ISOLATION AND SIMPLICITY FOR THE FIRST EIGENVALUE OF THE p-LAPLACIAN WITH A NONLINEAR BOUNDARY CONDITION

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Abstract. In this paper we prove the simplicity and isolation of the first eigenvalue for the problem, $\Delta_p u = |u|^{p-2}u$ in a bounded smooth domain $\Omega \subset {}^N$, with a nonlinear boundary condition given by $|\nabla u|^{p-2}\partial u/\partial \nu = \lambda |u|^{p-2}u$ on the boundary of the domain.

1. Introduction.

In this paper we study the first eigenvalue for the following problem

$$
\left\{
\begin{array}{ll}
\Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\[2ex]
|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \partial\Omega.
\end{array}
\right.
\tag{1.1}
$$

Here Ω is a bounded domain in ^{N} with smooth boundary, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. In the linear case, that is for $p = 2$, this eigenvalue problem is known as the *Steklov* problem, [3].

Problems of the form (1.1) appears in a natural way when one considers the Sobelev trace inequality. In fact, the immersion $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ is compact, hence there exits a constant λ_1 such that

$$\lambda_1^{1/p}\|u\|_{L^p(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}.$$

The extremals (functions where the constant is attained) are solutions of (1.1). This Sobolev trace constant λ_1 can be characterized as

$$
\lambda_1 = \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p + |u|^p \, dx, \qquad \int_{\partial\Omega} |u|^p = 1 \right\},
\tag{1.2}
$$

and is the first eigenvalue of (1.1) in the sense that $\lambda_1 \le \lambda$ for any other eigenvalue λ.

In [13] it is proved that there exists a sequence of eigenvalues λ_n of (1.1) such that $\lambda_n \to +\infty$ as $n \to +\infty$. This is donne using standard variational arguments together with the Sobolev trace immersion that provide the necessary compactness.

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Indeed, for solutions of (1.1) we can understand critical points of the associated energy functional

$$\mathcal{F}(u)=\frac{1}{p}\int_{\Omega}|\nabla u|^{p}+\frac{1}{p}\int_{\Omega}|u|^{p}-\frac{\lambda}{p}\int_{\partial\Omega}|u|^{p},$$

This functional $\mathcal{F}$ is well defined and C^1 in $W^{1,p}(\Omega)$ and the usual min-max techniques can be applied, see [13]. Also see [15] for similar results for the $p-$Laplacian with Dirichlet boundary conditions.

In this paper we prove the following result,

Theorem 1.1. *λ_1 is isolated and simple.*

We remark that this theorem says that the extremals of the Sobolev trace inequality are unique up to multiplication by a real number. In the special case of a ball, $\Omega = B(0, R)$, our result implies that the first eigenfunction is radial. In fact, if $u_1(x)$ is an eigenfunction associated to λ_1 and $R(x)$ is any rotation then $u_1(R(x))$ is also an eigenfunction, by our result we have that $u_1(x) = u_1(R(x))$. We conclude that u_1 must be radial. Also from our results it follows that any other eigenvalue has non-radial eigenfunctions as they have to change sign on the boundary (see Lemma 2.4).

The study of the eigenvalue problem when the nonlinear term is placed in the equation, that is when one consider a quasilinear problem of the form $-\Delta_p u = \lambda u$ with Dirichlet boundary conditions, has received considerable attention, see for example [1], [2], [14], [15], [17], etc.

However, nonlinear boundary conditions have only been considered in recent years. For the Laplace operator with nonlinear boundary conditions see for example [6], [7], [8], [16], [19]. For elliptic systems with nonlinear boundary conditions see [11], [12]. For previous work for the $p-$Laplacian with nonlinear boundary conditions of different type see [5], [13] and [18]. Also, one is lead to nonlinear boundary conditions in the study of conformal deformations on Riemannian manifolds with boundary, see for example [4], [9] and [10].

2. Proof of the main result

In this section we prove that the first eigenvalue λ_1 is isolated and simple. To clarify the exposition, we will divide the proof in several lemmas.

Lemma 2.1. *Let u_1 be an eigenfunction with eigenvalue λ_1, then u_1 does not change sign on Ω. Moreover, if u_1 is $C^{1,\alpha}$, it does not vanish on $\overline{\Omega}$.*

Proof. We have that $|u_1|$ is also a minimizer of (1.2). By the maximum principle (see [20]) we have that $|u_1| > 0$ in Ω. Assume that u_1 is regular and that there exists $x_0 \in \partial\Omega$ such that $u_1(x_0) = 0$, by the Hopf Lemma (see [20]) we have that the normal derivative has strict sign, $\frac{\partial|u_1|}{\partial\nu}(x_0) < 0$, but the boundary condition imposes $\frac{\partial|u_1|}{\partial\nu}(x_0) = 0$, a contradiction that proves that $|u_1| > 0$ in $\overline{\Omega}$. The result follows. □

Now we state a auxiliary lemma,

Lemma 2.2. *a) Let $p \geq 2$, then for all ξ_1, $\xi_2 \in {}^N$*

$$|\xi_2|^p \geq |\xi_1|^p + p|\xi_1|^{p-2}\langle \xi_1, \xi_2 - \xi_1\rangle + C(p)|\xi_1 - \xi_2|^p,$$

b) Let $p < 2$, then for all ξ_1, $\xi_2 \in {}^N$

$$|\xi_2|^p \geq |\xi_1|^p + p|\xi_1|^{p-2}\langle \xi_1, \xi_2 - \xi_1\rangle + C(p)\frac{|\xi_1 - \xi_2|^p}{(|\xi_2| + |\xi_1|)^{2-p}},$$

where $C(p)$ is a constant depending only on p.

Proof. See [17]. □

Lemma 2.3. *λ_1 is simple. Let u, v be two eigenfunctions associated with λ_1, then there exists c such that $u = cv$.*

Proof. By lemma (2.1) we can assume that u, v are positive in Ω. We perform the following calculations assuming that u, v are strictly positive in $\overline{\Omega}$, to obtain our result we can consider $u+\varepsilon$ and $v+\varepsilon$ and let $\varepsilon \to 0$ at the end as in [17]. Therefore we can take $\eta_1 = (u^p - v^p)/u^{p-1}$ and $\eta_2 = (v^p - u^p)/v^{p-1}$ as test functions in the weak form of (1.1) satisfied by u and v respectively. We have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) = \lambda\int_{\partial\Omega}|u|^{p-2}u\left(\frac{u^p - v^p}{u^{p-1}}\right) - \int_\Omega |u|^{p-2}u\left(\frac{u^p - v^p}{u^{p-1}}\right),$$

and

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\left(\frac{v^p - u^p}{v^{p-1}}\right) = \lambda\int_{\partial\Omega}|v|^{p-2}v\left(\frac{v^p - u^p}{v^{p-1}}\right) - \int_\Omega |v|^{p-2}v\left(\frac{v^p - u^p}{v^{p-1}}\right).$$

Adding both equations we get

$$0 = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) + \int_\Omega |\nabla v|^{p-2}\nabla v \nabla\left(\frac{v^p - u^p}{v^{p-1}}\right). \tag{2.1}$$

Using that

$$\nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) = \nabla u - p\frac{v^{p-1}}{u^{p-1}}\nabla v + (p-1)\frac{v^p}{u^p}\nabla u,$$

we obtain that the first term of (2.1) is

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) =$$

$$\int_\Omega |\nabla u|^p - p\int_\Omega \frac{v^{p-1}}{u^{p-1}}|\nabla u|^{p-2}\nabla v \nabla u + \int_\Omega (p-1)\frac{v^p}{u^p}|\nabla u|^p =$$

$$\int_\Omega |\nabla \ln u|^p u^p - p\int_\Omega v^p|\nabla \ln u|^{p-2}\langle \nabla \ln u, \nabla \ln v\rangle uv + \int_\Omega (p-1)|\nabla \ln u|^p v^p.$$

We also have an analogous expression for the second term of equation (2.1). Using both expressions we get that equation (2.1) becomes,

$$0 = \int_\Omega (u^p - v^p)(|\nabla \ln u|^p - |\nabla \ln v|^p) - p\int_\Omega v^p(|\nabla \ln u|^{p-2}\langle \nabla \ln u, \nabla \ln v - \nabla \ln u\rangle -$$

$$p\int_\Omega u^p(|\nabla \ln v|^{p-2}\langle \nabla \ln v, \nabla \ln u - \nabla \ln v\rangle.$$

Taking $\xi_1 = |\nabla \ln u|$ and $\xi_2 = |\nabla \ln v|$ and using Lemma 2.2 we get, for $p \geq 2$,

$$0 \geq \int_\Omega C(p)|\nabla \ln u - \nabla \ln v|^p(u^p + v^p).$$

Hence,

$$0 = |\nabla \ln u - \nabla \ln v|.$$

This implies that $u = kv$, as we wanted to prove. For $p < 2$ we use the second part of Lemma 2.2 as above. □

Now we turn our attention to the proof of the isolation of the first eigenvalue, in order to prove this we need the following nodal result,

Lemma 2.4. *Let w be an eigenfunction corresponding to $\lambda \neq \lambda_1$, then w changes sign on $\partial\Omega$, that is $w^+ \mid_{\partial\Omega} \neq 0$ and $w^- \mid_{\partial\Omega} \neq 0$. Moreover there exists a constant C such that*

$$|\partial\Omega^+| \geq C\lambda^{-\beta}, \qquad |\partial\Omega^-| \geq C\lambda^{-\beta} \tag{2.2}$$

where $\partial\Omega^+ = \partial\Omega \cap \{w > 0\}$, $\partial\Omega^+ = \partial\Omega \cap \{w < 0\}$, $\beta = (N-1)/(p-1)$ if $1 < p < N$ and $\beta = 2$ if $p \geq N$. Here $|A|$ denotes the $N-1$ dimensional measure of a subset A of the boundary.

Proof. Assume that w does not change sign in Ω, then we can assume that $w > 0$ in Ω using ideas similar to those of Lemma 2.1. Let u_1 be a positive eigenfunction associated to λ_1. Making similar computations as the ones performed in the proof of Lemma 2.3 we arrive at

$$(\lambda_1 - \lambda) \int_{\partial\Omega} (u_1^p - w^p) \geq C \int_{\Omega} |\nabla \ln w - \nabla \ln u_1|^p (u_1^p + w^p) \geq 0.$$

Therefore if we take kw instead of w we get that for every $k > 0$ we have

$$\int_{\partial\Omega} (u_1^p - k^p w^p) \leq 0$$

a contradiction if we take

$$k^p \left(\int_{\partial\Omega} w^p \right) < \left(\int_{\partial\Omega} u_1^p \right).$$

Therefore w changes sign in Ω and by the maximum principle, [20], also w changes sign in $\partial\Omega$.

Let us use w^- as test function in the weak form of (1.1) satisfied by w to obtain

$$\int_{\Omega} |\nabla w^-|^p + \int_{\Omega} |w^-|^p = \lambda \int_{\partial\Omega \cap \{w<0\}} |w^-|^p.$$

Hence,

$$\|w^-\|^p_{W^{1,p}(\Omega)} \leq \lambda \left(\int_{\partial\Omega} |w^-|^{p\alpha} \right)^{1/\alpha} |\partial\Omega^-|^{1/\beta}.$$

If $1 < p < N$ we choose $\alpha = (N-1)/(N-p)$ and $\beta = (N-1)/(p-1)$. Now we use the trace theorem to get that there exists a constant C such that

$$\|w^-\|^p_{L^{p\alpha}(\partial\Omega)} \leq C \|w^-\|^p_{W^{1,p}(\Omega)}.$$

If $p \geq N$ we choose $\alpha = \beta = 2$ and we argue as before using that $W^{1,p}(\Omega) \hookrightarrow L^{2p}(\partial\Omega)$. A similar argument works for w^+. □

Lemma 2.5. *Let $\phi \in W^{1,p}(\Omega)'$. Then there exists a unique weak solution $u \in W^{1,p}(\Omega)$ of*

$$-\Delta_p u + |u|^{p-2}u = \phi.$$

Moreover, the operator $A_p : \phi \mapsto u$ is continuous.

Proof. See [13]. □

With these lemmas we can prove the isolation of λ_1.

Lemma 2.6. *λ_1 is isolated, that is there exists $a > \lambda_1$ such that λ_1 is the unique eigenvalue in $[0, a]$.*

Proof. From the characterization of λ_1 it is easy to see that $\lambda_1 \leq \lambda$ for every eigenvalue λ. Assume that λ_1 is not isolated, then there exists a sequence λ_k with $\lambda_k > \lambda_1$, $\lambda_k \searrow \lambda_1$. Let w_k be an eigenfunction associated to λ_k, we can assume that $\|w_k\|_{W^{1,p}(\Omega)} = 1$. Therefore we can extract a subsequence (that we still denote by w_k) such that $w_k \to u_1$ in $L^p(\partial\Omega)$. Let us define $\phi_k \in (W^{1,p}(\Omega))'$ as

$$\phi_k(u) = \lambda_k \int_{\partial\Omega} |w_k|^{p-2} w_k u$$

and $\phi \in (W^{1,p}(\Omega))'$ by

$$\phi(u) = \lambda_1 \int_{\partial\Omega} |u_1|^{p-2} u_1 u.$$

From the $L^p(\partial\Omega)$ convergence of w_k to u_1 we get that ϕ_k converges to ϕ in $(W^{1,p}(\Omega))'$. Using the continuity of A_p given by Lemma 2.5 we get that the sequence w_k converge strongly in $W^{1,p}(\Omega)$. Therefore, passing to the limit in the weak form of (1.1) we get that u_1 is an eigenfunction with eigenvalue λ_1. By lemma 2.1 we can assume that $u_1 > 0$ on $\partial\Omega$. By Egorov's Theorem we can find a subset A_ε of $\partial\Omega$ such that $|A_\varepsilon| < \varepsilon$ and $w_k \to u_1 > 0$ uniformly in $\partial\Omega \setminus A_\varepsilon$. This contradicts the fact that, by (2.2), we have, for every k

$$|\partial\Omega_k^-| = \partial\Omega \cap \{w_k < 0\} \geq C\lambda_k^{-(N-1)/(p-1)}.$$

This completes the proof. □

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References

[1] W. Allegretto and Y. X. Huang, *A picone's identity for the p−laplacian and applications.* Nonlinear Anal. TM&A. Vol. 32 (7) (1998), 819-830.

[2] A. Anane, *Simplicité et isolation de la premiere valeur propre du p−laplacien avec poids.* C. R. Acad. Sci. Paris, 305 (I), (1987), 725-728.

[3] I. Babuska and J. Osborn, Eigenvalue Problems, Handbook of Numer. Anal., Vol. II (1991). North-Holland.

[4] P. Cherrier, *Problèmes de Neumann non linéaires sur les variétés Riemanniennes.* J. Funct. Anal. Vol. 57 (1984), 154-206.

[5] F.-C. St. Cîrstea and V. Radulescu, *Existence and non-existence results for a quasilinear problem with nonlinear boundary conditions.* J. Math. Anal. Appl. 244 (2000), 169-183.

[6] M. Chipot, I. Shafrir and M. Fila, *On the solutions to some elliptic equations with nonlinear boundary conditions.* Adv. Differential Equations. Vol. 1 (1) (1996), 91-110.

[7] M. Chipot, M. Chlebík, M. Fila and I. Shafrir, *Existence of positive solutions of a semilinear elliptic equation in* $^{N}_{+}$ *with a nonlinear boundary condition.* J. Math. Anal. Appl. 223 (1998), 429-471.
[8] M. del Pino and C. Flores, *Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains.* Preprint.
[9] J. F. Escobar, *Uniqueness theorems on conformal deformations of metrics, Sobolev inequalities, and an eigenvalue estimate.* Comm. Pure Appl. Math. Vol. 43 (1990), 857-883.
[10] J. F. Escobar, *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature.* Ann. of Math. (2). Vol. 136 (1992), 1-50.
[11] J. Fernández Bonder, J.P. Pinasco and J.D. Rossi, *Existence results for a Hamiltonian elliptic system with nonlinear boundary conditions*, Electron. J. Differential Equations. Vol. 1999 (1999), No. 40, 1-15.
[12] J. Fernández Bonder and J.D. Rossi, *Existence for an elliptic system with nonlinear boundary conditions via fixed point methods.* Adv. Differential Equations. Vol. 6, (2001), 1-20.
[13] J. Fernández Bonder and J.D. Rossi, *Existence results for the* $p-$*Laplacian with nonlinear boundary conditions.* To appear in J. Math. Anal. Appl.
[14] J. Garcia-Azorero and I. Peral, *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term.* Trans. Amer. Math. Soc. Vol. 323 (2) (1991), 877-895.
[15] J. Garcia-Azorero and I. Peral, *Existence and non-uniqueness for the* $p-$*Laplacian: nonlinear eigenvalues.* Comm. Partial Differential Equations. Vol. 12 (1987), 1389-1430.
[16] B. Hu, *Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition*, Differential Integral Equations. Vol. 7, Nro. 2 (1994), 301-313.
[17] P. Lindqvist, *On the equation* $\Delta_p u + \lambda |u|^{p-2} u = 0$. Procc. A.M.S., 109-1, (1990), 157–164.
[18] K. Pflüger, *Existence and multiplicity of solutions to a* $p-$*Laplacian equation with nonlinear boundary condition.* Electron. J. Differential Equations 10 (1998), 1-13.
[19] S. Terraccini, *Symmetry properties of positive solutions to some alliptic equations with nonlinear boundary conditions.* Differential Integral Equations. Vol. 8 (8) (1995), 1911-1922.
[20] J. L. Vazquez, *A strong maximum principle for some quasilinear elliptic equations.* Appl. Math. Optim. (1984), 191–202.

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