γ_s respectively. Given a function ϕ defined on $\bar{\Omega}$, and given $y \in \Gamma_s$ (with s small) , there exists $z \in \partial\Omega$ such that $y = z - s\nu(z)$. Hence, we can change variables:

$$\int_{\Gamma_s} \phi(y) d\sigma_s = \int_{\partial\Omega} \phi(z - s\nu(z)) G(s,z)\, d\sigma$$

where $G(s,z)$ depends on Ω (more precisely, it depends on the surface measures $d\sigma$ and $d\sigma_s$), and by the regularity of $\partial\Omega$, $G(s,z) \to 1$ as $s \to 0$ uniformly for $z \in \partial\Omega$.

Using these ideas, we define the following extension of g in Ω. Consider $\eta : [0,\infty) \to [0,\infty)$ a $\mathcal{C}^\infty$ such that $\eta(s) = 1$ if $0 \le s \le \frac{1}{2}$, $\eta(s) = 0$ if $s > 1$, $0 \le \eta(s) \le 1$ and $\int_0^\infty \eta(s)\, ds = A$. Defining $\eta_\varepsilon(s) = \frac{1}{A\varepsilon}\eta\left(\frac{s}{\varepsilon}\right)$, we get $\int_0^\infty \eta_\varepsilon(s)\, ds = 1$. For $\delta < \varepsilon$ consider Γ_s and

$$g_\varepsilon(y) = \eta_\varepsilon(s) \frac{g(z)}{G(s,z)}, \qquad y = z - s\nu(z).$$

We have $g_\varepsilon \equiv 0$ in $\Omega - \omega_\varepsilon$ and $g_\varepsilon \in \mathcal{C}(\Omega)$. Moreover,

$$\begin{aligned}
\int_\Omega g_\varepsilon(x)\, dx &= \int_0^\varepsilon \int_{\Gamma_s} g_\varepsilon(y)\, d\sigma_s\, ds \\
&= \int_0^\varepsilon \int_{\partial\Omega} g_\varepsilon(z - s\nu(z)) G(s,z)\, d\sigma\, ds \\
&= \int_0^\varepsilon \eta_\varepsilon(s) \int_{\partial\Omega} g(z)\, d\sigma\, ds = 0.
\end{aligned}$$

Associated to this extension we could consider the following two variational problems. First, the maximization problem in $W^{1,p}(\Omega)$,

$$\max\left\{ \int_{\omega_\varepsilon} w g_\varepsilon \colon w \in W^{1,p}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^p(\Omega)} \le 1 \right\}, \tag{10}$$

and the maximization problem in $W^{1,\infty}(\Omega)$,

$$\max\left\{ \int_{\omega_\varepsilon} w g_\varepsilon \colon w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^\infty(\Omega)} \le 1 \right\}. \tag{11}$$

We call $u_{p,\varepsilon}$ a solution to (10) and $u_{\infty,\varepsilon}$ a solution to (11).

Our first result says that we can take the limits as $\varepsilon \to 0$ and $p \to \infty$ in these variational problems. With the above notations we have the following commutative diagram

$$\begin{array}{ccccc} & u_{\infty,\varepsilon} & \rightarrow & u_{\infty,0} \\ p\to\infty & \uparrow & & \uparrow \\ & u_{p,\varepsilon} & \underset{\varepsilon\to 0}{\rightarrow} & u_{p,0} \end{array} \tag{12}$$

This diagram can be understood in two senses, either taking into account the variational properties satisfied by the functions, or considering the corresponding PDEs that the functions satisfy.

From the variational viewpoint, first we can state the following result:

Theorem 4. *Diagram* (12) *is commutative in the following sense:*

(1) *Maximizers of* (10)*,* $u_{p,\varepsilon}$*, converge along subsequences uniformly in* $\overline{\Omega}$ *to* $u_{p,0}$ *a maximizer of* (2) *as* $\varepsilon \to 0$*.*
(2) *Maximizers of* (10)*,* $u_{p,\varepsilon}$*, converge along subsequences uniformly in* $\overline{\Omega}$ *to* $u_{\infty,\varepsilon}$ *a maximizer of* (11) *as* $p \to \infty$*.*
(3) *Maximizers of* (11)*,* $u_{\infty,\varepsilon}$*, converge along subsequences uniformly in* $\overline{\Omega}$ *to* $u_{\infty,0}$ *a maximizer of* (3) *as* $\varepsilon \to 0$*.*
(4) *Maximizers of* (2)*,* $u_{p,0}$*, converge along subsequences uniformly in* $\overline{\Omega}$ *to* $u_{\infty,0}$ *a maximizer of* (3) *as* $p \to \infty$*.*

Proof of Theorem 4. The proof of the uniform convergence (along subsequences) of $u_{p,0}$ to $u_{\infty,0}$ is contained in [13].

Let us prove that $u_{p,\varepsilon}$ converges to $u_{p,0}$ as $\varepsilon \to 0$. We have

$$\|Du_{p,\varepsilon}\|_{L^p(\Omega)} \le 1.$$

Therefore we can extract a subsequence (that we still call $u_{p,\varepsilon}$) such that

$$u_{p,\varepsilon} \rightharpoonup v, \qquad \text{as } \varepsilon \to 0,$$

weakly in $W^{1,p}(\Omega)$ and, since $p > N$,

$$u_{p,\varepsilon} \to v, \qquad \text{as } \varepsilon \to 0,$$

uniformly in Ω (in fact, convergence holds in C^β). This limit v verifies the normalization constraint

$$\int_\Omega v = 0$$

and moreover

$$\|Dv\|_{L^p(\Omega)} \le 1.$$

On the other hand, thanks to the uniform convergence and to the definition of the extension g_ε we obtain,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\omega_\varepsilon} g_\varepsilon u_{p,\varepsilon} &= \lim_{\varepsilon\to 0}\int_0^\varepsilon \int_{\Gamma_s} g_\varepsilon(y)u_{p,\varepsilon}(y)\,d\sigma_s\,ds\\ &= \lim_{\varepsilon\to 0}\int_0^\varepsilon \int_{\partial\Omega} g_\varepsilon(z-s\nu(z))u_{p,\varepsilon}(z-s\nu(z))G(s,z)\,d\sigma\,ds\\ &= \lim_{\varepsilon\to 0}\int_0^\varepsilon \eta_\varepsilon(s)\int_{\partial\Omega} g(z)u_{p,\varepsilon}(z-s\nu(z))\,d\sigma\,ds\\ &= \int_{\partial\Omega} gv\,d\sigma\end{aligned}$$

and hence

$$\int_\Omega |Dv|^p - \int_{\partial\Omega} gv\,d\sigma \le \liminf_{\varepsilon\to 0}\left(\int_\Omega |Du_{p,\varepsilon}|^p - \int_{\omega_\varepsilon} g_\varepsilon u_{p,\varepsilon}\right). \tag{13}$$

On the other hand for every $w\in C^1(\overline{\Omega})$ we have

$$\int_\Omega |Dw|^p - \int_{\partial\Omega} gw\,d\sigma = \lim_{\varepsilon\to 0}\int_\Omega |Dw|^p - \int_{\omega_\varepsilon} g_\varepsilon w.$$

Hence, the extremal characterization of $u_{p,\varepsilon}$ implies

$$\inf_{u\in W^{1,p}(\Omega),\int_\Omega u=0}\left\{\int_\Omega |Du|^p - \int_{\partial\Omega} gud\sigma\right\} \ge \liminf_{\varepsilon\to 0}\int_\Omega |Du_{p,\varepsilon}|^p - \int_{\omega_\varepsilon} g_\varepsilon u_{p,\varepsilon}.$$

And by (13) we obtain

$$\inf_{u\in W^{1,p}(\Omega),\ \int_\Omega u=0}\left\{\int_\Omega |Du|^p - \int_{\partial\Omega} gu\,d\sigma\right\} = \int_\Omega |Dv|^p - \int_{\partial\Omega} gv\,d\sigma,$$

and therefore all possible limits $v = u_{p,0}$ satisfy the extremal property (2).

Now, let us prove that $u_{\infty,\varepsilon}$ converges to $u_{\infty,0}$, a maximizer of (3). Recall that $u_{\infty,\varepsilon}$ is a solution to the problem

$$M_\varepsilon = \max\left\{\int_{\omega_\varepsilon} wg_\varepsilon\colon w\in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^\infty(\Omega)} \le 1\right\}.$$

That is,

$$M_\varepsilon = \int_{\omega_\varepsilon} u_{\infty,\varepsilon} g_\varepsilon.$$

Therefore $u_{\infty,\varepsilon}$ is bounded in $W^{1,\infty}(\Omega)$ and then there exists a subsequence (that we still denote by $u_{\infty,\varepsilon}$) such that,

$$\begin{aligned}&u_{\infty,\varepsilon} \overset{*}{\rightharpoonup} v \text{ weakly-* in } W^{1,\infty}(\Omega) \text{ and}\\ &u_{\infty,\varepsilon} \to v \text{ uniformly in } \overline{\Omega},\end{aligned} \tag{14}$$

as $\varepsilon\to 0$. Hence

$$\lim_{\varepsilon\to 0}\int_{\omega_\varepsilon} u_{\infty,\varepsilon} g_\varepsilon = \int_{\partial\Omega} vg\,d\sigma.$$

On the other hand, for every $z\in C^1(\overline{\Omega})$ it holds that

$$\lim_{\varepsilon\to 0}\int_{\omega_\varepsilon} g_\varepsilon z = \int_{\partial\Omega} gz\,d\sigma.$$

Hence, if we call

$$M = \max \left\{ \int_{\partial\Omega} wg \, d\sigma \colon w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^\infty(\Omega)} \le 1 \right\}, \tag{15}$$

we obtain, from (14),

$$M \le \liminf_{\varepsilon \to 0} M_\varepsilon = \int_{\partial\Omega} vg \, d\sigma.$$

Therefore $v = u_{\infty,0}$ is a maximizer of (15), as we wanted to prove.

Finally, let us prove that $u_{p,\varepsilon} \to u_{\infty,\varepsilon}$. Recall that

$$\int_{\omega_\varepsilon} u_{p,\varepsilon} g_\varepsilon = \max \left\{ \int_{\omega_\varepsilon} wg_\varepsilon \colon w \in W^{1,p}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^p(\Omega)} \le 1 \right\}.$$

Therefore, for any $q < p$

$$\|Du_{p,\varepsilon}\|_{L^q(\Omega)} \le \left(\|Du_{p,\varepsilon}\|_{L^p(\Omega)} |\Omega|^{\frac{p-q}{p}} \right)^{1/q} \le |\Omega|^{\frac{p-q}{pq}}.$$

Hence, we can extract a subsequence (still denoted by $u_{p,\varepsilon}$) such that,

$$u_{p,\varepsilon} \to u, \qquad \text{uniformly in } \overline{\Omega},$$

as $p \to \infty$ with

$$\|Du\|_{L^\infty(\Omega)} \le 1.$$

Then

$$\int_{\omega_\varepsilon} u_{p,\varepsilon} g_\varepsilon \to \int_{\omega_\varepsilon} ug_\varepsilon, \qquad \text{as } p \to \infty.$$

This limit u verifies that

$$\int_{\omega_\varepsilon} ug_\varepsilon \le \max \left\{ \int_{\omega_\varepsilon} wg_\varepsilon \colon w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^\infty(\Omega)} \le 1 \right\}.$$

Let us prove that we have an equality here. If not, there exists a function v such that $v \in W^{1,\infty}(\Omega)$, $\int_\Omega v = 0$, $\|Dv\|_{L^\infty(\Omega)} \le 1$ with

$$\int_{\omega_\varepsilon} ug_\varepsilon < \int_{\omega_\varepsilon} vg_\varepsilon.$$

If we normalize, taking $\varphi = v/|\Omega|^{1/p}$, we obtain a function in $W^{1,p}(\Omega)$ with $\int_\Omega \varphi = 0$, $\|D\varphi\|_{L^p(\Omega)} \le 1$ and such that

$$\lim_{p\to\infty} \int_{\omega_\varepsilon} u_{p,\varepsilon} g_\varepsilon = \int_{\omega_\varepsilon} ug_\varepsilon < \int_{\omega_\varepsilon} vg_\varepsilon = \lim_{p\to\infty} |\Omega|^{1/p} \int_{\omega_\varepsilon} \varphi g_\varepsilon.$$

This contradiction proves that

$$\int_{\omega_\varepsilon} ug_\varepsilon = \max \left\{ \int_{\omega_\varepsilon} wg_\varepsilon \colon w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_{L^\infty(\Omega)} \le 1 \right\}.$$

This ends the proof. □

Now, we turn our attention to the PDE verified by the limits in the viscosity sense (see Section 3 for the precise definition) or in the weak sense.

Up to a Lagrange multiplier λ_p the functions $u_{p,0}$ are viscosity (and weak) solutions to the problem,

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ |Du|^{p-2} \frac{\partial u}{\partial \nu} = \lambda_p \, g & \text{on } \partial\Omega. \end{cases} \tag{16}$$

Let us to point out that it is easily seen that $\lambda_p \to 1$ as $p \to \infty$. Hence, to simplify the notation, we will drop this Lagrange multiplier in the sequel.

In the previous section, see [13] and also [14], the limit as $p \to \infty$ of the family $u_{p,0}$ is studied in the viscosity setting. It is proved that the problem that is satisfied by a uniform limit $u_{\infty,0}$ in the viscosity sense is (7) that we recall below,

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } \Omega, \\ B(x,u,Du) = 0, & \text{on } \partial\Omega, \end{cases} \tag{17}$$

where

$$B(x,u,Du) \equiv \begin{cases} \min\left\{|Du| - 1\ ,\ \frac{\partial u}{\partial \nu}\right\} & \text{if } g > 0, \\ \max\{1 - |Du|\ ,\ \frac{\partial u}{\partial \nu}\} & \text{if } g < 0, \\ H(|Du|)\frac{\partial u}{\partial \nu} & \text{if } g = 0, \end{cases}$$

and $H(a)$ is given by

$$H(a) = \begin{cases} 1 & \text{if } a \geq 1, \\ 0 & \text{if } 0 \leq a < 1. \end{cases}$$

Moreover, $u_{\infty,0}$ satisfies in the sense of viscosity the estimates:

$$|Du_{\infty,0}| \leq 1 \quad \text{and} \quad -|Du_{\infty,0}| \geq -1,$$

see [6].

On the other hand, when we deal with the problems in the strips, again up to a Lagrange multiplier that converges to one, the functions $u_{p,\varepsilon}$ are weak (and hence viscosity) solutions to the problem,

$$\begin{cases} -\Delta_p u = g_\varepsilon & \text{in } \Omega, \\ |Du|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

Passing to the limit as $p \to \infty$ in these problems we get that the function $u_{\infty,\varepsilon}$ satisfy the following properties in the viscosity sense (see again [6]):

$$\begin{cases} |Du| \leq 1 & \text{in } \Omega, \\ -|Du| \geq -1 & \text{in } \Omega, \end{cases} \tag{19}$$

and, in the different regions determined by g_ε:

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } \Omega \setminus \omega_\epsilon, \\ \min\{|Du| - 1, -\Delta_\infty u\} = 0 & \text{in } \{g_\epsilon > 0\}, \\ \max\{1 - |Du|, -\Delta_\infty u\} = 0 & \text{in } \{g_\epsilon < 0\}, \\ -\Delta_\infty u \geq 0 & \text{in } \Omega \cap \partial\{g_\epsilon > 0\} \cap (\partial\{g_\epsilon < 0\})^c, \\ -\Delta_\infty u \leq 0 & \text{in } \Omega \cap \partial\{g_\epsilon < 0\} \cap (\partial\{g_\epsilon > 0\})^c. \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

Notice that the equations in $\{g_\epsilon > 0\}$ and $\{g_\epsilon < 0\}$ can be simplified by the estimate (19), however to understand the boundary condition in viscosity sense it is necessary to consider such equations in its full generality.

We split our following results in two theorems.

First, we have,

Theorem 5.

(1) *The limit* $u_{p,0}$ *of a uniformly converging sequence* $u_{p,\varepsilon}$ *of weak solutions to* (18) *as* $\varepsilon \to 0$ *is a weak solution to* (16) *(and hence a viscosity solution).*
(2) *The limit* $u_{\infty,0}$ *of a uniformly converging sequence* $u_{p,0}$ *of viscosity solutions to* (16) *as* $p \to \infty$ *is a viscosity solution to* (17).

Let us to point out that when $\varepsilon \to 0$, g_ε concentrates on the boundary, and therefore the sequence $\{g_\varepsilon\}$ is not uniformly bounded. This makes difficult to give a sense to pass to the limit in the viscosity framework when $\varepsilon \to 0$. Hence in this case we consider the variational characterization of the sequence $\{u_{p,\varepsilon}\}$ (that is equivalent to the fact of being a weak solution). To the best of our knowledge, it is not known that the notions of viscosity and weak solutions coincide for solutions to (18), cf. [19] where such equivalence is only proved for Dirichlet boundary conditions.

Now, we deal with the rest of the commutative diagram. To pass to the limit in the sequence $u_{\infty,\varepsilon}$ we need the variational characterization, and we also need a uniqueness result for the limit problem, which has been proved in [13]. This uniqueness result says that:

If Ω *is convex and* $\{g = 0\}^o = \emptyset$*, then there is a unique function which satisfies the extremal property* (3).

Let us to point out that the hypothesis $\{g = 0\}^o = \emptyset$ implies also the uniqueness of the extremals to (11). Therefore, under this hypothesis there exists a unique $u_{\infty,\varepsilon}$ reached as a limit of the solutions $u_{p,\varepsilon}$ as $p \to \infty$.

Now, we can state our second theorem, see [15] for the proof.

Theorem 6.

(1) *The limit* $u_{\infty,\varepsilon}$ *of a uniformly converging sequence* $u_{p,\varepsilon}$ *of viscosity solutions to* (18) *as* $p \to \infty$ *is a viscosity solution to* (19)-(20).
(2) *Assume that* Ω *is convex and* $\{g = 0\}^o = \emptyset$*. Consider the viscosity solutions* $u_{\infty,\varepsilon}$ *to* (19)-(20)*, obtained as a uniform limit as* $p \to \infty$ *of the solutions* $u_{p,\varepsilon}$*. Then, the sequence* $\{u_{\infty,\varepsilon}\}$ *converges uniformly to a viscosity solution to* (17)*,* $u_{\infty,0}$*.*

5. Connections with game theory. Tug-of-War games

In this section we deal with an approach to these type of problems based on game theory.

A Tug-of-War is a two-person, zero-sum game, that is, two players are in contest and the total earnings of one are the losses of the other. Hence, one of them, say Player I, plays trying to maximize his expected outcome, while the other, say Player II is trying to minimize Player I's outcome (or, since the game is zero-sum, to maximize his own outcome). Recently, these type of games have been used in connection with some PDE problems, see [5], [20], [22], [23].

For the reader's convenience, let us first describe briefly the game introduced in [23] by Y. Peres, O. Schramm, S. Sheffield and D. Wilson. Consider a bounded domain $\Omega \subset \mathbb{R}^n$, and take $\Gamma_D \subset \partial\Omega$ and $\Gamma_N \equiv \partial\Omega \setminus \Gamma_D$. Let $F : \Gamma_D \to \mathbb{R}$ be a Lipschitz continuous function. At an initial time, a token is placed at a point $x_0 \in \overline{\Omega} \setminus \Gamma_D$. Then, a (fair) coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in \overline{B_\varepsilon(x_0)} \cap \overline{\Omega}$. At each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in \overline{B_\varepsilon(x_{k-1})} \cap \overline{\Omega}$. Once the token has reached some $x_\tau \in \Gamma_D$, the game ends and Player I earns $F(x_\tau)$ (while Player II earns $-F(x_\tau)$). This is the reason why we will refer to F as the *final payoff function.* In more general models, it is considered also a *running payoff* $f(x)$ defined on Ω, which represents the reward (respectively, the cost) at each intermediate state x, and gives rise to nonhomogeneous problems. We will assume throughout this paper that $f \equiv 0$. This procedure gives a sequence of game states $x_0, x_1, x_2, \dots, x_\tau$, where every x_k except x_0 are random variables, depending on the coin tosses and the strategies adopted by the players.

Now we want to give a definition of the *value of the game.* To this end we introduce some notation and the normal or strategic form of the game (see [22] and [21]). The initial state $x_0 \in \overline{\Omega} \setminus \Gamma_D$ is known to both players (public knowledge). Each player i chooses an *action* $a_0^i \in \overline{B_\varepsilon(0)}$; this defines an action profile $a_0 = \{a_0^1, a_0^2\} \in \overline{B_\varepsilon(0)} \times \overline{B_\varepsilon(0)}$ which is announced to the other player. Then, the new state $x_1 \in \overline{B_\varepsilon(x_0)}$ (namely, the current state plus the action) is selected according to the distribution $p(\cdot|x_0, a_0)$ in $\overline{\Omega}$. At stage k, knowing the history $h_k = (x_0, a_0, x_1, a_1, \dots, a_{k-1}, x_k)$, (the sequence of states and actions up to that stage), each player i chooses an action a_k^i. If the game terminated at time $j < k$, we set $x_m = x_j$ and $a_m = 0$ for $j \le m \le k$. The current state x_k and the profile $a_k = \{a_k^1, a_k^2\}$ determine the distribution $p(\cdot|x_k, a_k)$ of the new state x_{k+1}.

Denote $H_k = (\overline{\Omega} \setminus \Gamma_D) \times \left(\overline{B_\varepsilon(0)} \times \overline{B_\varepsilon(0)} \times \overline{\Omega}\right)^k$, the set of *histories up to stage* k, and by $H = \bigcup_{k\ge1} H_k$ the set of all histories. Notice that H_k, as a product space, has a measurable structure. The *complete history space* H_∞ is the set of plays defined as infinite sequences $(x_0, a_0, \dots, a_{k-1}, x_k, \dots)$ endowed with the product topology. Then, the final payoff for Player I, i.e. F, induces a Borel-measurable function on H_∞. A *pure strategy* S_i for Player i, is a mapping from histories to actions, namely, a mapping from H to $\overline{B_\varepsilon(0)}$ such that S_i^k is a Borel-measurable mapping from H_k to $\overline{B_\varepsilon(0)}$ that maps histories ending with x_k to elements of $\overline{B_\varepsilon(0)}$ (roughly speaking, at every stage the strategy gives the next movement for the player, provided he win the coin toss, as a function of the current state and the past history). The initial state x_0 and a profile of strategies $\{S_I, S_{II}\}$ define (by Kolmogorov's extension theorem) a unique probability $\mathbb{P}^{x_0}_{S_I,S_{II}}$ on the space of plays H_∞. We denote by $\mathbb{E}^{x_0}_{S_I,S_{II}}$ the corresponding expectation. Then, if S_I and S_{II} denote the strategies adopted by Player I and II respectively, we define the expected payoff for player I as

$$V_{x_0,I}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ -\infty, & \text{otherwise.} \end{cases}$$

Analogously, we define the expected payoff for player II as

$$V_{x_0,II}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ +\infty, & \text{otherwise.} \end{cases}$$

The ε-value of the game for Player I is given by

$$u_I^\varepsilon(x_0) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}_{S_I,S_{II}}^{x_0}[F(x_\tau)],$$

while the ε-value of the game for Player II is defined as

$$u_{II}^\varepsilon(x_0) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}_{S_I,S_{II}}^{x_0}[F(x_\tau)].$$

In some sense, $u_I^\varepsilon(x_0), u_{II}^\varepsilon(x_0)$ are the least possible outcomes that each player expects to get when the ε-game starts at x_0. As in [23], we penalize severely the games that never end. If the game does not stop then we define $u_I^\varepsilon(x_0) = -\infty$ and $u_{II}^\varepsilon(x_0) = +\infty$.

If $u_I^\varepsilon = u_{II}^\varepsilon := u_\varepsilon$, we say that *the game has a value.* In [23] it is shown that, under very general hypotheses, that are fulfilled in the present setting, the ε-Tug-of-War game has a value.

All these $\varepsilon-$values are Lipschitz functions which converge uniformly when $\varepsilon \to 0$. The uniform limit as $\varepsilon \to 0$ of the game values u_ε is called *the continuous value* of the game that we will denote by u. Indeed, see [23], it turns out that u is a viscosity solution to the problem

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ u(x) = F(x) & \text{on } \Gamma_D, \end{cases} \tag{21}$$

where $\Delta_\infty u = |\nabla u|^{-2} \sum_{ij} u_{x_i} u_{x_i x_j} u_{x_j}$ is the $1-$homogeneous infinity Laplacian.

When $\Gamma_D \equiv \partial\Omega$, it is known that problem (21) has a unique viscosity solution, (as proved in [17], and in a more general framework, in [23]). Moreover, it is the unique AMLE (absolutely minimal Lipschitz extension) of $F : \Gamma_D \to \mathbb{R}$ in the sense that $Lip_U(u) = Lip_{\partial U \cap \Omega}(u)$ for every open set $U \subset \overline{\Omega} \setminus \Gamma_D$. AMLE extensions were introduced by Aronsson, see the survey [3] for more references and applications of this subject.

When $\Gamma_D \neq \partial\Omega$ the PDE problem (21) is incomplete, since there is a missing boundary condition on $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Our main concern is to find the boundary condition that completes the problem. We have that it is in fact the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n}(x) = 0.$$

On the other hand, we give an alternative proof of the property $-\Delta_\infty u(x) = 0$ by using direct viscosity arguments. We have the following result:

Theorem 7. *Let $u(x)$ be the continuous value of the Tug-of-War game introduced in* [23]. *Then,*

i) *$u(x)$ is a viscosity solution to the mixed boundary value problem*

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{on } \Gamma_N, \\ u(x) = F(x) & \text{on } \Gamma_D. \end{cases} \tag{22}$$

ii) *Reciprocally, assume that* Ω *verifies for every* $z \in \overline{\Omega}$ *and every* $x^* \in \Gamma_N$ $z \neq x^*$ *that*
$$\Big\langle \frac{x^* - z}{|x^* - z|}; n(x^*) \Big\rangle > 0.$$
Then, if $u(x)$ *is a viscosity solution to* (22)*, it coincides with the unique continuous value of the game.*

The hypothesis imposed on Ω in part ii) holds for the case which Γ_N is strictly convex. The first part of the theorem comes as a consequence of the Dynamic Programming Principle read in the viscosity sense. To prove the second part we use that the continuous value of the game enjoys comparison with quadratic functions, and this property uniquely determine the value of the game.

We have found a PDE problem, (22), which allows to find both the continuous value of the game and the AMLE of the Dirichlet data F (which is given only on a subset of the boundary) to $\overline{\Omega}$. To summarize, we point out that a complete equivalence holds, in the following sense:

Theorem 8. *It holds*
$$u \text{ is AMLE of } F \Leftrightarrow u \text{ is the value of the game} \Leftrightarrow u \text{ solves } (22).$$

The first equivalence was proved in [23] and the second one is just Theorem 7.

Another consequence of Theorem 7 is the following:

Corollary 2. *There exists a unique viscosity solution to* (22).

The existence of a solution is a consequence of the existence of a continuous value for the game together with part i) in the previous theorem, while the uniqueness follows by uniqueness of the value of the game and part ii).

Note that to obtain uniqueness we have to invoke the uniqueness of the game value. It should be interesting to obtain a direct proof (using only PDE methods) of existence and uniqueness for (22) but we have not been able to find the appropriate perturbations near Γ_N to obtain uniqueness (existence follows easily by taking the limit as $p \to \infty$ in the mixed boundary value problem problem for the $p-$laplacian).

Remark 3. *Corollary 2 allows to improve the convergence result given in* [13] *for solutions to the Neumann problem for the* $p-$*laplacian as* $p \to \infty$*. The uniqueness of the limit holds under weaker assumptions on the data (for example,* Ω *strictly convex).*

6. Eigenvalue problems

We end this note by briefly describe a similar limit in an eigenvalue problem.

Eigenvalues of $-\Delta_p u = \lambda |u|^{p-2} u$ with Dirichlet boundary conditions, $u = 0$ on $\partial\Omega$, have been extensively studied since [12]. The limit as $p \to \infty$ was studied in [18].

Our last aim here is to state a result concerning the limit as $p \to \infty$ for the Steklov eigenvalue problem

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda |u|^{p-2} u & \text{on } \partial\Omega. \end{cases} \tag{23}$$

Theorem 9. *For the first eigenvalue of* (23) *we have,*

$$\lim_{p\to\infty} \lambda_{1,p}^{1/p} = \lambda_{1,\infty} = 0,$$

with eigenfunction given by $u_{1,\infty} = 1$.

For the second eigenvalue, it holds

$$\lim_{p\to\infty} \lambda_{2,p}^{1/p} = \lambda_{2,\infty} = \frac{2}{\operatorname{diam}(\Omega)}.$$

Moreover, given $u_{2,p}$ *eigenfunctions of* (23) *of* $\lambda_{2,p}$ *normalized by* $\|u_{2,p}\|_{L^\infty(\partial\Omega)} = 1$, *there exits a sequence* $p_i \to \infty$ *such that* $u_{2,p_i} \to u_{2,\infty}$, *in* $C^\alpha(\overline{\Omega})$. *The limit* $u_{2,\infty}$ *is a solution of*

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ \Lambda(x,u,\nabla u) = 0, & \text{on } \partial\Omega, \end{cases} \tag{24}$$

in the viscosity sense, where

$$\Lambda(x,u,\nabla u) \equiv \begin{cases} \min\left\{|\nabla u| - \lambda_{2,\infty}|u| \ , \ \frac{\partial u}{\partial \nu}\right\} & \text{if } u > 0, \\ \max\{\lambda_{2,\infty}|u| - |\nabla u| \ , \ \frac{\partial u}{\partial \nu}\} & \text{if } u < 0, \\ \frac{\partial u}{\partial \nu} & \text{if } u = 0. \end{cases}$$

For the k*-th eigenvalue we have that if* $\lambda_{k,p}$ *is the* k*-th variational eigenvalue of* (23) *with eigenfunction* $u_{k,p}$ *normalized by* $\|u_{k,p}\|_{L^\infty(\partial\Omega)} = 1$, *then for every sequence* $p_i \to \infty$ *there exists a subsequence such that*

$$\lim_{p_i\to\infty} \lambda_{k,p}^{1/p} = \lambda_{*,\infty}$$

and $u_{k,p_i} \to u_{*,\infty}$ *in* $C^\alpha(\overline{\Omega})$, *where* $u_{*,\infty}$ *and* $\lambda_{*,\infty}$ *is a solution of* (24).

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