ON NONLINEAR NONLOCAL DIFFUSION PROBLEMS

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ABSTRACT. We summarize in this paper some of our recent results on the nonlocal, nonlinear evolution problem given by

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy, \\ u(x,0) = u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega,\ t > 0. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain and $1 \le p \le +\infty$. We deal with existence, uniqueness and the asymptotic behaviour of solutions. In addition, we show a convergence results of solutions to nonlocal problems to the solution to the local $p-$Laplacian evolution equation, $v_t(x,t) = \mathrm{div}(|\nabla v|^{p-2}\nabla v)$ with Neumann boundary conditions $|\nabla v|^{p-2}\nabla v \cdot \eta = 0$ and the same initial condition when the kernel J is rescaled in an appropriate way.

1. INTRODUCTION

The goal of this article is to present recent results concerning existence, uniqueness and asymptotic behaviour of solutions of the nonlocal nonlinear diffusion problems, called *nonlocal p-Laplacian problems*, with homogeneous Neumann boundary conditions (see [7] and the recent book [10]):

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy, \\ u(x,0) = u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. The kernel $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous radial function with compact support, $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)dx = 1$ (this last condition is imposed only for normalization) and p is a fixed but arbitrary number between 1 and $+\infty$.

To make this note short and avoid entering into subtle technicalities we present only a brief sketch of each proof and refer to [7, 10].

For other problems involving nonlocal $p-$Laplacian problems, including problems with weights, with nonhomogeneous Dirichlet boundary conditions, etc, see [11, 9, 10].

We also give some results concerning limits of such solutions when a rescaling parameter goes to zero ([7, 10]), recovering the well-known local diffusion model of the p-Laplacian evolution

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equation with Neumann boundary conditions, that is,

$$\begin{cases} v_t = \Delta_p v & \text{in } \Omega \times (0,T), \\ |\nabla v|^{p-2}\nabla v \cdot \eta = 0 & \text{on } \partial\Omega \times (0,T), \\ v(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

The limit cases $p = 1$ and $p = +\infty$ will be also treated ([7, 8, 10]). They correspond, respectively, to a nonlocal version of the total variation flow, which has been used as a model in image processing, and to a nonlocal model for the evolution of sandpiles.

For a different approach to this kind of nonlinear problems, with integral equations with a degenerate kernel, see [35].

To finish this introduction, let us briefly introduce some references for the prototype of nonlocal problem considered along this work. Nonlocal evolution equations of the form $u_t(t,x) = \int_{\mathbb{R}^N} J(x-y)u(t,y)\,dy - u(t,x)$, and variations of it, have been recently widely used to model diffusion processes. More precisely, as stated in [31], if $u(t,x)$ is thought of as a density at the point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} J(y-x)u(t,y)\,dy = (J * u)(t,x)$ is the rate at which individuals are arriving at position x from all other places and $-u(t,x) = -\int_{\mathbb{R}^N} J(y-x)u(t,x)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies the equation $u_t = J * u - u$. These kind of equations are called nonlocal diffusion equations since in them the diffusion of the density u at a point x and time t depends not only on $u(x,t)$ and its derivatives, but also on all the values of u in a neighbourhood of x through the convolution term $J * u$. This equation shares many properties with the classical heat equation, $u_t = \Delta u$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and, even if J is compactly supported, perturbations propagate with infinite speed, [31]. However, there is no regularizing effect in general.

Let us now fix a bounded domain Ω in $\mathbb{R}^N$. When looking at boundary conditions for nonlocal problems, one has to modify the usual formulations for local problems. As an analog for nonlocal problems of Neumann boundary conditions we consider

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)(u(y,t) - u(x,t))\,dy, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

In this model, the integral term takes into account the diffusion inside Ω. In fact, as we have explained, the integral $\int J(x-y)(u(y,t) - u(x,t))\,dy$ takes into account the individuals arriving at or leaving position x from other places. Since we are integrating over Ω, we are assuming that diffusion takes place only in Ω. The individuals may not enter or leave the domain. This is analogous to what is called homogeneous Neumann boundary conditions in the literature. Note that this problem is just (1.1) for $p = 2$.

For the linear case see [23, 25, 26] and [10]. For a general vector calculus of this kind of nonlocal problems, in the linear case, see [34, 27].

Nonlocal diffusion equations have been recently widely studied and have connections with probability theory (for example, Levy processes are related to the fractional Laplacian), see [6],

[7], [8], [14], [21], [22], [23], [24], [25], [26], [31], [44], the book [10] and references therein. They have also been used to model very different applied situations, for example in biology ([22], [39]), image processing ([33], [38]), particle systems ([19]), coagulation models ([32]), nonlocal anisotropic models for phase transition ([1], [2]), mathematical finances using optimal control theory ([17], [36]), etc.

2. Existence and uniqueness results

A *solution* of (1.1) in $[0,T]$ is a function $u \in W^{1,1}(0,T;L^1(\Omega))$ that satisfies $u(x,0) = u_0(x)$ a.e. $x \in \Omega$ and

$$u_t(x,t) = \int_\Omega J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy \quad \text{a.e. in } \Omega \times (0,T).$$

Let us note that the evolution problem (1.1) is the gradient flow associated to the functional

$$G_p^J(u) = \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|^p\,dy\,dx,$$

which is the nonlocal analog of the energy functional associated to the local p-Laplacian

$$F_p(v) = \frac{1}{p}\int_\Omega |\nabla v|^p.$$

Theorem 2.1. *Suppose $p>1$ and let $u_0 \in L^p(\Omega)$. Then, for any $T>0$, there exists a unique solution to (1.1). Moreover, if $u_{i0} \in L^1(\Omega)$, and u_i is a solution in $[0,T]$ of (1.1) with initial data u_{i0}, $i=1,2$, respectively, then we have the following contraction principle:*

$$\int_\Omega (u_1(t)-u_2(t))^+ \le \int_\Omega (u_{10}-u_{20})^+ \quad \textit{for every } t \in [0,T].$$

Sketch of proof. Let B_p^J be defined by

$$B_p^J u(x) = -\int_\Omega J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, \quad x \in \Omega.$$

We prove that B_p^J is completely accretive (see [15]) and satisfies the range condition $L^p(\Omega) \subset \mathrm{R}(I+B_p^J)$. In short, this means that for any $\phi \in L^p(\Omega)$ there is a unique solution of the problem $u + B_p^J u = \phi$ and the resolvent $(I+B_p^J)^{-1}$ is a contraction in $L^q(\Omega)$ for all $1 \le q \le +\infty$. Therefore, the Nonlinear Semigroup Theory (see, *e.g.*,[16]) gives us the existence of solutions and the contraction principle.

3. Rescaling

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. For fixed $p>1$ and J we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right),$$

where $C_{J,p}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|^p\,dz$ is a normalizing constant. Associated with these rescaled kernels we have solutions u_ε of problem (1.1) with J replaced by $J_{p,\varepsilon}$ and the same initial condition u_0.

The main result now states that these functions u_ε converge strongly in $L^p(\Omega)$ to the solution of the local p-Laplacian Neumann problem with homogeneous Neumann boundary conditions

$$\begin{cases} v_t = \Delta_p v & \text{in } \Omega \times (0,T), \\ |\nabla v|^{p-2}\nabla v \cdot \eta = 0 & \text{on } \partial\Omega \times (0,T), \\ v(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.1}$$

where η is the unit outward normal on $\partial\Omega$ and $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$ is the so-called p-Laplacian of v.

Theorem 3.1. *Assume that $J(x) \geq J(y)$ if $|x| \leq |y|$. Let $T > 0$, $u_0 \in L^p(\Omega)$ and let u_ε be the unique solution of* (1.1) *with J replaced by $J_{p,\varepsilon}$. Then, if v is the unique solution of* (3.1),

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_\varepsilon(\cdot,t) - v(\cdot,t)\|_{L^p(\Omega)} = 0.$$

A formal calculation: Let us perform a formal calculation just to convince the reader that the above convergence result is true. Let $N = 1$. Let $u(x)$ be a smooth function and consider

$$A_\varepsilon(u)(x) = \frac{1}{\varepsilon^{p+1}} \int_{\mathbb{R}} J\left(\frac{x-y}{\varepsilon}\right) |u(y) - u(x)|^{p-2}(u(y) - u(x))\, dy.$$

Changing variables, $y = x - \varepsilon z$, we get

$$A_\varepsilon(u)(x) = \frac{1}{\varepsilon^p} \int_{\mathbb{R}} J(z)|u(x-\varepsilon z) - u(x)|^{p-2}(u(x-\varepsilon z) - u(x))\, dz. \tag{3.2}$$

Now, we expand in powers of ε to obtain

$$|u(x-\varepsilon z) - u(x)|^{p-2} = \varepsilon^{p-2}\left|u'(x)z + \tfrac{u''(x)}{2}\varepsilon z^2 + O(\varepsilon^2)\right|$$

$$= \varepsilon^{p-2}|u'(x)|^{p-2}|z|^{p-2} + \varepsilon^{p-1}(p-2)|u'(x)z|^{p-4}u'(x)z\frac{u''(x)}{2}z^2 + O(\varepsilon^p),$$

and

$$u(x-\varepsilon z) - u(x) = \varepsilon u'(x)z + \frac{u''(x)}{2}\varepsilon^2 z^2 + O(\varepsilon^3).$$

Hence, (3.2) becomes

$$A_\varepsilon(u)(x) = \frac{1}{\varepsilon}\int_{\mathbb{R}} J(z)|z|^{p-2}z\, dz|u'(x)|^{p-2}u'(x)$$

$$+\frac{1}{2}\int_{\mathbb{R}} J(z)|z|^p\, dz\left((p-2)|u'(x)|^{p-2}u''(x) + |u'(x)|^{p-2}u''(x)\right) + O(\varepsilon).$$

Using that J is radially symmetric, the first integral vanishes and therefore

$$\lim_{\varepsilon\to 0} A_\varepsilon(u)(x) = C_{J,p}^{-1}(|u'(x)|^{p-2}u'(x))'.$$

A way of making this formal calculation rigorous is using the Nonlinear Semigroup Theory. More precisely, since the solutions of the problems are obtained via Crandall-Liggett's Theorem, by a classical result of Brezis-Pazy ([20]) it is enough to prove the convergence of the resolvents, that is the following result.

Theorem 3.2. *Suppose $J(x) \geq J(y)$ if $|x| \leq |y|$. For any $\phi \in L^\infty(\Omega)$,*

$$\left(I + B_p^{J_{p,\varepsilon}}\right)^{-1} \phi \to (I + B_p)^{-1} \phi \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0.$$

The main ingredient for the proof of convergence to the local problems is the following precompactness lemma inspired by a result due to Bourgain, Brezis and Mironescu, [18, Theorem 4].

For a function g defined in a set Ω, we define

$$\overline{g}(x) = \begin{cases} g(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $BV(\Omega)$ the space of functions of bounded variation.

Theorem 3.3. *Let $1 \leq q < +\infty$ and $\Omega \subset \mathbb{R}^N$ open. Let $\rho : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous radial function with compact support, non identically zero, and $\rho_n(x) := n^N \rho(nx)$. Let $\{f_n\}$ be a sequence of functions in $L^q(\Omega)$ such that*

$$\int_\Omega \int_\Omega |f_n(y) - f_n(x)|^q \rho_n(y - x)\, dx\, dy \leq \frac{M}{n^q}.$$

1. *If $\{f_n\}$ is weakly convergent in $L^q(\Omega)$ to f, then*

 (i) *For $q > 1$, $f \in W^{1,q}(\Omega)$, and moreover*

$$(\rho(z))^{1/q} \chi_\Omega \left(x + \frac{1}{n} z\right) \frac{\overline{f}_n\left(x + \frac{1}{n}z\right) - f_n(x)}{1/n} \rightharpoonup (\rho(z))^{1/q}\, z \cdot \nabla f(x)$$

weakly in $L^q(\Omega) \times L^q(\mathbb{R}^N)$.

 (ii) *For $q = 1$, $f \in BV(\Omega)$, and moreover*

$$\rho(z)\chi_\Omega \left(\cdot + \frac{1}{n} z\right) \frac{\overline{f}_n\left(\cdot + \frac{1}{n}z\right) - f_n(\cdot)}{1/n} \rightharpoonup \rho(z) z \cdot Df$$

weakly in the sense of measures.

2. *Suppose Ω is a smooth bounded domain in $\mathbb{R}^N$ and $\rho(x) \geq \rho(y)$ if $|x| \leq |y|$. Then $\{f_n\}$ is relatively compact in $L^q(\Omega)$, and consequently, there exists a subsequence $\{f_{n_k}\}$ such that*

 (i) *if $q > 1$, $f_{n_k} \to f$ in $L^q(\Omega)$ with $f \in W^{1,q}(\Omega)$;*

 (ii) *if $q = 1$, $f_{n_k} \to f$ in $L^1(\Omega)$ with $f \in BV(\Omega)$.*

4. The nonlocal total variation flow

Motivated by problems in image processing, the Neumann problem for the total variation flow is studied in [4])

$$\begin{cases} v_t = \mathrm{div}\left(\dfrac{Dv}{|Dv|}\right) & \text{in } \Omega \times (0, +\infty), \\ \dfrac{Dv}{|Dv|} \cdot \eta = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ v(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

In the literature the operator $\operatorname{div}\left(\frac{Dv}{|Dv|}\right)$ is also called the 1-Laplacian. Problem (4.1) appears when one uses the steepest descent method to minimize the total variation, a method introduced by L. Rudin, S. Osher and E. Fatemi [40] in the context of image denoising and reconstruction. Then solving (4.1) amounts to regularizing or, in other words, filtering the initial datum u_0. This filtering process has less destructive effect on the edges than filtering with a Gaussian, i.e., than solving the heat equation with initial condition u_0. In this context the given *image* u_0 is a function defined on a bounded smooth or piecewise smooth open subset Ω of $\mathbb{R}^N$; typically, Ω will be a rectangle in $\mathbb{R}^2$.

The nonlocal version of problem (4.1) can be written formally as

$$\left\{\begin{array}{ll} u_t(x,t) = \displaystyle\int_\Omega J(x-y)\frac{u(y,t)-u(x,t)}{|u(y,t)-u(x,t)|}\,dy, & x\in\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega, \end{array}\right. \tag{4.2}$$

J as above. This problem is the gradient flow associated to the functional

$$G_1^J(u) = \frac{1}{2}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|\,dy\,dx,$$

which is the nonlocal analog of the energy functional associated to the total variation: $F_1(v) = \int_\Omega |Dv|$.

A *solution* of (4.2) in $[0,T]$ is a function $u\in W^{1,1}(0,T;L^1(\Omega))$ which satisfies $u(x,0)=u_0(x)$ for a.e. $x\in\Omega$ and

$$u_t(x,t) = \int_\Omega J(x-y)g(x,y,t)\,dy \quad \text{a.e. in } \Omega\times(0,T),$$

for some $g\in L^\infty(\Omega\times\Omega\times(0,T))$ with $\|g\|_\infty\le 1$ such that $g(x,y,t)=-g(y,x,t)$ and

$$J(x-y)g(x,y,t)\in J(x-y)\operatorname{sgn}(u(y,t)-u(x,t)).$$

Here,

$$\operatorname{sgn}(r) := \left\{\begin{array}{ll} \dfrac{r}{|r|} & \text{if } r\neq 0,\\ {[-1,1]} & \text{if } r=0. \end{array}\right.$$

Observe that the formal expression $\frac{u(y,t)-u(x,t)}{|u(y,t)-u(x,t)|}$ in the evolution equation has to be interpreted as an L^∞ function $g(x,y,t)$ antisymmetric in the space variables and such that it is related to the above expression by using the multivalued function sgn.

To prove the existence and uniqueness of this kind of solutions, the idea is to take the limit as $p\searrow 1$ of the solutions of (1.1) studied in Section 2, see [10] for details.

Rescaling. Let now Ω be a smooth bounded domain in $\mathbb{R}^N$ and set

$$J_{1,\varepsilon}(x) := \frac{C_{J,1}}{\varepsilon^{1+N}}J\left(\frac{x}{\varepsilon}\right),$$

with $C_{J,1}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|\,dz$. Associated with these rescaled kernels are the solutions u_ε of the equation in (4.2) with J replaced by $J_{1,\varepsilon}$ and the same initial condition u_0. Then we prove, using again the Nonlinear Semigroup Theory and Theorem 3.3, the following result:

Theorem 4.1. *Suppose $J(x) \geq J(y)$ if $|x| \leq |y|$. Let $T > 0$ and $u_0 \in L^1(\Omega)$. Let u_ε be the unique solution in $[0, T]$ of (4.2) with J replaced by $J_{1,\varepsilon}$ and v the unique weak solution of (4.1). Then*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_\varepsilon(\cdot, t) - v(\cdot, t)\|_{L^1(\Omega)} = 0.$$

5. Asymptotic behaviour

In this section we focus our attention on the behaviour of the solutions as t goes to infinity. For this study the following *Poincaré type inequality* plays a crucial role.

Proposition 5.1. *Given $q \geq 1$, J as above and Ω a bounded domain in $\mathbb{R}^N$, there exists $c_q > 0$ such that:*

$$c_q \int_\Omega \left| u - \frac{1}{|\Omega|} \int_\Omega u \right|^q \leq \frac{1}{2} \int_\Omega \int_\Omega J(x-y)|u(y) - u(x)|^q \, dy \, dx,$$

for every $u \in L^q(\Omega)$.

Using the above Poincaré's inequality we show that the solution of this nonlocal problem converges to the mean value of the initial condition as $t \to \infty$.

Theorem 5.2. *Let $u_0 \in L^\infty(\Omega)$. Let u be the solution of (1.1); then*

$$\left\| u(t) - \frac{1}{|\Omega|} \int_\Omega u_0(x) \, dx \right\|_{L^p(\Omega)}^p \leq C \frac{||u_0||_{L^2(\Omega)}^2}{t} \quad \forall \, t > 0,$$

where $C = C(J, \Omega, p)$.

6. A nonlocal model for sandpiles

In the last years an increasing attention has been paid to the study of differential models in granular matter theory (see, e.g., [12] for an overview of different theoretical approaches and models). This field of research, which is of course of strong relevance in applications, has also been the source of many new and challenging problems in the theory of partial differential equations. In this context, the continuous models for the dynamics of a sandpile, introduced, independently, by L. Prigozhin ([42], [43]) and by G. Aronsson, L. C. Evans and Y. Wu ([13]) have been of special interest. These two pile growth models, obtained using different arguments, yield to a model in the form of a variational inequality. Our next purpose now is to present the nonlocal version of the Aronsson-Evans-Wu model (see [9, 10] for a nonlocal model of the Prigozhin model).

6.1. The Aronsson-Evans-Wu model for sandpiles.

In [29], [13] and [28] the authors investigated the limiting behaviour as $p \to \infty$ of solutions to the quasilinear parabolic problem

$$\begin{cases} (v_p)_t - \Delta_p v_p = f & \text{in } \mathbb{R}^N \times (0, T), \\ v_p(x, 0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{6.1}$$

where f is a nonnegative function that represents a given source term, which is interpreted physically as adding material to an evolving system, within which mass particles are continually rearranged by diffusion.

Let us define for $1 < p < \infty$ the functional

$$F_p(v) = \begin{cases} \displaystyle\frac{1}{p}\int_{\mathbb{R}^N} |\nabla v(y)|^p\, dy & \text{if } v \in L^2(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \\ +\infty & \text{if } v \in L^2(\mathbb{R}^N) \setminus W^{1,p}(\mathbb{R}^N). \end{cases}$$

Then the PDE problem (6.1) can be written as the abstract Cauchy problem associated to the subdifferential of F_p, that is,

$$\begin{cases} f(\cdot,t) - (v_p)_t(\cdot,t) = \partial F_p(v_p(\cdot,t)) & \text{a.e. } t \in (0,T), \\ v_p(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

In [13], assuming that u_0 is a Lipschitz function with compact support such that $\|\nabla u_0\|_\infty \le 1$, and f is a smooth nonnegative function with compact support in $\mathbb{R}^N \times (0,T)$, it is proved that there exist a sequence $p_i \to +\infty$ and a limit function v_∞ such that, for each $T > 0$,

$$\begin{cases} v_{p_i} \to v_\infty & \text{in } L^2(\mathbb{R}^N \times (0,T)) \text{ and a.e.,} \\ \nabla v_{p_i} \rightharpoonup \nabla v_\infty,\ (v_{p_i})_t \rightharpoonup (v_\infty)_t & \text{weakly in } L^2(\mathbb{R}^N \times (0,T)). \end{cases}$$

Moreover, the limit function v_∞ satisfies

$$\begin{cases} f(\cdot,t) - (v_\infty)_t(.,t) \in \partial F_\infty(v_\infty(\cdot,t)) & \text{a.e. } t \in (0,T), \\ v_\infty(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{6.2}$$

where the limit functional is given by

$$F_\infty(v) = \begin{cases} 0 & \text{if } v \in L^2(\mathbb{R}^N),\ |\nabla v| \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

This limit problem (6.2) is interpreted in [13] to explain the movement of a sandpile ($v_\infty(x,t)$ describes the amount of the sand at the point x at time t), the main assumption being that the sandpile is stable when the slope is less than or equal to one and unstable if not. So, this local model are based on the requirement that the slope of sandpiles is at most one. However, a more realistic model would require the slope constraint only on a larger scale, with no slope requirements on a smaller scale. This is exactly the case for the nonlocal models presented here.

6.2. **The nonlocal model.** Let Ω be a convex domain in $\mathbb{R}^N$ and consider the evolution problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J_{p,\varepsilon}(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))dy + f(x,t), \\ u(x,0) = u_0(x), \qquad\qquad x \in \Omega,\ t > 0. \end{cases} \tag{6.3}$$

Associated to this problem is the energy functional $G_p^{J_{p,\varepsilon}}$ given in Section 2. With a formal computation, taking limits as $p \to +\infty$, we arrive at the limit problems

$$\begin{cases} f(\cdot,t) - u_t(\cdot,t) \in \partial G_\infty^\varepsilon(u(.,t)) & \text{a.e. } t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{6.4}$$

with associated functionals

$$G_\infty^\varepsilon(u) = \begin{cases} 0 & \text{if } |u(x)-u(y)| \le \varepsilon, \text{ for } |x-y| \le \varepsilon;\ \ x,y \in \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

The limit problem for the local model is

$$\left\{\begin{array}{ll} f(\cdot,t)-(v_\infty)_t(\cdot,t)\in\partial F_\infty(v_\infty(\cdot,t)) & \text{a.e. } t\in(0,T),\\ v_\infty(x,0)=g(x) & \text{in } \Omega, \end{array}\right. \tag{6.5}$$

where the functional F_∞ is defined in $L^2(\Omega)$ by

$$F_\infty(v)=\left\{\begin{array}{ll} 0 & \text{if } \ |\nabla v|\le 1,\\ +\infty & \text{otherwise.}\end{array}\right.$$

Observe that in these limit problems we assume that the material is confined in a domain Ω; thus we are looking at models for sandpiles inside a container (see [30] for a local model).

The main results concerning these problems are stated in the following theorem.

Theorem 6.1. *Let Ω be a convex domain in $\mathbb{R}^N$.*

(1) *Let $T>0$, $u_0\in L^\infty(\Omega)$ such that $|u_0(x)-u_0(y)|\le 1$ for $x-y\in\Omega\cap\operatorname{supp}(J)$. Take $f\in L^2(0,T;L^\infty(\Omega))$ and let u_p be the unique solution of* (6.3)*. Then, if u_∞ is the unique solution of* (6.4) *with $\varepsilon=1$,*

$$\lim_{p\to\infty}\sup_{t\in[0,T]}\|u_p(\cdot,t)-u_\infty(\cdot,t)\|_{L^2(\Omega)}=0.$$

(2) *Let $T>0$, $u_0\in W^{1,\infty}(\Omega)$ such that $|\nabla u_0|\le 1$, take $f\in L^2(0,T;L^2(\Omega))$ and consider $u_{\infty,\varepsilon}$, the unique solution of* (6.4)*. Then, if v_∞ is the unique solution of* (6.5)*, we have*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\|u_{\infty,\varepsilon}(\cdot,t)-v_\infty(\cdot,t)\|_{L^2(\Omega)}=0.$$

Observe the statement (1) proves the formal computation resulting in (6.4), and (2) states that when the scale converges to zero we cover the local model.

A mass transport interpretation. If we define

$$K_\infty:=\left\{u\in L^2(\Omega)\ :\ |u(x)-u(y)|\le 1,\ \text{for }\ x-y\in\operatorname{supp}(J)\right\},$$

we have that the above functional G^1_∞ is given by the indicator function of K_∞, that is, $G^J_\infty=I_{K_\infty}$. Then the *nonlocal limit problem* (6.5) can be written as

$$\left\{\begin{array}{ll} f(\cdot,t)-u_t(\cdot,t)\in\partial I_{K_\infty}(u(\cdot,t)) & \text{a.e. } t\in(0,T),\\ u(x,0)=u_0(x). & \end{array}\right. \tag{6.6}$$

Then , we can also give an interpretation of the limit problem (6.6) in terms of Monge-Kantorovich mass transport theory as in [29], [30] (see [45] for a general introduction to mass transportation problems, and [5] for a detailed study of this situation). Let us see this. Consider the distance

$$d_1(x,y)=\left\{\begin{array}{ll} 0 & \text{if } \ x=y,\\ 1 & \text{if } \ 0<|x-y|\le 1,\\ 2 & \text{if } \ 1<|x-y|\le 2,\\ \vdots & \end{array}\right.$$

where $|\,.\,|$ denotes the Euclidean norm. Note that this function d measures distances with jumps of length one. Then, given two measures f_+, f_- in $L^1(\Omega)$, and supposing the overall condition of mass balance

$$\int_\Omega f_+ \, dx = \int_\Omega f_- \, dy, \tag{6.7}$$

the Monge problem associated to the cost function the distance d is given by

$$\text{minimize} \int d_1(x, s(x)) \, f_+(x) dx$$

among the set of maps s that transport f_+ into f_-, which means that

$$\int_{\mathbb{R}^N} h(s(x)) f_+(x) \, dx = \int_{\mathbb{R}^N} h(y) f_-(y) \, dy$$

for each continuous function $h : \mathbb{R}^N \to \mathbb{R}$.

The original problem studied by Monge corresponds to the cost function $d(x,y) := |x-y|$ the Euclidean distance. In general, the Monge problem is ill-posed. To overcome the difficulties, in 1942 L. V. Kantorovich, [37], proposed to study a relaxed version of the Monge problem and, what is more relevant here, introduced a dual variational principle.

Denote by $\pi_i : \mathbb{R}^N \times \mathbb{R}^N$ the projections, $\pi_1(x,y) := x$, $\pi_2(x,y) := y$. Given a Radon measure μ in $\Omega \times \Omega$, its marginals are defined by $proj_x(\mu) := \pi_1 \# \mu$, $proj_y(\mu) := \pi_2 \# \mu$. Let $\pi(f^+, f^-)$ the set of transport plans between f^+ and f^-, that is the set of nonnegative Radon measures μ in $\Omega \times \Omega$ such that $\mathrm{proj}_x(\mu) = f^+(x) \, dx$ and $\mathrm{proj}_y(\mu) = f^-(y) \, dy$. The Monge-Kantorovich relaxed problem for d_1 consists in finding a measure $\mu^* \in \pi(f^+, f^-)$ which minimizes the cost functional

$$\mathcal{K}_{d_1}(\mu) := \int_{\Omega \times \Omega} d_1(x,y) \, d\mu(x,y),$$

in the set $\pi(f^+, f^-)$. A minimizer μ^* is called an *optimal transport plan* between f^+ and f^-. In general (see [3, Propostion 2.1]), $\inf\{\mathcal{K}_{d_1}(\mu) \ : \ \mu \in \pi(f^+, f^-)\} \leq \inf\{\mathcal{F}_{d_1}(T) \ : \ T \in \mathcal{A}(f^+, f^-)\}$.

Since d_1 is a lower semi-continuous metric, it is well known the existence of an optimal transport plan (see [3, 41] and the references therein), and what is quite interesting (see for instance [45, Theorem 1.14]):

$$\min\{\mathcal{K}_{d_1}(\mu) \ : \ \mu \in \pi(f^+, f^-)\} = \sup\left\{\mathcal{P}_{f^+, f^-}(u) \ : \ u \in K_{d_1}(\Omega)\right\}, \tag{6.8}$$

where

$$\mathcal{P}_{f^+, f^-}(u) := \int_\Omega u(x)(f^+(x) - f^-(x)) \, dx,$$

and $K_{d_1}(\Omega)$ is the set of 1-Lipschitz funtions w.r.t. d_1,

$$K_{d_1}(\Omega) := \left\{u \in L^2(\Omega) \ : \ |u(x) - u(y)| \leq d_1(x,y) \ \text{ for all } x, y \in \Omega\right\}.$$

The maximizers u^* of the right hand side of (6.8) are called *Kantorovich (transport) potentials.*

With these definitions and notation we have the following result.

Theorem 6.2. *The solution $u_\infty(\cdot, t)$ of the limit problem* (6.6) *is a solution of the dual problem*

$$\max_{u \in K_\infty} \int_{\mathbb{R}^N} u(x)(f_+(x) - f_-(x)) dx$$

when the involved measures are the source term $f_+ = f(x,t)$ and the time derivative of the solution $f_- = u_t(x,t)$.

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