SYMMETRY PROPERTIES FOR THE EXTREMALS OF THE SOBOLEV TRACE EMBEDDING

J. FERNÁNDEZ BONDER, E. LAMI DOZO AND J.D. ROSSI

ABSTRACT. In this article we study symmetry properties of the extremals for the Sobolev trace embedding $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$ with $1 \le q \le 2(N-1)/(N-2)$ for different values of μ. These extremals u are solutions of the problem

$$\begin{cases} \Delta u = u & \text{in } B(0,\mu), \\ \frac{\partial u}{\partial \eta} = \lambda |u|^{q-2} u & \text{on } \partial B(0,\mu). \end{cases}$$

We find that, for $1 \le q < 2(N-1)/(N-2)$, there exists a unique normailzed extremal u, which is positive and has to be radial, for μ small enough. We also find an asymptotic expansion for u in powers of μ and in powers of $|x|$ that provides some monotonicity properties for the best Sobolev trace constant in small balls and a uniform in q estimate for the radius μ_0 below which the extremals are radial. For the critical case, $q = 2(N-1)/(N-2)$, as a consequence of the symmetry properties for small balls, we conclude the existence of extremals. Finally we show that for large balls, there is no extremal in this critical case.

1. INTRODUCTION.

The aim of this article is to study of the following problem: Given a ball of radius μ, $B(0,\mu)$, in ^{N}, decide wether or not there exists a radial extremal for the embedding

$$H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu)).$$

First, let us introduce our motivation. Let $\Omega \subset {}^N$ be a bounded smooth domain. Relevant for the study of boundary value problems for differential operators are the two following Sobolev inequalities. For each $1 \le q \le 2(N-1)/(N-2) \equiv \tilde{2}$, we have a continuous inclusion $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$, and for each $1 \le p \le 2N/(N-2) \equiv 2^*$, $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, hence the following inequalities hold:

$$S\|u\|^2_{L^q(\partial\Omega)} \le \|u\|^2_{H^1(\Omega)}, \qquad \bar{S}\|u\|^2_{L^p(\Omega)} \le \|u\|^2_{H_0^1(\Omega)}.$$

These inequalities are known as the Sobolev trace theorem and the Sobolev embedding theorem respectively. The best constants for these embeddings are the largest S and $\bar{S}$ such that the above inequalities hold, that is,

$$S = \inf_{v \in H^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla v|^2 + |v|^2\,dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,d\sigma\right)^{2/q}} \tag{1.1}$$

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and

$$\bar{S} = \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla v|^2\,dx}{\left(\displaystyle\int_\Omega |v|^p\,dx\right)^{2/p}}. \tag{1.2}$$

The main difference between these two quantities, is the fact that $\bar{S}$ is homogeneous under dilatations of the domain, that is, if we define $\mu\Omega = \{\mu x \mid x \in \Omega\}$, taking $v(x) = u(\mu x)$ in (1.2) and changing variables we get

$$\bar{S}(\Omega) = \mu^{(2p+Np-2N)/p}\bar{S}(\mu\Omega).$$

On the other hand, S is not homogeneous under dilatations. In fact we have

$$S_\mu \equiv S(\mu\Omega) = \mu^\beta \inf_{v \in H^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \mu^{-2}|\nabla v|^2 + |v|^2\,dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,d\sigma\right)^{2/q}}, \tag{1.3}$$

where $\beta = (Nq - 2N + 2)/q$.

For $1 \le q < 2(N-1)/(N-2)$ and $1 \le p < 2N/(N-2)$ the embeddings are compact, so we have existence of extremals, i.e. functions where the infimum is attained. These extremals are weak solutions of the following problems

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = \lambda |u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\frac{\partial}{\partial \eta}$ is the outer unit normal derivative, and

$$\begin{cases} -\Delta u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

The asymptotic behavior of $S(\mu\Omega)$ in expanding ($\mu \to \infty$) and contracting domains ($\mu \to 0$), was studied in [2] and [4]. In [2] it is proved that for expanding domains and $q > 2$, $S(\mu\Omega) \to S(\ {}^N_+)$. In [4] it is shown that

$$\lim_{\mu \to 0+} \frac{S_\mu}{\mu^\beta} = \frac{|\Omega|}{|\partial\Omega|^{2/q}}. \tag{1.6}$$

The behavior of the extremals for (1.1) in expanding and contracting domains is also studied in [2] and [4]. For expanding domains, it is proved in [2] that the extremals develop a peak near a point where the mean curvature of the boundary maximizes. For contracting domains, we have that the extremals, when rescaled to the original domain as $v(x) = u(\mu x)$, $x \in \Omega$, and normalized with $\|v\|_{L^q(\partial\Omega))} = 1$, are nearly constant in the sense that

$$\lim_{\mu \to 0} v = \frac{1}{|\partial\Omega|^{1/q}} \qquad \text{in } H^1(\Omega).$$

Another big difference between the Sobolev trace theorem and the Sobolev embedding theorem arises in the behavior of extremals. Namely, if Ω is a ball, $\Omega = B(0,\mu)$, as the extremals do not change sign, from results of [5] the extremals for (1.2) are radial while, at least for large μ, extremals for (1.1) are not, since they develop peaking concentration phenomena as is described in [2].

The above discussion leads naturally to the purpose of this article: the study of the symmetry properties for the extremals of the Sobolev trace embedding in small balls.

Surprisingly, we find that the symmetry properties of the extremals depend on the size of the ball. Our first result says that for small balls there exists a unique extremal that has to be radial.

Theorem 1.1. *There exists $\mu_0 > 0$ such that for every $\mu < \mu_0$ there exists a unique positive extremal, u, for the embedding $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$ normalized such that $\|u(\mu x)\|_{L^q(\partial B(0,1))} = 1$. Moreover this extremal is a radial function.*

The main ingredient of the proof is the implicit function theorem. We remark that the moving planes technique cannot be applied to obtain symmetry results in this case, as the extremals for large μ are not radial.

This method also works for any domain Ω and gives the following result

Theorem 1.2. *There exists $\mu_0 > 0$ such that for every $\mu < \mu_0$ there exists a unique positive extremal, u, for the embedding $H^1(\mu\Omega) \hookrightarrow L^q(\partial\mu\Omega)$ normalized such that $\|u(\mu x)\|_{L^q(\partial\Omega)} = 1$.*

Once we know that the extremals are radial we can obtain a finer description of their shape expanding u as a power series. This is the content of our second result.

Theorem 1.3. *There exists μ_0 such that for every $\mu < \mu_0$ the extremal u for the embedding $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$ verifies, once rescaled as $v(x) = u(\mu x)$,*

$$\begin{cases} \Delta v = \mu^2 v & \text{in } B(0,1), \\ v = |\partial B(0,1)|^{-1/q} & \text{on } \partial B(0,1). \end{cases} \tag{1.7}$$

Hence, it has an expansion in powers of μ

$$v(r) = \sum_{j=0}^{\infty} a_j(r)\mu^{2j} \tag{1.8}$$

where $a_0 \equiv |\partial B(0,1)|^{-1/q}$ and the rest of the a_j are obtained as solutions of

$$\begin{cases} \Delta a_i = a_{i-1} & \text{in } B(0,1), \\ a_i = 0 & \text{on } \partial B(0,1). \end{cases} \tag{1.9}$$

Also, v has an expansion in powers of r given by

$$v(r) = \frac{\varphi(r,\mu)}{\varphi(1,\mu)} |\partial B(0,1)|^{-1/q}, \tag{1.10}$$

where

$$\varphi(r,\mu) = \sum_{k=0}^{\infty} \frac{\mu^{2k} r^{2k}}{(2k)!!(2k-2+N)!!}$$

and $m!! = m(m-2)\dots 4\cdot 2$.

As an immediate consequence, we get an explicit expression for S_μ.

Corollary 1.1. *For small values of μ, S_μ is given by*

$$S_\mu = \frac{\varphi_r(1,\mu)}{\varphi(1,\mu)} \mu^{\beta-2} |\partial B(0,1)|^{(q-2)/q}. \tag{1.11}$$

Observe that $b_j(\mu) \sim \mu^{2j}$ so (1.11) gives the analytic expansion of $S_\mu/\mu^{\beta-2}$ in powers of μ.

Also as a corollary of Theorem 1.3 we can obtain some monotonicity properties for small balls.

Corollary 1.2. *For small values of μ, the normalized extremals are strictly decreasing as functions of μ, that is, if $\mu_1 < \mu_2$ then $v_{\mu_1} > v_{\mu_2}$ in $B(0,1)$. Moreover, S_μ/μ^β is decreasing as a function of μ, that is, if $\mu_1 < \mu_2$ then*

$$\frac{S_{\mu_1}}{\mu_1^\beta} > \frac{S_{\mu_2}}{\mu_2^\beta}.$$

Theorem 1.1 says that for small balls (μ small) the extremals are radial, while the results of [2] say that this is not the case for large balls. Therefore we can define

$$\mu_0 = \sup\{\mu \,:\, u_\theta \text{ is radial for every } \theta < \mu\}. \tag{1.12}$$

This value μ_0 is the critical size where we pass from radial extremals to nonradial ones. Our next result is an estimate on the value of μ_0

Theorem 1.4. *The critical radius μ_0 defined by (1.12) verifies*

$$\mu_0 \geq \frac{|\partial B(0,1)|^{1/q}}{|B(0,1)|^{1/2}} \sqrt{\frac{\bar{c}_1}{q-1}}, \tag{1.13}$$

where

$$\bar{c}_1 = \inf \left\{ \frac{\displaystyle\int_{B(0,1)} |\nabla v|^2}{\|v\|^2_{L^2(\partial B(0,1))}} \;:\; v \in H^1(B(0,1)), \int_{\partial B(0,1)} v = 0, \right\} > 0. \tag{1.14}$$

Moreover, the set of parameters (μ, q) such that there is no radial extremal is open.

With this bound on μ_0 we can prove existence of extremals for the critical exponent $\tilde{2} = 2(N-1)/(N-2)$ for small balls. These extremals are obtained as limits of the extremals for $q < \tilde{2}$. We remark that the existence of extremals for the critical exponent is not trivial, this is due to the lack of compactness.

Theorem 1.5. *For every μ small enough there exists a radial extremal for the immersion*

$$H^1(B(0,\mu)) \to L^{2(N-1)/(N-2)}(\partial B(0,\mu)).$$

Next we present a nonexistence result for large balls

Theorem 1.6. *For every μ large enough there is no extremal for the immersion*

$$H^1(B(0,\mu)) \to L^{2(N-1)/(N-2)}(\partial B(0,\mu)).$$

These results has to be compared with the case of the immersion $H_0^1(B(0,\mu)) \to L^{2^*}(B(0,\mu))$ where it is well known that, by Pohozaev identity, there is no extremal regardless the size of the ball for the critical exponent $2^* = 2N/(N-2)$. However, there exist extremals for topological nontrivial domains. This is a remarkable and unexpected difference with the fact that, as suggested by our results, for the trace immersion with critical exponent, the existence of extremals depends on the size of the domain and not on the shape.

Finally we see that if there exists a radial extremal for some q_1 then there exists a radial extremal for every $1 < q < q_1$.

Theorem 1.7. *If there exists a radial extremal of the embedding $H^1(B(0,\mu)) \hookrightarrow L^{q_1}(\partial B(0,\mu))$ and $1 < q \le q_1$, then there exists a radial extremal for $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$. Moreover, these extremals are multiples of each other.*

Hence, using that the extremal is radial for $q = 2$, see [6], we get the following result.

Corollary 1.3. *For every $q \le 2$ and every $\mu > 0$ there exists a radial extremal for the embedding $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$.*

The rest of the paper is organized as follows: in section 2 we prove our symmetry result, Theorem 1.1, in section 3 we expand the radial extremal as a power series, Theorem 1.3, and prove our monotonicity results, Corollary 1.2, in section 4 we find a bound on the critical radius r_0, Theorem 1.4, in section 5 we deal with the existence of extremals with critical exponent and finally in section 6 we prove Theorem 1.7

2. Proof of Theorem 1.1

In this section we use the implicit function theorem to show that there exists a unique minimizer for μ small.

As observed in the introduction, making the change of variables $v(x) = u(\mu x)$, we get

$$S_\mu = \mu^\beta \inf_{v \in H^1(B(0,1))\setminus\{0\}} \frac{\displaystyle\int_{B(0,1)} \mu^{-2}|\nabla v|^2 + |u|^2\,dx}{\left(\displaystyle\int_{\partial B(0,1)} |v|^q\,d\sigma\right)^{2/q}},$$

where $\beta = (qN - 2N + 2)/q$. Therefore the extremals are solutions of

$$(2.1)\qquad \begin{cases} \Delta v = \mu^2 v & \text{in } B(0,1), \\ \frac{\partial v}{\partial \eta} = \mu^2 \frac{S_\mu}{\mu^\beta} |v|^{q-2} v & \text{on } \partial B(0,1). \end{cases}$$

We denote

$$\mathcal{S} = \left\{ v \in H^1(B(0,1))\,;\ \int_{\partial B(0,1)} v^q = 1 \right\}.$$

Let us consider the functional

$$F : \mathcal{S} \times [0,1] \to (H^1(B(0,1)))^*$$

given by

$$F(v,\mu)(\phi) = \int_{B(0,1)} \nabla v \nabla \phi + \mu^2 \int_{B(0,1)} v\phi - \mu^2 \frac{S_\mu}{\mu^\beta} \int_{\partial B(0,1)} v^{q-1}\phi.$$

This functional F is continuous and C^1 with respect to v.

We observe that

$$v_0 \equiv \frac{1}{|\partial B(0,1)|^{1/q}} \in \mathcal{S}$$

and satisfies

$$F\left(\frac{1}{|\partial B(0,1)|^{1/q}},0\right)=0.$$

We want to use the implicit function theorem to show that there exists a unique solution, $v=v(\mu)$ to the equation $F(v,\mu)=0$, defined for small values of μ near $v_0=\frac{1}{|\partial B(0,1)|^{1/q}}$.

To this end we state the following lemmas.

Lemma 2.1. *The tangent space to $\mathcal{S}$ at v_0 is given by*

$$T_{v_0}(\mathcal{S})=\left\{z\in H^1(B(0,1))\,;\ \int_{\partial B(0,1)} z=0\right\}.$$

Proof. First let us prove that

$$T_{v_0}(\mathcal{S})\subset\left\{z\in H^1(B(0,1))\,;\ \int_{\partial B(0,1)} z=0\right\}.$$

Each curve $\gamma:(-1,1)\to\mathcal{S}$ with $\gamma(0)=v_0$ satisfies

$$\int_{\partial B(0,1)}|\gamma(t)|^q=1.$$

Differentiating at $t=0$ we get

$$q\int_{\partial B(0,1)}\gamma(0)^{q-1}\gamma'(0)=qv_0^{q-1}\int_{\partial B(0,1)}\gamma'(0)=0.$$

Now let us prove the reverse inclusion

$$T_{v_0}(\mathcal{S})\supset\left\{z\in H^1(B(0,1))\,;\ \int_{\partial B(0,1)} z=0\right\}.$$

Let z be such that

$$\int_{\partial B(0,1)} z=0,$$

and consider the following curve

$$\gamma(t)=\frac{v_0+tz}{\left(\displaystyle\int_{\partial B(0,1)}|v_0+tz|^q\right)^{1/q}}.$$

This curve verifies $\gamma(t)\in\mathcal{S}$, $\gamma(0)=v_0$, $\gamma'(0)=z$. This ends the proof. $\square$

Lemma 2.2. *Let*

$$A=\{\varphi\in(H^1(B(0,1)))^*:\ \langle\varphi,1\rangle=0\}.$$

The derivative of F with respect to v at the point $(v_0,0)$ is given by

$$\frac{\partial F}{\partial v}(v_0,0):T_{v_0}(\mathcal{S})\to A$$

$$\frac{\partial F}{\partial v}(v_0,0)(w)(\phi)=\int_{B(0,1)}\nabla w\nabla\phi.$$

Proof. The result follows directly from the fact that

$$F(v,0)(\phi) = \int_{B(0,1)} \nabla v \nabla \phi.$$

□

Next we prove that $\frac{\partial F}{\partial v}(v_0,0)$ has a continuous inverse.

Lemma 2.3. *Given $\varphi \in A$ there exists a unique $w \in T_{v_0}(\mathcal{S})$ such that*

$$\int_{B(0,1)} \nabla w \nabla \phi = \langle \varphi, \phi \rangle, \qquad \forall\, \phi \in H^1(B(0,1)).$$

Moreover the map $\varphi \mapsto w$ is continuous.

Proof. Observe that $T_{v_0}(\mathcal{S})$ is a Hilbert space with the inner product given by

$$(u,v) = \int_{B(0,1)} \nabla u \nabla v.$$

Then the lemma follows from the Riesz representation Theorem. □

Now we can use the implicit function theorem to obtain Theorem 1.1.

Proof of Theorem 1.1. From the previous lemmas we get that F verifies the hypothesis of the implicit function theorem, see for example [1]. Hence there exists μ_0 such that for every $\mu \in [0,\mu_0]$ there exists a unique solution $v = v(\mu)$ of

$$F(v,\mu) = 0$$

with v near v_0. Therefore there exists a unique weak solution of (2.1) near v_0 for small values of μ. By the results of [4] the extremals are weak solutions of (2.1) that converges, as μ goes to zero, to v_0 in $H^1(B(0,1))$, the uniqueness of the extremal follows.

Now take an extremal u_1 in $B(0,\mu)$ and let R be any rotation, then $u_2(x) = u_1(Rx)$ is also an extremal. Since there is a unique extremal we must have $u_1 = u_2$ and we conclude that the unique extremal must be a radial function. □

Remark 2.1. *We remark that the same method can be applied to any domain Ω when is contracted multiplying by a small μ, proving Theorem 1.2.*

3. Asymptotic expansion in powers of μ

In this Section we prove Theorem 1.3, Corollary 1.1 and Corollary 1.2.

Proof of Theorem 1.3. As the extremals for small μ are radial, we look for an expansion in powers of μ of a solution of

$$\begin{cases} \Delta v = \mu^2 v & \text{in } B(0,1), \\ v = |\partial B(0,1)|^{-1/q} & \text{on } \partial B(0,1). \end{cases} \tag{3.1}$$

Let

$$w(r) = \sum_{j=0}^{\infty} a_j(r) \mu^{2j}$$

where the coefficients a_j are given by $a_0 = |\partial B(0,1)|^{-1/q}$ and

$$\begin{cases} \Delta a_i = a_{i-1} & \text{in } B(0,1), \\ a_i = 0 & \text{on } \partial B(0,1). \end{cases} \tag{3.2}$$

By standard regularity theory, the a_i's are uniformly bounded in $C^2(\overline{B(0,1)})$ norm, then w is a well defined C^2 function for μ small. To conclude the first part of the proof of Theorem 1.3, we only have to observe that w verifies (3.1) and also, by the results of the previous section, the extremal v is radial and is a solution of (3.1). The result follows by uniqueness of solutions of (3.1).

To obtain the expansion of u in powers of r we write

$$u(r) = \sum_{j=0}^{\infty} b_j(\mu) r^{2j}$$

and we get the formula for the coefficients imposing that u verifies

$$\begin{aligned} & u'' + \frac{N-1}{r} u' = \mu^2 u \\ & u'(0) = 0, \quad u(1) = |\partial B(0,1)|^{-1/q}, \end{aligned}$$

where $'$ denotes differentiation with respect to r. Finally we have

$$b_j(\mu) = \frac{\mu^{2j}}{(2j)!!(2j-2+N)!!} \frac{|\partial B(0,1)|^{-1/q}}{\sum_{k=0}^{\infty} \frac{\mu^{2k}}{(2k)!!(2k-2+N)!!}}$$

where $m!! = m(m-2)\dots 4 \cdot 2$. This ends the proof of Theorem 1.3. $\square$

Remark 3.1. *We can also obtain the same expansion in powers of* μ *considering the problem* (2.1)*, that is,*

$$\begin{cases} \Delta v = \mu^2 v & \text{in } B(0,1), \\ \frac{\partial v}{\partial \eta} = \mu^2 \frac{S_\mu}{\mu^\beta} |v|^{q-2} v & \text{on } \partial B(0,1), \end{cases}$$

even if we do not know a priori that the solution is radial. The resulting coefficients a_i *turn out to be radial functions. This fact confirms Theorem 1.1.*

Proof of Corollary 1.1. To compute S_μ, we observe that

$$u'(1) = \mu^2 \frac{S_\mu}{\mu^\beta} |\partial B(0,1)|^{-(q-1)/q}.$$

As $u'(1) = \sum_{j=1}^{\infty} 2j b_j(\mu) = |\partial B(0,1)|^{-1/q} \varphi_r(1,\mu)/\varphi(1,\mu)$, we get

$$S_\mu = \sum_{j=1}^{\infty} 2j b_j(\mu) \mu^{\beta-2} |\partial B(0,1)|^{(q-1)/q} = \frac{\varphi_r(1,\mu)}{\varphi(1,\mu)} \mu^{\beta-2} |\partial B(0,1)|^{(q-2)/q}.$$

This finishes the proof. $\square$

Proof of Corollary 1.2. Let us compute an asymptotic expansion for S_μ/μ^β. Let $u \in H^1(B(0,\mu))$ be an extremal for (1.1), then if $v(x) = u(\mu x)$ we have that

$$\frac{S_\mu}{\mu^\beta} = \frac{\displaystyle\int_{B(0,1)} \mu^{-2}|\nabla v|^2 + |v|^2\,dx}{\left(\displaystyle\int_{\partial B(0,1)} |v|^q\,d\sigma\right)^{2/q}}, \tag{3.3}$$

and, by Theorem 1.3, $v = a_0 + a_1(r)\mu^2 + o(\mu^2)$ with $a_1 = 0$ on $\partial B(0,1)$. Hence

$$\int_{\partial B(0,1)} |v|^q\,d\sigma = \int_{\partial B(0,1)} |a_0 + o(\mu^2)|^q\,d\sigma = \int_{\partial B(0,1)} |a_0|^q + o(\mu^2) = 1 + o(\mu^2).$$

Moreover

$$\begin{aligned}\int_{B(0,1)} \mu^{-2}|\nabla u|^2 + |u|^2\,dx &= \int_{B(0,1)} \mu^2|\nabla a_1|^2 + |a_0 + \mu^2 a_1|^2\,dx + o(\mu^2)\\ &= \int_{B(0,1)} a_0^2 + \mu^2\left(|\nabla a_1|^2 + 2a_0a_1\right)\,dx + o(\mu^2).\end{aligned}$$

Therefore from (3.3) we get

$$\frac{S_\mu}{\mu^\beta} = \int_{B(0,1)} a_0^2 + \mu^2\left(|\nabla a_1|^2 + 2a_0a_1\right)\,dx + o(\mu^2).$$

Multiplying (3.2) for $i = 1$ by a_1 and integrating by parts we get

$$-\int_{B(0,1)} |\nabla a_1|^2 = \int_{B(0,1)} a_0a_1,$$

hence we arrive at

$$\frac{S_\mu}{\mu^\beta} = \left(\frac{|B(0,1)|}{|\partial B(0,1)|^{2/q}}\right) - \mu^2\left(\int_{B(0,1)} |\nabla a_1|^2\,dx\right) + o(\mu^2).$$

Therefore, we get that S_μ/μ^β is decreasing as a function of μ, for small values of μ.

Now, we have proved that

$$v = a_0 + a_1\mu^2 + f(x,\mu).$$

By the maximum principle, $a_1(r) < 0$ and, moreover $\partial a_1/\partial\eta > 0$ at the boundary. Also, the remainder $f(x,\mu)$ is C^2 up to the boundary and verifies $f(x,\mu) = o(\mu^2)$ uniformly in x. Therefore $v_\mu(r)$ is decreasing for small values of μ. $\square$

Remark 3.2. *The monotonicity of S_μ/μ^β can also be obtained from the explicit formula for S_μ given in Corollary 1.1.*

4. Estimate for the critical radius μ_0.

From now on we use the notation

$$Q(v,\mu,q) = \frac{\displaystyle\int_{B(0,1)} \mu^{-2}|\nabla v|^2 + |v|^2\,dx}{\left(\displaystyle\int_{\partial B(0,1)} |v|^q\,d\sigma\right)^{2/q}},$$

for $v \in H^1(B(0,1)) \setminus \{0\}$, $\mu > 0$ and $1 < q \leq \tilde{2}$.

Proof of Theorem 1.4. Let us look at the linear part of the problem near any positive solution v of (1.4). The kernel of the linear part are the solutions of

$$\begin{cases} \Delta z = \mu^2 z & \text{in } B(0,1), \\ \frac{\partial z}{\partial \eta} = \mu^2 \frac{S_\mu}{\mu^\beta}(q-1)v^{q-2}z & \text{on } \partial B(0,1), \end{cases} \tag{4.1}$$

with

$$\int_{\partial B(0,1)} z = 0.$$

Let us look for a bound on the value $\tilde{\mu}$ such that (4.1) has a nontrivial solution.

This value $\tilde{\mu}$ can be estimated as follows, multiply by z (4.1) and integrate by parts to get

$$\int_{B(0,1)} |\nabla z|^2 + \mu^2 \int_{B(0,1)} z^2 = \mu^2 \frac{S_\mu}{\mu^\beta}(q-1) \int_{\partial B(0,1)} v^{q-2} z^2.$$

Now, by Hölder inequality,

$$\int_{B(0,1)} |\nabla z|^2 + \mu^2 \int_{B(0,1)} z^2 \le \mu^2 \frac{S_\mu}{\mu^\beta}(q-1) \|v\|^{q-2}_{L^q(\partial B(0,1))} \|z\|^2_{L^q(\partial B(0,1))}.$$

We define $c_1 = c_1(q)$ as

$$c_1(q) = \inf \left\{ \frac{\displaystyle\int_{B(0,1)} |\nabla v|^2}{\|v\|^2_{L^q(\partial B(0,1))}} \ : \ v \in H^1(B(0,1)), \int_{\partial B(0,1)} v = 0, \right\},$$

and we get

$$c_1(q) \le \mu^2 \frac{S_\mu}{\mu^\beta}(q-1).$$

Also we have that, by [4] and Corollary 1.2,

$$\frac{S_\mu}{\mu^\beta} \le \frac{|B(0,1)|}{|\partial B(0,1)|^{2/q}}.$$

Then, if

$$c_1(q) > \mu^2 \frac{|B(0,1)|}{|\partial B(0,1)|^{2/q}}(q-1),$$

we can not have a nontrivial solution v. Therefore we get that

$$\tilde{\mu} \ge \frac{|\partial B(0,1)|^{1/q}}{|B(0,1)|^{1/2}} \sqrt{\frac{c_1(q)}{q-1}}.$$

We observe that $c_1(q) \ge c_1(\tilde{2}) = \bar{c}_1$.

To finish the proof of the estimate it remains to show that $\tilde{\mu} \le \mu_0$.

Assume that $\mu_0 < \tilde{\mu}$. Then, there exists a sequence $\mu_n \in$ and $v_n \in H^1(B(0,1))$ such that $\mu_n \to \mu_0 < \tilde{\mu}$, v_n is an extremal for S_{μ_n} with $\|v_n\|_{L^q(B(0,1))} = 1$. Then, it follows that $\|v_n\|_{H^1(B(0,1))} \le C$ and so, there exists a subsequence (that we still call v_n) and a function $v \in H^1(B(0,1))$ such that

$$v_n \rightharpoonup v \text{ weakly in } H^1(B(0,1)), \tag{4.2}$$

$$v_n \to v \text{ in } L^2(B(0,1)), \tag{4.3}$$

$$v_n \to v \text{ in } L^q(\partial B(0,1)). \tag{4.4}$$

Let us see that v is an non-radial extremal for S_{μ_0}. By (4.4), $\|v\|_{L^q(\partial B(0,1))} = 1$ so $v \neq 0$.

Now, if v is a radial function, as there exists a unique radial solution of (1.4), (v, μ_0) must be a bifurcation point from the branch that starts with $(v_0, 0)$ which contradicts the definition of $\tilde{\mu}$, then v is not a radial function.

By (4.2)-(4.3), we have

$$Q(v, \mu_0, q) = \int_{B(0,1)} \mu_0^{-2}|\nabla v|^2 + |v|^2\, dx \le \liminf Q(v_n, \mu_n, q).$$

Also, if there exists a function $\tilde{v} \in H^1(B(0,1))$ with $Q(\tilde{v}, \mu_0, q) < Q(v, \mu_0, q)$, we get a contradiction from the fact that $Q(\tilde{v}, \mu_n, q) < Q(v_n, \mu_n, q)$. Therefore, the limit v must be a non-radial extremal.

By the implicit function theorem, and by our assumption $\tilde{\mu} < \mu_0$ there exists a branch of non-radial solutions, v_μ, of (2.1) that passes through v and verify $\|v_\mu\|_{L^q(\partial B(0,1))} = 1$. Therefore, $\|v_\mu\|_{H^1(B(0,1))}$ is uniformly bounded, hence the branch cannot go to infinity for $\mu < \tilde{\mu}$ and cannot go to zero. The only possibility that remains is that for every $0 < \mu < \mu_0$ there exists a nonradial solution of (2.1), but this is a contradiction with the results of section 2 that proves that the unique solution of (2.1) must be radial for μ small enough.

To finish the proof of the Theorem, it remains to show that the set of parameters where the extremals are non-radial functions is open.

Let us define the set

$$A = \{(\mu, q) : \text{ there is no radial extremal of (1.1)}\}$$

We denote by $H^1_{rad}(B(0,1))$ the set of radial functions in $H^1(B(0,1))$. Let $(\mu_0, q_0) \in A$. We have that

$$\inf_{v \in H^1(B(0,1))} Q(v, \mu_0, q_0) < \inf_{v \in H^1_{rad}(B(0,1))} Q(v, \mu_0, q_0).$$

Now, if (μ, q) is close to (μ_0, q_0) by continuity of Q, we get that

$$\inf_{v \in H^1(B(0,1))} Q(v, \mu, q) < \inf_{v \in H^1_{rad}(B(0,1))} Q(v, \mu, q).$$

Hence $(\mu, q) \in A$ for every (μ, q) close to (μ_0, q_0) and the result follows. □

5. Extremals for the critical exponent.

In this section we focus on the existence of extremals for the critical exponent $\tilde{2} = 2(N-1)/(N-2)$.

First we prove Theorem 1.5.

Proof of Theorem 1.5. From (1.13) we get that there exists μ_1 such that for every $\mu < \mu_1$ the extremals are radial for every $2 < q < \tilde{2} = 2(N-1)/(N-2)$. In order to prove existence of extremals in the critical case, the main idea is to take the limit as $q \nearrow \tilde{2}$ of a sequence of radial extremals for $H^1(B(0,\mu)) \hookrightarrow L^q(\partial B(0,\mu))$.

Let $\mu < \mu_1$ be fixed, $q_j < \tilde{2}$ be any sequence such that $q_j \nearrow \tilde{2}$ and let v_j a radial extremal defined in $B(0,1)$. By Theorem 1.3, these v_j are solutions of

$$\begin{cases} \Delta v_j = v_j & \text{in } B(0,1), \\ v_j = |\partial B(0,1)|^{-1/q_j} & \text{on } \partial B(0,1). \end{cases} \tag{5.1}$$

As the boundary values converges uniformly

$$v_j \mid_{\partial B(0,1)} = |\partial B(0,1)|^{-1/q_j} \to |\partial B(0,1)|^{-1/\tilde{2}}$$

we get that the sequence v_j converges strongly in $H^1(B(0,1))$ and uniformly to some function v^*.

We claim that v^* is an extremal for $\tilde{2}$. In fact, assume that there exists $w \in H^1(B(0,1))$ such that $Q(w,\mu,\tilde{2}) < Q(v^*,\mu,\tilde{2})$. We arrive at a contradiction noticing that, $v_j \to v^*$ and $q_j \to \tilde{2}$ imply, by the continuity of Q, $Q(w,\mu,q_j) < Q(v_j,\mu,q_j)$ for j large enough. $\square$

Now we prove that there are no extremals for $H^1(B(0,\mu)) \hookrightarrow L^{\tilde{2}}(\partial B(0,\mu))$, if μ is large enough.

Proof of Theorem 1.6. Assume by contradiction that there exists a sequence $\mu_j \to \infty$ with extremals u_j. We can normalize the sequence u_j to ensure that

$$\int_{B(0,\mu_j)} |\nabla u_j|^2 + |u_j|^2 \, dx = \int_{\partial B(0,\mu_j)} |u_j|^{\tilde{2}} \, d\sigma.$$

From [2] we know that the following exponential decay holds:

$$u_j(x) \le c_1 \exp(-c_2|x - x_j|) \tag{5.2}$$

for all $x \in B(0,\mu_j)$. Here x_j denotes a point where u_j attains its maximum. Also, from [2], we have a uniform bound in $C^{2+\alpha}_{loc}$ for the sequence u_j. We remark that the results in [2] are stated for subcritical exponents, however the proofs of the exponential decay and the $C^{2+\alpha}_{loc}$ estimates still hold for $\tilde{2}$.

We define $w_j(z) = u_j(R(z + x_j))$ where R is a suitable rotation so that the functions w_j are defined for a ball of radius μ_j contained in $\{z_n > 0\}$.

Observe that $\max w_j = w_j(0)$ and that

$$w_j(z) \le c_1 \exp(-c_2|z|).$$

By means of this exponential decay and the uniform local bounds in $C^{2+\alpha}$, we can pass to the limit, taking a subsequence if necessary as in [2], to obtain $w_j \to w$ and this limit function w is a least energy solution of

$$\begin{cases} \Delta w = w & \text{in } {}^N_{+}, \\ \frac{\partial w}{\partial \eta} = w^{\tilde{2}-1} & \text{on } \partial \ {}^N_{+}. \end{cases}$$

Hence w is a minimizer of

$$\inf_{v \in H^1({}^N_{+})} \frac{\displaystyle\int_{{}^N_{+}} |\nabla v|^2 + |v|^2 \, dx}{\left(\displaystyle\int_{\partial \ {}^N_{+}} |v|^{\tilde{2}} \, d\sigma \right)^{2/\tilde{2}}}.$$

Then

$$\inf_{v\in H^1({}^N_+)} \frac{\displaystyle\int_{{}^N_+} |\nabla v|^2 + |v|^2\, dx}{\left(\displaystyle\int_{\partial\ {}^N_+} |v|^{\tilde 2}\, d\sigma\right)^{2/\tilde 2}} > \inf_{v\in H^1({}^N_+)} \frac{\displaystyle\int_{{}^N_+} |\nabla v|^2\, dx}{\left(\displaystyle\int_{\partial\ {}^N_+} |v|^{\tilde 2}\, d\sigma\right)^{2/\tilde 2}}. \tag{5.3}$$

Let

$$\phi_\varepsilon(x) = \left(\frac{\varepsilon}{|x'|^2 + (\varepsilon + x_N)^2}\right)^{(N-2)/2},$$

where $x = (x', x_N)$. This function ϕ_ε minimizes

$$\inf_{v\in H^1({}^N_+)} \frac{\displaystyle\int_{{}^N_+} |\nabla v|^2\, dx}{\left(\displaystyle\int_{\partial\ {}^N_+} |v|^{\tilde 2}\, d\sigma\right)^{2/\tilde 2}},$$

for any $\varepsilon > 0$, see [3]. We have that

$$\frac{\displaystyle\int_{{}^N_+} |\nabla \phi_\varepsilon|^2 + |\phi_\varepsilon|^2\, dx}{\left(\displaystyle\int_{\partial\ {}^N_+} |\phi_\varepsilon|^{\tilde 2}\, d\sigma\right)^{2/\tilde 2}} = \frac{\displaystyle\int_{{}^N_+} |\nabla \phi_\varepsilon|^2\, dx + \varepsilon^2 \int_{{}^N_+} |\phi_1|^2\, dx}{\left(\displaystyle\int_{\partial\ {}^N_+} |\phi_\varepsilon|^{\tilde 2}\, d\sigma\right)^{2/\tilde 2}}.$$

As

$$\varepsilon^2 \int_{{}^N_+} |\phi_1|^2\, dx \to 0, \qquad \varepsilon \to 0,$$

and

$$\left(\int_{\partial\ {}^N_+} |\phi_\varepsilon|^{\tilde 2}\, d\sigma\right)^{2/\tilde 2} = \left(\int_{\partial\ {}^N_+} |\phi_1|^{\tilde 2}\, d\sigma\right)^{2/\tilde 2}$$

we get that

$$\inf_{v\in H^1({}^N_+)} \frac{\displaystyle\int_{{}^N_+} |\nabla v|^2 + |v|^2\, dx}{\left(\displaystyle\int_{\partial\ {}^N_+} |v|^{\tilde 2}\, d\sigma\right)^{2/\tilde 2}} = \inf_{v\in H^1({}^N_+)} \frac{\displaystyle\int_{{}^N_+} |\nabla v|^2\, dx}{\left(\displaystyle\int_{\partial\ {}^N_+} |v|^{\tilde 2}\, d\sigma\right)^{2/\tilde 2}}.$$

a contradiction with (5.3). This proves that such a sequence of extremals in expanding balls, w_j, cannot exist, proving that there is no extremal for large balls. □

Remark 5.1. *The same argument shows that for a general bounded smooth domain Ω, for the critical exponent $\tilde 2$ there is no extremal for the embedding $H^1(\mu\Omega) \hookrightarrow L^{\tilde 2}(\partial\mu\Omega)$ for large enough μ.*

6. Proof of Theorem 1.7

From Hölder's inequality

$$\left(\int_{\partial\Omega} |u|^{q_1}\right)^{1/q_1} \le |\partial\Omega|^{\frac{1}{q_1}-\frac{1}{q_2}} \left(\int_{\partial\Omega} |u|^{q_2}\right)^{1/q_2}$$

for $1 < q_1 < q_2$, we get that $S_{q_2} \le S_{q_1} |\partial\Omega|^{\frac{1}{q_1}-\frac{1}{q_2}}$.

Now assume that there exists a radial extremal, u_r, for $q = q_2$. Using that u_r is constant on the boundary $\partial B(0,\mu)$

$$S_{q_2} = \frac{\displaystyle\int_{B(0,\mu)} |\nabla u_r|^2 + |u_r|^2\, dx}{\left(\displaystyle\int_{\partial B(0,\mu)} |u_r|^{q_2}\, d\sigma\right)^{2/q_2}}$$

$$= |\partial B(0,\mu)|^{\frac{1}{q_1}-\frac{1}{q_2}} \frac{\displaystyle\int_{B(0,\mu)} |\nabla u_r|^2 + |u_r|^2\, dx}{\left(\displaystyle\int_{\partial B(0,\mu)} |u_r|^{q_1}\, d\sigma\right)^{2/q_1}} \le |\partial B(0,\mu)|^{\frac{1}{q_1}-\frac{1}{q_2}} S_{q_1}.$$

Therefore

$$\frac{\displaystyle\int_{B(0,\mu)} |\nabla u_r|^2 + |u_r|^2\, dx}{\left(\displaystyle\int_{\partial B(0,\mu)} |u_r|^{q_1}\, d\sigma\right)^{2/q_1}} \le S_{q_1}.$$

This finishes the proof. □

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Departamento de Matemática, FCEyN
UBA (1428) Buenos Aires, Argentina.
E-mail address: jfbonder@dm.uba.ar, jrossi@dm.uba.ar

CONICET-Univ. de Buenos Aires and Univ. Libre de Bruxelles.
E-mail address: elami@dm.uba.ar