OPTIMAL REGULARITY AT THE FREE BOUNDARY FOR THE INFINITY OBSTACLE PROBLEM

J.D. ROSSI, E.V. TEIXEIRA AND J.M. URBANO

Abstract. This paper deals with the obstacle problem for the infinity Laplacian. The main results are a characterization of the solution through comparison with cones that lie above the obstacle and the sharp $C^{1,\frac{1}{3}}$–regularity at the free boundary.

1. Introduction

The regularity of infinity harmonic functions is an outstanding issue in the theory of nonlinear partial differential equations. The belief that viscosity solutions of $\Delta_\infty u = 0$ are of class $C^{1,\frac{1}{3}}$ has hitherto remained unproven despite some recent exciting developments. The flatland example of Aronsson

$$u(x,y) = |x|^{\frac{4}{3}} - |y|^{\frac{4}{3}}$$

sets the framework to what can be expected: the first derivatives of u are Hölder continuous with exponent $1/3$, whereas its second derivatives do not exist on the lines $x = 0$ and $y = 0$. The sharpest results to date are due to Evans and Savin, who prove in [8] that infinity harmonic functions in the plane are of class $C^{1,\alpha}$, building upon Savin's breakthrough in [17] (the optimal α remains unknown even in 2-D), and to Evans and Smart, who recently obtained in [9] the everywhere differentiability, irrespective of the dimension.

This paper addresses the obstacle problem for the infinity Laplacian and its most striking results concern the behaviour at the free boundary. We prove, under natural assumptions on the obstacle, that the solution leaves the obstacle as a $C^{1,\frac{1}{3}}$–function and that this regularity is optimal. The sharp estimates we derive are yet another conspicuous hint towards the optimal regularity for infinity harmonic functions.

Obstacle problems in infinite dimensional spaces, where operators are naturally degenerate, are studied in [18]. A prototype example is given by $F(D^2u) = \mathrm{Trace}(AD^2u)$, for $A \in \mathcal{S}(H)$ in the trace class, *i.e.*,

Date: June 24, 2012.

2000 *Mathematics Subject Classification.* 35B65, 35R35, 35J70.

Key words and phrases. Obstacle problem; infinity Laplacian; free boundary; optimal regularity.

$\sum |\lambda_j| < \infty$, where λ_j are the eigenvalues of A. The main result in [18] is that the solution of the obstacle problem satisfies the bounds

$$|\lambda_j||D_{jj}u| < C, \quad \forall j,$$

provided the obstacle is semi-concave. It is a perfect generalization of the optimal $C^{1,1}$–regularity for the obstacle problem in finite dimensional spaces. For problems governed by the infinity Laplacian, a naive inference indicates that $|Du_\infty|^2|D^2u_\infty|$ should remain bounded for points at the free boundary. Such observation brings us to the recent work [1], where it is proven that

$$|D^2v| \lesssim |Dv|^{-\delta} \quad \Longrightarrow \quad v \in C^{1,\frac{1}{1+\delta}}.$$

Taking $\delta = 2$ discloses the optimal regularity at the free boundary for the infinity obstacle problem, ultimately proven in this paper.

The heuristics behind the proof is the following: showing that a given function v is of class $C^{1,\frac{1}{3}}$ at a point x_0 amounts to finding an affine function ℓ for which

$$|v(x) - \ell(x)| = \mathrm{O}(|x - x_0|^{4/3}).$$

As mentioned above, such a task remains unaccomplished for an arbitrary infinity harmonic function; however, for a solution u_∞ of the infinity obstacle problem, it is expected that at a free boundary point $x_0 \in \partial\{u_\infty > \Psi\}$,

$$\nabla(u_\infty - \Psi)(x_0) = 0.$$

In such a geometric scenario, establishing the $C^{1,\frac{1}{3}}$–regularity of $u_\infty - \Psi$ reduces to proving

$$(u_\infty - \Psi)(x) = \mathrm{O}(|x - x_0|^{4/3}).$$

Thus, a scaling-sharp flatness improvement, in the same spirit as in [20], gives the full optimal regularity for $u_\infty - \Psi$; this, in turn, implies

$$u_\infty \in C^{1,\frac{1}{3}}$$

along the free boundary, provided the obstacle Ψ is smooth enough.

The obstacle problem for elliptic operators has been extensively studied. The classical setting amounts at minimizing the energy

$$E(u) = \int_\Omega |Du|^2$$

among the functions that coincide with a given function F at the boundary of $\Omega \subset \mathbb{R}^d$ and remain above a prescribed obstacle Ψ. Such a problem is motivated by the description of the equilibrium position of a membrane (the graph of the solution) attached at level F along the boundary of Ω and that is forced to remain above the obstacle in the interior of Ω. The same mathematical framework appears in many other contexts: fluid filtration in porous media, elasto-plasticity, optimal control or financial mathematics, to name just a few.

On the other hand, if we pass to the limit, as $p \to \infty$, in a sequence (u_p) of $p-$harmonic functions, that is, solutions of $\Delta_p u_p = 0$, with given boundary values, the limit exists (in the uniform topology) and is a solution of the infinity Laplace equation (see [3])

$$\Delta_\infty u = \sum_{i,j=1}^{d} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0.$$

The infinity Laplacian is connected with the optimal Lipschitz extension problem [11], and arises also in the context of certain random tug-of-war games [2, 16], mass transportation problems [10] and several other applications, such as image reconstruction and enhancement [6]. See also the recent approach of [14] to a two-phase problem of mixed type.

In the next section, we introduce the infinity obstacle problem and obtain a solution u_∞, passing to the limit, as $p \to \infty$, in a sequence of solutions u_p to the obstacle problem for the p-Laplacian. We gather a few elementary properties of the solution and study a radially symmetric explicit example. Let us remark that the limit obtained here does not necessarily coincide with the solution of the infinity obstacle problem obtained by direct methods in [4].

Section 3 of the paper deals with characterizations of the limit. We first show that u_∞ is the smallest infinity superharmonic function in Ω that is above the obstacle and equals F on the boundary, a result that implies its uniqueness. Then we establish a sort of comparison with cones that lie above the obstacle. This characterization is interesting in its own right but it also implies a regularity result at the free boundary, a warm-up for what will come later. The section closes with the analysis of the behaviour at infinity of the coincidence sets for the p-obstacle problem and its relation with the coincidence set of the limiting problem.

The heart of the paper is Section 4, where the behaviour of the solution at the free boundary is analyzed. We establish the optimal asymptotic profile near the free boundary, showing u_∞ behaves as a $C^{1,\frac{1}{3}}$–function. We use this sharp information to deduce the uniform positive density of the region $\{u_\infty > \Psi\}$. In particular, the free boundary does not develop cusps pointing inwards to the coincidence set.

2. The limit as $p \to \infty$ for the p–obstacle problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded smooth domain, F a Lipschitz function on $\partial\Omega$ and $1 < p < \infty$. Given an obstacle $\Psi \colon \overline{\Omega} \to \mathbb{R}$, with

$$\sup_{\partial\Omega} \Psi < \inf_{\partial\Omega} F, \tag{2.1}$$

the p-degenerate obstacle problem for Ψ refers to the minimization problem

$$\mathrm{Min}\left\{ \int_\Omega |Dv(x)|^p dx \;\Big|\; v \in W^{1,p}_F \text{ and } v \geq \Psi \right\}. \tag{2.2}$$

Here $W^{1,p}_F$ means the set of functions in $W^{1,p}(\Omega)$ with trace F on $\partial\Omega$.

Simple soft functional analysis arguments assure that (2.2) has a unique solution u_p. Let z be a Lipschitz extension of F such that $z \geq \Psi$ (for the proof of the existence of such z see Proposition 3.3). Since z competes in the minimization problem (2.2) for every p, if L denotes the Lipschitz norm of z, we have

$$\left(\int_\Omega |Du_p|^p \right)^{1/p} \leq L|\Omega|^{1/p}.$$

For a fixed q, we can write

$$\left(\int_\Omega |Du_p|^q \right)^{1/q} \leq \left(\int_\Omega |Du_p|^p \right)^{1/p} |\Omega|^{\frac{p-q}{pq}} \leq L|\Omega|^{1/p}|\Omega|^{\frac{p-q}{pq}}.$$

Hence, we have a uniform bound for the sequence (u_p) in every $W^{1,q}(\Omega)$. Taking the limit as $p \to \infty$, we conclude that there exists a function u_∞ such that, up to a subsequence, $u_p \to u_\infty$, locally uniformly in $\overline{\Omega}$ and weakly in every $W^{1,q}(\Omega)$. Clearly, $u_\infty \geq \Psi$ pointwise. Also,

$$\left(\int_\Omega |Du_\infty|^q \right)^{1/q} \leq L|\Omega|^{\frac{1}{q}} \qquad \forall q > 1.$$

We then conclude that u_∞ is a Lipschitz function, with

$$\|Du_\infty\|_{L^\infty(\Omega)} \leq L.$$

Since this holds being L the Lipschitz constant of any extension of F that is above Ψ, we conclude that u_∞ is a solution of the minimization problem

$$\min_{w|_{\partial\Omega}=F;\ w\geq\Psi \text{ in } \Omega} \mathrm{Lip}(w). \tag{2.3}$$

The minimizers u_p are weak, and hence viscosity, solutions (see [10]) of the following obstacle problem:

$$\begin{cases} u_p(x) & = & F(x) & \text{on } \partial\Omega, \\ u_p(x) & \geq & \Psi(x) & \text{in } \Omega, \\ -\Delta_p u_p & = & 0 & \text{in } \Omega \setminus A_p := \{u_p > \Psi\}, \\ -\Delta_p u_p & \geq & 0 & \text{in } \Omega. \end{cases}$$

Concerning the PDE problem satisfied by u_∞, we verify that it is a viscosity solution to the obstacle problem for the infinity Laplacian:

$$\begin{cases} u_\infty(x) & = & F(x) & \text{on } \partial\Omega, \\ u_\infty(x) & \geq & \Psi(x) & \text{in } \Omega, \\ -\Delta_\infty u_\infty & = & 0 & \text{in } \Omega \setminus A_\infty = \{u_\infty > \Psi\}, \\ -\Delta_\infty u_\infty & \geq & 0 & \text{in } \Omega. \end{cases}$$

Indeed, fix a point y in the set $\{u_\infty > \Psi\}$. From the uniform convergence, $u_p > \Psi$ in a neighbourhood of y, provided $p \gg 1$. Hence, taking the limit as $p \to \infty$ in the viscosity sense, we obtain

$$-\Delta_\infty u_\infty = 0 \quad \text{in } \{u_\infty > \Psi\}.$$

On the other hand, a uniform limit of u_p verifies

$$-\Delta_\infty u_\infty \geq 0, \qquad \text{in } \Omega,$$

since for every p, u_p verifies

$$-\Delta_p u_p \geq 0, \qquad \text{in } \Omega$$

in the viscosity sense.

To gain some insight on the problem, we next construct a radially symmetric explicit example. Let us consider the p-obstacle problem in $B_2 \subset \mathbb{R}^d$, with zero boundary data and the spherical cap

$$\psi(x) = 1 - |x|^2$$

as the obstacle. It is formulated as the following minimization problem:

$$\text{Min} \left\{ \int_{B_2} |Dv(x)|^p dx \;\middle|\; v \in W_0^{1,p}(B_2) \text{ and } v(x) \geq \psi(x) \right\}.$$

As mentioned before, the above minimization problem has a unique minimizer u_p. By symmetry, we conclude u_p is radially symmetric, *i.e.*, $u_p(x) = u_p(|x|)$. By the geometry of the obstacle problem, as well as its regularity theory, we know that there exists an $h = h(p, d)$, that depends on p and dimension, such that

$$\begin{cases} u_p(x) = \psi(x) & \text{in} \quad |x| \leq h \\ \Delta u_p = 0 & \text{in} \quad 2 > |x| > h \\ u_p \in C^{1,\alpha_p} & \text{in} \quad B_2 \\ \|Du_p\|_{L^\infty(B_\rho)} \leq C(\rho, d), & \end{cases} \tag{2.4}$$

for a constant $C(\rho, d)$ independent of p. Such an estimate has been obtained in the previous section. In particular, as observed before, up to a subsequence, u_p converges locally uniformly to a function u_∞. Furthermore, u_∞ solves $\Delta_\infty u_\infty = 0$ within $\{u_\infty > \psi\}$ in the viscosity sense.

Our goal is to solve the p-obstacle problem explicitly and then analyze the limiting function u_∞. In view of the properties listed in (2.4), we are initially led to search for p-harmonic radially symmetric functions. If $g(x) = f(r)$, then

$$\Delta_p g = |f'(r)|^{p-2}\left\{(p-1)f''(r) + \frac{d-1}{r}f'(r)\right\}. \tag{2.5}$$

Solving the homogeneous ODE, we obtain

$$f(r) = \begin{cases} a + b \cdot r^{\frac{1-d}{p-1}+1} & \text{if} \quad p \neq d \\ a + b \cdot \ln r & \text{if} \quad p = d, \end{cases} \tag{2.6}$$

for any constants $a, b \in \mathbb{R}$. Returning to the obstacle problem (we will only deal with the case, $p \neq d > 1$, as we are interested in the limiting problem as $p \to \infty$), by regularity considerations, we end up with the following system of equations:

$$a + b \cdot h^{-\alpha+1} = 1 - h^2 \quad \text{and} \quad b \cdot (-\alpha+1)h^{\alpha} = -2h, \tag{2.7}$$

where the exponent $\alpha = \alpha(p)$ is given by

$$\alpha(p) = \frac{d-1}{p-1}$$

and verifies

$$\lim_{p\to\infty} \alpha(p) = 0. \tag{2.8}$$

The first equation in (2.7) comes from continuity and the second from C^1–estimates. By the boundary condition, we have

$$a + b \cdot 2^{-\alpha+1} = 0.$$

Subtracting the first equality from the above equation, we obtain

$$b \cdot (2^{-\alpha+1} - h^{-\alpha+1}) = -1 + h^2,$$

which simplifies out to

$$(-\alpha+1)b \cdot h^{-\alpha} = -2h.$$

Combining the above with the second equation in (2.7), we end up with

$$\frac{2}{1-\alpha}(2^{-\alpha+1}h^{1+\alpha} - h^2) = 1 - h^2,$$

that is,

$$\left(\frac{2}{1-\alpha} - 1\right)h^2 - 4\left(\frac{2^{-\alpha}}{1-\alpha}\right)h^{1+\alpha} + 1 = 0.$$

Now, we observe that, from (2.8), this equation converges to

$$h^2 - 4h + 1 = 0,$$

which has as solution in $(0,1)$ (the free boundary must lie in this interval)

$$h_\infty = \frac{4-\sqrt{12}}{2}.$$

With this limit, we can also compute the limit of

$$f_p(r) = a_p + b_p r^{-\frac{d-1}{p-1}+1} = a_p + b_p r^{-\alpha(p)+1}$$

that is given by

$$f_\infty(r) = a_\infty + b_\infty r,$$

with

$$a_\infty = 4h_\infty$$

and

$$b_\infty = -2h_\infty.$$

Note that $f_\infty(r)$ is infinity harmonic in $B_2 \setminus B_{h_\infty}$ and verifies

$$f_\infty(h_\infty) = 1 - h_\infty^2$$

and

$$f'_\infty(h_\infty) = -2h_\infty.$$

It is the solution of the limit obstacle problem.

3. Characterizations of the limit

A crucial issue, with striking implications, is to characterize the limit u_∞. We give two characterizations, one involving supersolutions of the infinity Laplacian, the other making use of appropriately defined cones. From both we will derive important properties of the limit.

Theorem 3.1. *The limit u_∞ is the smallest infinity superharmonic function in Ω that is above the obstacle and equals F on the boundary.*

Proof. Let $\mathcal{F}$ be the set of all functions v that are infinity superharmonic in Ω and satisfy $v \geq \Psi$ in Ω and $v = F$ on $\partial\Omega$. This set is not empty because $u_\infty \in \mathcal{F}$. Let

$$v_\infty := \inf_{v \in \mathcal{F}} v,$$

which is upper semicontinuous (as it is the infimum of continuous functions) and infinity superharmonic in Ω. Since $u_\infty \in \mathcal{F}$, it is obvious that

$$u_\infty \geq v_\infty \quad \text{in} \quad \overline{\Omega}.$$

Now, define the open set

$$W = \{x \in \Omega : u_\infty(x) > v_\infty(x)\}.$$

On $\partial W \subset \overline{\Omega}$, we have $v_\infty = u_\infty$. Moreover,

$$u_\infty > v_\infty \geq \Psi \quad \text{in} \quad W$$

so $W \subset \{u_\infty > \Psi\}$ and u_∞ is infinity harmonic in W. Thus, by the comparison principle,

$$u_\infty \leq v_\infty \quad \text{in} \quad W,$$

a contradiction that shows that $W = \emptyset$. Consequently, $u_\infty \equiv v_\infty$. □

Corollary 3.2. *The limit u_∞ is unique.*

Proof. Suppose we have two limits, say $u_{1,\infty}$ and $u_{2,\infty}$. Then

$$v = u_{1,\infty} \wedge u_{2,\infty}$$

is also an infinity superharmonic function in Ω that is above the obstacle and equals F on the boundary. By the theorem, we have

$$u_{i,\infty} \le v, \quad i = 1, 2$$

and since, trivially, $v \le u_{i,\infty}$, $i = 1, 2$, we conclude that

$$u_{1,\infty} = v = u_{2,\infty}.$$

□

Let's now turn to our second characterization of the limit. For this, consider the family of cones with vertex at a boundary point and positive opening, which lie above both the obstacle and the boundary data. For more on comparison with cones and the characterization of infinity harmonic functions see [7].

To be concrete, for $y \in \partial\Omega$ and $b = (b_1, b_2)$, with $b_1 \ge 0$, we consider the cones

$$K_y^b(x) = b_1|x - y| + b_2$$

such that

$$K_y^b(x) \ge F(x), \qquad x \in \partial\Omega$$

and

$$K_y^b(x) \ge \Psi(x), \qquad x \in \Omega.$$

Note that, since the vertex of the cone is at the boundary of Ω, these cones are infinity harmonic in Ω, that is, $-\Delta_\infty K_y^b = 0$ in Ω. We denote by $\mathcal{K}$ the family of all such cones.

Now, we define

$$K_\infty(x) := \inf_{\mathcal{K}} K_y^b(x), \quad x \in \overline{\Omega}.$$

It is obvious that

$$K_\infty(x) \ge F(x), \quad x \in \partial\Omega$$

and

$$K_\infty(x) \ge \Psi(x), \quad x \in \Omega.$$

Proposition 3.3. *The function K_∞ is Lipschitz continuous in $\overline{\Omega}$ and infinity superharmonic in Ω. Moreover,*

$$K_\infty(y) = F(y), \quad y \in \partial\Omega.$$

Proof. Since we assume that F is Lipschitz, we have that for every point $y \in \partial\Omega$, there exists a constant L such that, for every $b_1 > L$ and every $b_2 > L$,

$$K_y^b(x) \ge F(x) \qquad \text{and} \qquad K_y^b(x) \ge \Psi(x).$$

Hence, when computing the infimum that defines $K_\infty(x)$, we can restrict to cones with $b = (b_1, b_2)$ in a compact set and since $y \in \partial\Omega$

(which is also compact), we conclude that the infimum is in fact a minimum. This means that, for every $x \in \overline{\Omega}$, there exists a $y \in \partial\Omega$ and a $b = (b_1, b_2)$, with $|b_i| \leq L$, depending on x, such that

$$K_\infty(x) = K_{y(x)}^{b(x)}(x).$$

From this fact, it follows that K_∞ is Lipschitz continuous in $\overline{\Omega}$. Let's show why. Take any two points $\hat{x}, \tilde{x} \in \overline{\Omega}$; we have

$$K_\infty(\hat{x}) = K_{y(\hat{x})}^{b(\hat{x})}(\hat{x}) \quad \text{and} \quad K_\infty(\tilde{x}) = K_{y(\tilde{x})}^{b(\tilde{x})}(\tilde{x}).$$

From the definition, it is clear that $K_\infty(\hat{x}) \leq K_{y(\tilde{x})}^{b(\tilde{x})}(\hat{x})$ and thus

$$\begin{aligned} K_\infty(\hat{x}) - K_\infty(\tilde{x}) &\leq K_{y(\tilde{x})}^{b(\tilde{x})}(\hat{x}) - K_{y(\tilde{x})}^{b(\tilde{x})}(\tilde{x}) \\ &= b_1(\tilde{x})\left(|\hat{x} - y(\tilde{x})| - |\tilde{x} - y(\tilde{x})|\right) \\ &\leq L\,|\hat{x} - \tilde{x}|\,. \end{aligned}$$

Reversing the role of $\hat{x}$ and $\tilde{x}$ gives the desired Lipschitz regularity.

Moreover, as the infimum of infinity harmonic functions, K_∞ is infinity superharmonic, *i.e.*,

$$-\Delta_\infty K_\infty \geq 0 \ \text{ in } \Omega. \tag{3.1}$$

Finally, by taking b_1 large enough and $b_2 = F(y)$, we also have, recalling (2.1),

$$F(y) \leq K_\infty(y) \leq K_y^b(y) = F(y)$$

and, hence, $K_\infty(y) = F(y)$, for $y \in \partial\Omega$. □

Theorem 3.4. *The limit u_∞ is such that*

$$u_\infty(x) \leq K_\infty(x), \quad x \in \overline{\Omega}. \tag{3.2}$$

Equality holds if, and only if, $K_\infty(x)$ is infinity harmonic outside of its coincidence set $\{K_\infty = \Psi\}$.

Proof. Inequality (3.2) follows immediately from Proposition 3.3 and Theorem 3.1. If we have an equality it is also immediate that $K_\infty(x)$ is infinity harmonic outside of its coincidence set $\{K_\infty = \Psi\}$ So we are left to prove the other implication.

Arguing by contradiction, assume that

$$W = \{x \in \Omega : K_\infty(x) > u_\infty(x)\} \neq \emptyset.$$

Note that W is open because u_∞ and K_∞ are continuous functions. Since $W \subset \{K_\infty > \Psi\}$, we deduce that $-\Delta_\infty K_\infty = 0$ in W. But $-\Delta_\infty u_\infty \geq 0$ in Ω (thus in W) and $u_\infty = K_\infty$ on ∂W so, by the comparison principle for the infinity Laplacian, we conclude that

$$u_\infty \geq K_\infty \ \text{ in } W,$$

a contradiction that shows that $W = \emptyset$ and completes the proof. □

Remark 3.5. The condition that $K_\infty(x)$ is infinity harmonic outside of its coincidence set $\{K_\infty = \Psi\}$ strongly depends on the geometry of the problem. In the radial example explicitly computed in Section 2, the condition holds. However, in general, this is not the case, as the following example shows. Consider Ω to be the union of two disjoints balls connected by a narrow tube of width δ, an obstacle placed in one of the balls and boundary data $F = 0$. It can be readily checked that, as $\delta \to 0$, $u_\infty \to 0$ in the ball without obstacle. But K_∞ is uniformly bounded below inside this ball since the opening of the corresponding cones is uniformly bounded below (as these cones have to be above the obstacle).

Corollary 3.6. *Assume the obstacle Ψ is differentiable and equality holds in* (3.2)*. Then u_∞ is differentiable at the free boundary and*

$$Du_\infty(x_0) = D\Psi(x_0), \quad \forall x_0 \in \partial\{u_\infty = \Psi\}.$$

Proof. Let $x_0 \in \partial\{u_\infty = \Psi\}$. It follows from the previous results that there exists a cone $K^b_{y_0}$ such that

$$K^b_{y_0}(x_0) = K_\infty(x_0) = u_\infty(x_0) = \Psi(x_0) \tag{3.3}$$

and

$$K^b_{y_0}(x) \geq K_\infty(x) = u_\infty(x) \geq \Psi(x), \quad \forall x \in \Omega. \tag{3.4}$$

Hence, $K^b_{y_0}(x) - \Psi(x)$ attains a minimum at x_0 and, since it is differentiable,

$$DK^b_{y_0}(x_0) = D\Psi(x_0).$$

From (3.3) and (3.4), we conclude that u_∞ is also differentiable at x_0, with

$$Du_\infty(x_0) = D\Psi(x_0),$$

as claimed. □

Remark 3.7. As a consequence of this corollary, we conclude that u_∞ is differentiable everywhere in Ω. In fact, in the interior of the coincidence set, it coincides with the differentiable obstacle and, in the interior of the non-coincidence set, it is infinity harmonic, thus differentiable everywhere by the results of [9]. Also note that the radial solution constructed in Section 2 is a C^1–solution that can be characterized by the equality in (3.2).

We close this section with the analysis of the behaviour at infinity of the coincidence sets for the p-obstacle problem and relate it with the coincidence set of the limiting problem. We recall that

$$\limsup_{p\to\infty} A_p = \bigcap_{p=1}^{\infty} \bigcup_{n\geq p} A_n \qquad \text{and} \qquad \liminf_{p\to\infty} A_p = \bigcup_{p=1}^{\infty} \bigcap_{n\geq p} A_n.$$

Theorem 3.8. *Let $A_p = \{u_p = \Psi\}$ be the coincidence sets of the p-obstacle problems and $A_\infty = \{u_\infty = \Psi\}$ be the coincidence set of the limiting problem. Then*

$$\overline{\mathrm{int}(A_\infty)} \subset \liminf_{p\to\infty} A_p \subset \limsup_{p\to\infty} A_p \subset A_\infty. \tag{3.5}$$

Proof. Given a neighbourhood V of A_∞, $\Omega \setminus V$ is a closed set contained in $\{u_\infty > \Psi\}$. Thus, the continuity of $u_\infty - \Psi$ gives us a $\eta > 0$ such that $u_\infty - \Psi > \eta$ in $\Omega \setminus V$. Using the uniform convergence of u_p to u_∞, we conclude that, for p large enough, we also have $u_p - \Psi > \eta$ in $\Omega \setminus V$. Therefore, we conclude that $\Omega \setminus V \subset \{u_p > \Psi\}$ and, consequently, that

$$A_p \subset V,$$

for every large enough p. This shows that

$$\limsup_{p\to\infty} A_p \subset V,$$

for any neighbourhood V of A_∞, and since A_∞ is compact, we also obtain

$$\limsup_{p\to\infty} A_p \subset A_\infty.$$

Next, assume that Ψ is smooth and verifies

$$-\Delta_\infty \Psi > 0.$$

Then, given $x_0 \in \mathrm{int}(A_\infty)$, if we have

$$u_{p_j}(x_0) > \Psi(x_0),$$

for a subsequence $p_j \to \infty$, then

$$-\Delta_{p_j} u_{p_j}(x_0) = 0.$$

Passing to the limit as before, we conclude that

$$-\Delta_\infty \Psi(x_0) = -\Delta_\infty u_\infty(x_0) = 0,$$

a contradiction with $-\Delta_\infty \Psi > 0$. Therefore, we conclude that for every $x_0 \in \mathrm{int}(A_\infty)$, there exists $p_0 = p_0(x_0)$ such that

$$u_n(x_0) = \Psi(x_0),$$

for every $n \geq p_0$. This means that

$$x_0 \in \bigcap_{n \geq p_0} A_n$$

and consequently

$$\mathrm{int}(A_\infty) \subset \liminf_{p\to\infty} A_p.$$

Since the larger set is closed, we also obtain

$$\overline{\mathrm{int}(A_\infty)} \subset \liminf_{p\to\infty} A_p$$

and the proof is complete. □

4. $C^{1,\frac{1}{3}}$–BEHAVIOUR AT THE FREE BOUNDARY

In this section, we show that, along the free boundary, u_∞ behaves as a $C^{1,\frac{1}{3}}$–function. This result is in connection with the celebrated optimal regularity conjecture for infinity harmonic functions. The ultimate goal is to show that $u_\infty - \Psi$ grows precisely as

$$[\mathrm{dist}(x, \partial\{u_\infty - \Psi\})]^{4/3}$$

away from the free boundary. We shall use this sharp information to establish the uniform positive density of the region $\{u_\infty > \Psi\}$. In particular, it follows that the free boundary does not develop cusps pointing inwards to the coincidence set.

The assumptions we shall impose on the obstacle in this section are the following:

$$\Psi \in C^{1,1}; \tag{4.1}$$

$$\sup_{\Omega} |\Delta_\infty (u_\infty - \Psi)| \leq M; \tag{4.2}$$

$$\inf_{\{u_\infty > \Psi\}} \Delta_\infty (u_\infty - \Psi) =: \nu > 0. \tag{4.3}$$

Both (4.2) and (4.3) are to be understood in the viscosity sense.

Condition (4.2) is rather natural in the context of obstacle-type problems, namely for

$$Lv = f(x)\chi_{\{v>0\}}, \tag{4.4}$$

and it concerns the boundedness of the function $f(x)$ (*cf.* [15]). In the linear case, the physical obstacle problem is transformed into an *obstacle-type equation* of the form (4.4) by defining v as the difference between the membrane and the obstacle. In this case, $f(x)$ is the negative of the operator L applied to the obstacle; it is then bounded provided the obstacle is of class $C^{1,1}$.

Condition (4.3), in turn, refers to the appropriate infinity concavity of the obstacle. We recall it has been well established that in order to study geometric properties of the free boundary, a sort of concavity-type non-degeneracy condition on the obstacle is needed. In fact, if no such assumption is imposed, one could produce arbitrary contact sets, just by lifting up subregions of the obstacle previously below the membrane, making them touch the original solution.

Our first result in this section gives the optimal regularity estimate for solutions of the infinity obstacle problem along the free boundary.

Theorem 4.1. *Let $x_0 \in \partial\{u_\infty > \Psi\}$ be a generic free boundary point. Then*

$$\sup_{B_r(x_0)} |u_\infty - \Psi| \leq C\, r^{4/3}, \tag{4.5}$$

for a constant C that depends only upon the data of the problem.

Proof. For simplicity, and without loss of generality, assume $x_0 = 0$, and denote $v := u_\infty - \Psi$. By combining discrete iterative techniques and a continuous reasoning (see, for instance, [5]), it is well established that proving estimate (4.5) is equivalent to verifying the existence of a constant $C > 0$, such that

$$\mathfrak{s}_{j+1} \le \max\left\{C\, 2^{-4/3(j+1)},\ 2^{4/3}\mathfrak{s}_j\right\}, \quad \forall\, j \in \mathbb{N}, \tag{4.6}$$

where

$$\mathfrak{s}_j = \sup_{B_{2^{-j}}} |v|.$$

Let us suppose, for the sake of contradiction, that (4.6) fails to hold, *i.e.*, that for each $k \in \mathbb{N}$, there exists $j_k \in \mathbb{N}$ such that

$$\mathfrak{s}_{j_k+1} > \max\left\{k\, 2^{-4/3(j_k+1)},\ 2^{4/3}\mathfrak{s}_{j_k}\right\}. \tag{4.7}$$

Now, for each k, define the rescaled function $v_k \colon B_1 \to \mathbb{R}$ by

$$v_k(x) := \frac{v(2^{-j_k}x)}{\mathfrak{s}_{j_k+1}}.$$

One easily verifies that

$$0 \le v_k(x) \le 2^{-4/3}, \quad \forall x \in B_1; \tag{4.8}$$

$$v_k(0) = 0; \tag{4.9}$$

$$\sup_{B_{\frac{1}{2}}} v_k = 1. \tag{4.10}$$

Moreover, we formally have

$$\begin{aligned}
\Delta_\infty v_k(x) &= \frac{2^{-j_k}}{\mathfrak{s}_{j_k+1}} Dv(2^{-j_k}x) \cdot \left(\frac{2^{-2j_k}}{\mathfrak{s}_{j_k+1}} D^2 v(2^{-j_k}x)\right) \cdot \frac{2^{-j_k}}{\mathfrak{s}_{j_k+1}} Dv(2^{-j_k}x) \\
&= \frac{2^{-4j_k}}{\mathfrak{s}^3_{j_k+1}} \Delta_\infty v(2^{-j_k}x) =: f_k
\end{aligned}$$

and, using assumption (4.2) and (4.7), we conclude

$$|f_k| \le \frac{2^{-4j_k}}{2^{-4(j_k+1)}\, k^3}\, M = \frac{16M}{k^3} \le 16M. \tag{4.11}$$

It is a matter of routine to rigorously justify the above calculations using the language of viscosity solutions (see, *e.g.*, [19, section 2]).

Combining the uniform bounds (4.8) and (4.11), and local Lipschitz regularity results for the inhomogeneous infinity Laplace equation (cf., for example, [12, Corollary 2]), we obtain both the equiboundedness and the equicontinuity of the sequence $(v_k)_k$. By Ascoli's theorem, and passing to a subsequence if need be, we conclude that v_k converges locally uniformly to a infinity harmonic function v_∞ in B_1 such that

$$0 \le v_\infty \le 2^{-4/3} \quad \text{and} \quad v_\infty(0) = 0.$$

We now use Harnack's inequality for infinity harmonic functions (see [13, Corollary 2]) to obtain the bound

$$v_\infty(x) \le e^{2|x|}\, v_\infty(0) = 0, \quad \forall\, x \in B_{1/2}.$$

It follows that $v_\infty \equiv 0$ in $B_{1/2}$, which contradicts (4.10). The theorem is proven.

□

An immediate consequence of Theorem 4.1 is that u_∞ is $C^{1,\frac{1}{3}}$ along the free boundary, *i.e.*, the membrane leaves the obstacle as a $C^{1,\frac{1}{3}}$–function.

Corollary 4.2 (Sharp $C^{1,\frac{1}{3}}$–regularity at the free boundary)**.** *The function u_∞ is $C^{1,\frac{1}{3}}$ at any point of the free boundary. That is, there exists a constant $\Lambda > 0$, depending only upon the data of the problem, such that*

$$|u_\infty(x) - [u_\infty(x_0) + Du_\infty(x_0) \cdot (x - x_0)]| \le \Lambda |x - x_0|^{4/3},$$

for any point $x_0 \in \partial\{u_\infty > \Psi\}$ and $x \in B_r(x_0)$, for $r \ll 1$.

Proof. It readily follows from Theorem 4.1 that, for any free boundary point x_0 and x close to x_0, there holds

$$(u_\infty - \Psi)(x) \le \sup_{B_{2|x-x_0|}(x_0)} (u_\infty - \Psi) \le C\, 2^{4/3}\, |x - x_0|^{4/3}.$$

In particular, we have

$$u_\infty(x_0) = \Psi(x_0) \quad \text{and} \quad Du_\infty(x_0) = D\Psi(x_0).$$

Finally, using the $C^{1,1}$ regularity of the obstacle, we conclude

$$\begin{array}{l} |u_\infty(x) - [u_\infty(x_0) + Du_\infty(x_0) \cdot (x - x_0)]| \\ \quad \le \ |u_\infty(x) - \Psi(x)| + |\Psi(x) - [\Psi(x_0) + D\Psi(x_0) \cdot (x - x_0)]| \\ \quad \le \ C\, 2^{4/3}\, |x - x_0|^{4/3} + C'\, |x - x_0|^2 \\ \quad \le \ \Lambda\, |x - x_0|^{4/3} \end{array}$$

and the corollary is proven. □

Our next theorem establishes a $C^{1,\frac{1}{3}}$–estimate from below, which implies that u_∞ leaves the obstacle trapped by the graph of two functions of the order $\mathrm{dist}^{4/3}(x, \partial\{u_\infty > \Psi\})$.

Theorem 4.3. *Assume the non-degeneracy hypothesis* (4.3) *is in force. Let $y_0 \in \overline{\{u_\infty > \Psi\}}$ be a generic point in the closure of the non-coincidence set. Then*

$$\sup_{B_r(y_0)} |u_\infty - \Psi| \ge c\, r^{4/3},$$

for a constant $c > 0$ that depends only upon ν.

Proof. By continuity arguments, if is enough to prove the result for points in the non-coincidence set. For simplicity, and without loss of generality, take $y_0 = 0$. Define the barrier

$$\mathcal{B}_\infty(x) := \frac{3}{4}\sqrt[3]{3\nu}\,|x|^{4/3},$$

which satisfies, by direct computation,

$$\Delta_\infty \mathcal{B}_\infty = \nu.$$

Thus, by (4.3), there holds

$$\Delta_\infty (u_\infty - \Psi) \geq \nu = \Delta_\infty \mathcal{B}_\infty, \quad \text{in } \{u_\infty > \Psi\},$$

in the viscosity sense.

On the other hand,

$$u_\infty - \Psi \equiv 0 < \mathcal{B}_\infty \quad \text{on } \partial\{u_\infty > \Psi\} \cap B_r.$$

Therefore, for some point $y^\star \in \partial B_r \cap \{u_\infty > \Psi\}$, there must hold

$$u_\infty(y^\star) - \Psi(y^\star) > \mathcal{B}_\infty(y^\star); \tag{4.12}$$

otherwise, by Jensen's comparison principle for infinity harmonic functions [11], we would have, in particular,

$$0 < u_\infty(0) - \Psi(0) \leq \mathcal{B}_\infty(0) = 0.$$

Estimate (4.12) implies the thesis of the theorem. □

As usual, as soon as we establish the precise sharp asymptotic behaviour for a given free boundary problem, it becomes possible to obtain certain weak geometric properties of the phases. We conclude this section by proving that the region where the membrane is above the obstacle has uniform positive density along the free boundary, which is then inhibited to develop cusps pointing inwards to the coincidence set.

Corollary 4.4. *Let $x_0 \in \partial\{u_\infty > \Psi\}$ be a free boundary point. Then*

$$\mathscr{L}^n \left(B_\rho(x_0) \cap \{u_\infty > \Psi\}\right) \geq \delta_\star \rho^n,$$

for a constant $\delta_\star > 0$ that depends only upon the data of the problem.

Proof. It follows from Theorem 4.3 that there exists a point

$$z \in \partial B_\rho(x_0) \cap \{u_\infty > \Psi\}$$

such that $(u_\infty - \Psi)(z) \geq c\,\rho^{4/3}$. By $C^{1,\frac{1}{3}}$–bounds along the free boundary, Theorem 4.1, it follows that

$$B_{\lambda\rho}(z) \subset \{u_\infty > \Psi\},$$

where the constant

$$\lambda := \sqrt[4]{\left(\frac{c}{2C}\right)^3}$$

depends only on the data of the problem. In fact, if this were not true, there would exist a free boundary point $y \in B_{\lambda\rho}(z)$. From (4.5), we reach the absurd

$$c\,\rho^{4/3} \leq (u_\infty - \Psi)(z) \leq \sup_{B_{\lambda\rho}(y)} |u_\infty - \Psi| \leq C\,(\lambda\rho)^{4/3} = \frac{1}{2} c\,\rho^{4/3}.$$

Thus,

$$B_\rho(x_0) \cap B_{\lambda\rho}(z) \subset B_\rho(x_0) \cap \{u_\infty > \Psi\}$$

and, finally,

$$\mathscr{L}^n\left(B_\rho(x_0) \cap \{u_\infty > \Psi\}\right) \geq \mathscr{L}^n\left(B_\rho(x_0) \cap B_{\lambda\rho}(z)\right) \geq \delta_\star \rho^n.$$

□

Acknowledgments. JDR partially supported by DGICYT grant PB94-0153 MICINN, Spain. ET partially supported by CNPq-Brazil. JMU partially supported by CMUC/FCT and projects UTAustin/MAT/0035/2008 and PTDC/MAT/098060/2008.

References

[1] D. Araújo, G. Ricarte and E. Teixeira, *Optimal gradient continuity for degenerate elliptic equations*, preprint available at `arxiv.org/abs/1206.4089`.

[2] E.N. Barron, L.C. Evans and R. Jensen, *The infinity laplacian, Aronsson's equation and their generalizations*, Trans. Amer. Math. Soc. **360** (2008), 77–101.

[3] T. Bhattacharya, E. DiBenedetto and J.J. Manfredi, *Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems*, Rend. Sem. Mat. Univ. Politec. Torino **1989** (1991), 15–68.

[4] C. Bjorland, L. Caffarelli and A. Figalli, *Non-local tug-of-war and the infinity fractional Laplacian*, Comm. Pure. Appl. Math. **65** (2012), 337–380.

[5] L. Caffarelli, L. Karp and H. Shahgholian, *Regularity of a free boundary with application to the Pompeiu problem*, Ann. Math. **151** (2000), 269–292.

[6] V. Caselles, J.M. Morel and C. Sbert, *An axiomatic approach to image interpolation*, IEEE Trans. Image Process. **7** (1998), 376–386.

[7] M.G. Crandall, *A Visit with the ∞–Laplace Equation*, in Calculus of Variations and Nonlinear Partial Differential Equations, (C.I.M.E. Summer School, Cetraro, 2005), Lecture Notes in Math, vol. 1927, Springer, Berlin, 2008.

[8] L.C. Evans and O. Savin, *$C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions*, Calc. Var. Partial Differential Equations **32** (2008), 325–347.

[9] L.C. Evans and C.K. Smart, *Everywhere differentiability of infinity harmonic functions*, Calc. Var. Partial Differential Equations **42** (2011), 289–299.

[10] J. García-Azorero, J.J. Manfredi, I. Peral and J.D. Rossi, *The Neumann problem for the ∞–Laplacian and the Monge-Kantorovich mass transfer problem*, Nonlinear Anal. **66** (2007), 349–366.

[11] R. Jensen, *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient*, Arch. Ration. Mech. Anal. **123** (1993), 51–74.

[12] E. Lindgren, *A note on the regularity of the inhomogeneous infinity Laplace equation*, preprint.

[13] P. Lindqvist and J.J. Manfredi, *The Harnack inequality for ∞–harmonic functions*, Electron. J. Differential Equations **1995**, No. 04, 1–5.

[14] J.J. Manfredi, J.D. Rossi and J.M. Urbano, *$p(x)$–Harmonic functions with unbounded exponent in a subdomain*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), 2581–2595.

[15] J. Mu and W.P. Ziemer, *Smooth regularity of solutions of double obstacle problems involving degenerate elliptic equations*, Comm. Partial Differential Equations **16** (1991), 821–843.

[16] Y. Peres, O. Schramm, S. Sheffield and D. Wilson, *Tug-of-war and the infinity Laplacian*, J. Amer. Math. Soc. **22** (2009), 167–210.

[17] O. Savin, *C^1 regularity for infinity harmonic functions in two dimensions*, Arch. Ration. Mech. Anal. **176** (2005), 351–361.

[18] A. Swiech and E. Teixeira, *Regularity for obstacle problems in infinite dimensional Hilbert spaces*, Adv. Math. **220** (2009), 964–983.

[19] E. Teixeira, *Optimal regularity of viscosity solutions of fully nonlinear singular equations and their limiting free boundary problems*, Mat. Contemp. **30** (2006), 217–237.

[20] E. Teixeira, *Regularity for quasilinear equations on degenerate singular sets*, preprint available at `arxiv.org/abs/1204.6607`.

Julio D. Rossi
Department of Mathematical Analysis, University of Alicante
03080 Alicante, Spain.
E-mail address: `julio.rossi@ua.es`

Eduardo V. Teixeira
Universidade Federal do Ceará
Campus of Pici - Bloco 914, Fortaleza - Ceará - 60.455-760, Brazil.
E-mail address: `teixeira@mat.ufc.br`

José Miguel Urbano
CMUC, Department of Mathematics, University of Coimbra
3001-501 Coimbra, Portugal.
E-mail address: `jmurb@mat.uc.pt`