Non-simultaneous blow-up in a nonlinear parabolic system

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We prove the existence of a nontrivially coupled parabolic system such that one of its components becomes unbounded at a finite time while the other remains bounded, a situation that we denote as non-simultaneous blow-up.

Our system consists of two porous medium equations with coupled nonlinear flux boundary conditions. As a preliminary step we will obtain a necessary and sufficient condition for blow-up. Next we characterize completely, in the case of increasing in time solutions, the set of parameters appearing in the system for which non-simultaneous blow-up indeed occurs.

In the course of our proofs we will obtain a necessary and sufficient condition for the blow-up of solutions to general porous medium type equations on the half-line with a prescribed flux at the boundary blowing up at a finite time, a result of independent interest.

1. INTRODUCTION

The main goal of this paper is to prove the existence of a nontrivially coupled parabolic system such that one of its components becomes unbounded at a finite time while the other remains bounded. We will denote this phenomenon as *non-simultaneous blow-up*. This is, to the best of our knowledge, the first known nontrivial example of such a behaviour.

Our system will consist of two *porous medium equations*

$$\left\{\begin{array}{l} u_t = (u^m)_{xx}, \\ v_t = (v^n)_{xx}, \end{array}\right. \qquad (x,t) \in D_T = \{(x,t) : x > 0,\ 0 < t < T\}, \tag{1}$$

with nonlinear boundary conditions given by

$$\left\{\begin{array}{l} -(u^m)_x(0,t) = u^{p_{11}}(0,t)v^{p_{12}}(0,t), \\ -(v^n)_x(0,t) = u^{p_{21}}(0,t)v^{p_{22}}(0,t), \end{array}\right. \qquad t \in (0,T), \tag{2}$$

and initial data

$$\left\{\begin{array}{l} u(x,0) = u_0(x) \geq 0, \\ v(x,0) = v_0(x) \geq 0, \end{array}\right. \qquad x > 0, \tag{3}$$

where $m, n > 1$, $p_{ij} > 0$ are parameters. By a solution of the problem we mean a continuous pair (u,v) that verifies (1)–(3) in a weak sense. The constant T denotes the maximal existence time for the solution (u,v), which is finite if and only if any of the components, u or v, becomes unbounded in a finite time. We assume that u_0, v_0 are continuous, nonnegative, nontrivial ($u_0, v_0 \not\equiv 0$) and compactly supported in $\mathbb{R}_+ = [0,\infty)$.

Remark 1. 1. The function u is a supersolution to

$$\left\{\begin{array}{l} u_t = (u^m)_{xx}, \\ -(u^m)_x(0,t) = 0. \end{array}\right. \tag{4}$$

The well-known Barenblatt solutions,

$$U(x,t;C,x_0,\tau) = (t+\tau)^{-\frac{1}{m+1}} \left(C - \frac{m-1}{2m(m+1)}\,\xi^2 \right)_+^{\frac{1}{m-1}},$$

with $\xi = (x-x_0)/(t+\tau)^{1/(m+1)}$ and $x_0, \tau > 0$, are subsolutions to (4). As u_0 is nontrivial, it is possible to choose C, x_0 and τ so that $U \leq u_0$ at time $t = 0$. Hence comparison for (4) implies that $u(0,t)$ will eventually become positive. The same argument works for v. Therefore, we can assume without loss of generality that $u_0(0) > 0$ and $v_0(0) > 0$. We will make this assumption in the sequel.

Local in time existence of a nonnegative continuous weak solution to problem (1)–(3) is easily proved by a standard monotone iteration argument (see for example [19], where the limit case $p_{11} = p_{22} = 0$ is considered). As u and v are solutions to porous medium equations, and they have compactly supported initial data, they are compactly supported in space for all time $t \in [0, T)$.

For certain values of the exponents m, n, p_{ij} (see below), there are solutions (u, v) to problem (1)–(3) that *blow up* at a finite time $T < \infty$ in L^∞-norm. By this we mean that these solutions are defined for $t \in [0, T)$, while

$$\limsup_{t \nearrow T} \left\{ \|u(\cdot, t)\|_{L^\infty(\mathbb{R}_+)} + \|v(\cdot, t)\|_{L^\infty(\mathbb{R}_+)} \right\} = \infty.$$

If $T = \infty$ we say that the solution is global. Blow-up for parabolic systems has received some attention in recent years, see for example [1], [4], [7], [8], [18], [19], [20], [21], [22], [24], the surveys [16], [5], and the references therein.

The blow-up definition does not exclude the possibility of non-simultaneous blow-up. Our aim is to show that this possibility indeed occurs for certain values of the parameters.

Two special limit cases of system (1)–(3) are already well known. The first one is the case in which there is no coupling,

$$p_{12} = p_{21} = 0.$$

Each component lives on its own, not being affected by the other. Hence we are dealing with two uncoupled scalar problems of the form

$$\begin{cases} u_t = (u^m)_{xx}, & (x, t) \in D_T, \\ -(u^m)_x(0, t) = u^p(0, t), & t \in (0, T), \\ u(x, 0) = u_0(x), & x > 0, \end{cases} \tag{5}$$

where $m > 1$. Galaktionov and Levine [9] show that if $0 < p \leq (m + 1)/2$, then for arbitrary initial data the solution is global in time, while for $p > (m + 1)/2$ there are solutions u with finite time blow-up in the L^∞-norm, *i.e.*, such that there exists a finite time T for which

$$\limsup_{t \nearrow T} \|(u(\cdot, t)\|_\infty = \infty.$$

Thus, if $p_{11} > (m + 1)/2$ and $p_{22} \leq (n + 1)/2$ there are initial data u_0 and v_0 such that system (1)–(3) has non-simultaneous blow-up. However, this system does not serve for our purposes, as the coupling is trivial.

The second special limit case is

$$p_{11} = p_{22} = 0, \tag{6}$$

which is studied in [19]. The authors prove in this case that if

$$4p_{12}p_{21} \leq (m+1)(n+1), \tag{7}$$

then every nonnegative solution of (1)–(3) is global in time, while if (7) does not hold, then there are solutions with finite time blow-up. They also prove that there are values of the exponents m, n, p_{12}, p_{21} for which the coupling is weak, so that the *blow-up sets*

$$B(u) := \{x \in \mathbb{R}_+ : \limsup_{t \nearrow T} u(x,t) = \infty\},$$

$$B(v) := \{x \in \mathbb{R}_+ : \limsup_{t \nearrow T} v(x,t) = \infty\}$$

of both components are different for certain initial data. However, whenever there is blow-up, both components become unbounded at the same time. That is, u blows up in L^∞-norm at time T if and only if v also does so. Non-simultaneous blow-up is therefore not possible if (6) holds.

The case $p_{ij} > 0$ is somewhat in the middle of these two limit cases. The coupling is nontrivial, but weaker than in the case (6), thus allowing the possibility of non-simultaneous blow-up. We will show that there are indeed certain choices of the parameters leading to this phenomenon.

In order to have non-simultaneous blow-up, we need that one of the components blows up. Hence, a first step in our search is to determine the region of parameters for which blow-up occurs. This result, which is interesting on its own, is the content of our first theorem.

THEOREM 1.1. *If any of the following conditions*

$$2p_{11} > m+1, \tag{8}$$

$$2p_{22} > n+1, \tag{9}$$

$$4p_{12}p_{21} > (m+1-2p_{11})(n+1-2p_{22}), \tag{10}$$

holds, then there exist solutions (u, v) *of* (1)–(3) *that blow up in* L^∞*-norm in finite time* T*. Conversely, if none of the above conditions holds, then every solution of* (1)–(3) *is global.*

Remark 1. 2. If $p_{11} = p_{22} = 0$ these conditions are reduced to (7).

Next theorem gives a set of parameters for which non-simultaneous blow-up indeed occurs. This provides us with the example we were looking for.

THEOREM 1.2. *If both conditions,*

$$2p_{11} > m + 1, \tag{11}$$

$$2p_{21} < 2p_{11} - (m + 1), \tag{12}$$

hold, then there exist initial data (u_0, v_0) such that u has blow-up at a finite time T while v remains bounded up to that time.

Condition (11) guarantees blow-up for u, see Theorem 1.1, while condition (12) implies that the coupling between u and v is weak enough to have non-simultaneous blow-up.

Remark 1. 3. Though v remains bounded in L^∞-norm, $(v^m)_x(0, t)$ blows up at time T.

Conditions (11) and (12) turn out to be also necessary for the existence of non-simultaneous blow-up in the case of solutions that increase with time.

THEOREM 1.3. *Let (u, v) be a solution of* (1)–(3) *that increases with time. If u blows up at time T and v remains bounded up to time T, then both* (11) *and* (12) *hold.*

Remark 1. 4. The initial data (u_0, v_0) provided by Theorem 1.2 are such that the corresponding solution (u, v) increases with time. Hence we have completely characterized, in the case of increasing in time solutions, the range of parameters for which the non-simultaneous blow-up phenomenon occurs.

In the class of increasing in time solutions, the occurrence of non-simultaneous blow-up depends continuously on the initial data.

THEOREM 1.4. *If we restrict ourselves to initial data corresponding to solutions that increase with time, then the set of initial data (u_0, v_0) such that u has blow-up at a finite time while v remains bounded up to that time is an open set in L^∞.*

In order to prove some of the results concerning problem (1)–(3) we will need to consider the behaviour of nonnegative solutions of porous medium equations on the half-line with a prescribed *inwards flux* at the boundary which becomes unbounded at a finite time. We will deal with a more general problem. Namely, for $T \in \mathbb{R}$, $T > 0$, we consider solutions $u(x,t)$ to the following problem

$$\begin{cases} u_t = (\varphi(u))_{xx}, & (x,t) \in D_T, \\ -(\varphi(u))_x(0,t) = \psi(t), & t \in (0,T), \\ u(x,0) = u_0(x), & x > 0, \end{cases} \tag{13}$$

with ψ smooth, nonnegative, nondecreasing and such that

$$\lim_{t \nearrow T} \psi(t) = +\infty, \tag{14}$$

and φ smooth, nonnegative and such that

$$0 < c \le \varphi'(u) \le C \qquad \text{if } 0 < \delta \le u \le K. \tag{15}$$

The initial data u_0 are assumed to be continuous, nonnegative and compactly supported in $\mathbb{R}_+$. Existence and uniqueness for solutions of this problem was proved by Cortazar and Elgueta in [3].

In the special case in which

$$\varphi(u) = u^m, \tag{16}$$

we have a Neumann problem for the heat equation if $m = 1$, and for the porous medium equation if $m > 1$. The corresponding Dirichlet problem has already been considered for example in [11] and [12]. If $m = 1$, then u is the temperature and $-u_x$ is the heat flux. If $m > 1$, then u stands for the density of a fluid flowing in a porous medium and $-(\varphi(u))_x(x,t)$ is the *flux*, *i.e.*, the amount of fluid crossing the point x at time t in the positive direction per time unit, [2], [23]. Condition (14) says that the flux becomes unbounded in a finite time.

With a boundary condition such as (14), the solution u of (13) may blow up at time T in the L^∞-norm. However, there is another possible situation. The solution u may remain bounded up to time T, *i.e.*, there may exist a value $K < \infty$ such that

$$\sup_{0<t<T} \|u(\cdot,t)\|_\infty \le K.$$

The actual behaviour of a solution depends on the precise behaviour of ψ near $t = T$. Our first aim is to obtain necessary and sufficient conditions on ψ for the blow-up of u. This problem was tackled in [3] for the special case in which φ is a power, (16). The authors prove that in these cases u is bounded in D_T if and only if

$$\lim_{t \nearrow T} \int_0^t \frac{\psi(\tau)}{(t-\tau)^{\frac{1}{2}}}\, d\tau < \infty. \tag{17}$$

We prove that this result remains valid when φ is not a power.

THEOREM 1.5. *For all* u_0*, the solution of* (13) *remains bounded in* L^∞*-norm up to time* T *if and only if* (17) *holds.*

Remark 1. 5. In the special case where

$$\psi(t) = c(T-t)^{-s}, \qquad c > 0, \tag{18}$$

Theorem 1.5 says that u blows up at time T if and only if $s \geq 1/2$.

Remark 1. 6. The total mass of u satisfies

$$\int_0^\infty u(x,t)\,dx - \int_0^\infty u_0(x)\,dx = \int_0^t \psi(\tau)\,d\tau.$$

Hence, a solution of (13) remains bounded in L^1-norm up to time T if and only if

$$\lim_{t\nearrow T} \int_0^t \psi(\tau)\,d\tau < \infty.$$

This condition is different from the condition for blow-up in L^∞-norm (17).

The fact that the blow-up condition does not depend on φ is an interesting feature. This is not always the case in nonlinear diffusion problems with blow-up. Consider, for example, the porous medium equation with a nonlinear flux boundary condition, problem (5). As mentioned above, in this case the condition for the existence of blowing up solutions is $p > (m+1)/2$. Thus, the blow-up condition depends on φ (through m). This is not the case for problem (13).

We will also need to consider the problem

$$\begin{cases} u_t = (u^m)_{xx}, & (x,t) \in D_T, \\ -(u^m)_x(0,t) = cu^p(0,t)(T-t)^{-s}, & t \in (0,T), \\ u(x,0) = u_0(x), & x > 0, \end{cases} \tag{19}$$

with $c > 0$ and $m > 1$. As before we consider continuous weak solutions. Our main result concerning this problem, which is a corollary of Theorem 1.5, is a characterization of the exponents for which there is blow-up.

THEOREM 1.6. *a) If* $s \geq 1/2$*, then every nonnegative, nontrivial solution of* (19) *with* $u_0(0) > 0$ *blows up at a finite time* $T_0 \leq T$*.*
b) If $s < 1/2$*, there exist initial data* u_0 *with* $u_0(0) > 0$ *such that the solution of* (19) *remains bounded up to time* T*. Moreover, given* $\varepsilon > 0$*, we can find initial*

data with $u_0(0) > 0$ and a blow-up time for the flux, T, such that

$$\sup_{0<t<T} \|u(\cdot,t)\|_\infty \le \|u_0(\cdot)\|_\infty + \varepsilon.$$

Remark 1. 7. The assumption $u_0(0) > 0$ in Theorem 1.6 is relevant. If we do not assume it, part *a)* is false and part *b)* becomes trivial. To see this, take initial data such that the support of the solution of the Cauchy problem for the porous medium equation with that data does not reach $x = 0$ for $0 < t < T$.

ORGANIZATION OF THE PAPER. In Section 2 we study the scalar problems (13) and (19). Section 3 is devoted to the study of the blow-up conditions for the system (1)–(3). Finally, in Section 4 we prove the results concerning the occurrence of non-simultaneous blow-up. Throughout this paper C and c denote constants that may change from one line to another.

2. BLOW-UP RESULTS FOR THE SCALAR PROBLEMS

This section is devoted to the scalar problems (13) and (19). We will prove theorems 1.5 and 1.6. First we prove Theorem 1.5 in the linear case, $\varphi(u) = u$. Thus we are dealing with the heat equation

$$\begin{cases} u_t = u_{xx}, & (x,t) \in D_T, \\ -u_x(0,t) = \psi(t), & t \in (0,T), \\ u(x,0) = u_0(x), & x > 0. \end{cases} \tag{20}$$

This result is contained in [3] but we prove it here for the sake of completeness.

LEMMA 2.1. *Let u be a solution to* (20). *It remains bounded up to time T if and only if* (17) *holds.*

Proof. Let $\Gamma(x,t)$ be the fundamental solution of the heat equation in the half-line, namely

$$\Gamma(x,t) = \frac{1}{(\pi t)^{1/2}} \exp\left(-\frac{x^2}{4t}\right).$$

For $x \in \mathbb{R}_+$ we have

$$u(x,t) = \int_{\mathbb{R}_+} \Gamma(x-y,t)u(y,0)\,dy - \int_0^t \frac{\partial u}{\partial x}(0,\tau)\Gamma(x,t-\tau)\,d\tau \tag{21}$$

$$+ \int_0^t \frac{\partial \Gamma}{\partial x}(x,t-\tau)u(0,\tau)\,d\tau.$$

Taking limits as $x \to 0$ we get that

$$\frac{1}{2}u(0,t) = \int_{\mathbb{R}_+} \Gamma(-y,t)u_0(y)\,dy + \frac{1}{\sqrt{\pi}}\int_0^t \frac{\psi(\tau)}{(t-\tau)^{\frac{1}{2}}}\,d\tau.$$

The first term of the right-hand side is bounded. Hence, $u(0,t)$ is bounded if and only if (17) holds. ▮

Remark 2. 1. Let (17) hold. It follows from the representation formula (21) that,

$$|u(x,t) - u_0(x)| \le \left| \int_{\mathbb{R}_+} \Gamma(x-y,t)u_0(y)\,dy - u_0(x) \right| + \frac{1}{\sqrt{\pi}}\int_0^t \frac{\psi(\tau)}{(t-\tau)^{\frac{1}{2}}}\,d\tau$$

$$+ \int_0^t \left| \frac{\partial\Gamma}{\partial x}(x,t-\tau) \right| u(0,\tau)\,d\tau.$$

It is trivial to check that the first term of the right-hand side is small for all $0 < t < T$ if T is small. Assume that $\psi(\tau) = \tilde{\psi}(T-\tau)$, where $\tilde{\psi}$ does not depend on T. With this assumption, we can also prove that if T is small then the term

$$\frac{1}{\sqrt{\pi}}\int_0^t \frac{\psi(\tau)}{(t-\tau)^{\frac{1}{2}}}\,d\tau,$$

is small for every $0 < t < T$. We have

$$\int_0^T \frac{\psi(\tau)}{(T-\tau)^{\frac{1}{2}}}\,d\tau = \int_0^T \frac{\tilde{\psi}(T-\tau)}{(T-\tau)^{\frac{1}{2}}}\,d\tau = \int_0^T \frac{\tilde{\psi}(\omega)}{\omega^{\frac{1}{2}}}\,d\omega.$$

As we assume (17), $\tilde{\psi}(\omega)/\omega^{\frac{1}{2}} \in L^1(0,T)$ and the claim follows. Finally, since $u(0,t)$ is bounded up to time T (uniformly in T), the last term is also small provided that T is small.

Therefore, given $\varepsilon > 0$ there exists T small such that $|u(x,t) - u_0(x)| < \varepsilon$ for every $0 < t < T$. Hence,

$$\sup_{0<t<T} \|u(\cdot,t)\|_\infty \le \|u_0(\cdot)\|_\infty + \varepsilon.$$

We remark that, since the flux blows up, this result gives more than just the continuous dependence on the initial data.

Remark 2. 2. If the diffusion coefficient is a constant different from 1, *i.e.*, if the equation in (20) is $u_t = cu_{xx}$, then we still have that u remains bounded up to time T if and only if (17) holds. Again, $|u(x,t) - u_0(x)|$ is small if T is small.

To deal with a general φ, first we prove that either every solution is bounded or every solution blows up.

LEMMA 2.2. *Either for every u_0 the nonnegative solution of* (13) *blows up at time T or every for every u_0 the nonnegative solution is bounded up to time T.*

Proof. Let u_1 be any solution of (13) and u_2 be a solution with initial data $u_2(x,0) = 0$. A comparison argument shows that $u_1 \geq u_2$. We will prove that $\|u_1(\cdot,t) - u_2(\cdot,t)\|_\infty$ does not increase with time. Hence, if u_1 blows up at time $t = T$, so does u_2. Once we know that u_2 blows up at time T, to prove that every solution blows up is immediate just using comparison.

Thus, we only have to prove that $\|u_1(\cdot,t) - u_2(\cdot,t)\|_\infty$ does not increase with time. That is what we do next. Assume that it is not zero. Let $0 \leq t_1 < t_2 < T$. We define

$$\Phi(x,t) = \begin{cases} \dfrac{\varphi(u_1(x,t)) - \varphi(u_2(x,t))}{u_1(x,t) - u_2(x,t)} & \text{if } u_1(x,t) \neq u_2(x,t), \\ \\ \varphi'(u_1(x,t)) & \text{if } u_1(x,t) = u_2(x,t). \end{cases}$$

Then $w = u_1 - u_2 \geq 0$ is a solution of

$$\begin{cases} w_t = (\Phi(x,t)w)_{xx}, & x > 0,\ t \in (t_1,t_2), \\ -(\Phi(x,t)w)_x(0,t) = 0, & t \in (t_1,t_2), \\ w(x,t_1) = u_1(x,t_1) - u_2(x,t_1), & x > 0. \end{cases} \tag{22}$$

As (22) is a parabolic equation the maximum of $w = u_1 - u_2$ in $[t_1,t_2] \times \mathbb{R}_+$ must lie on the parabolic boundary, that is $t = t_1$ or $x = 0$. If the maximum lies somewhere at $x = 0$, then necessarily $u_1 > 0$ at that maximum point. Therefore, the equation (22) is uniformly parabolic in a neighbourhood of that point. Applying Hopf's Lemma we arrive to a contradiction. So we have

$$\|u_1(\cdot,t_2) - u_2(\cdot,t_2)\|_{L^\infty} \leq \|u_1(\cdot,t_1) - u_2(\cdot,t_1)\|_{L^\infty}.$$

This ends the proof. ▌

Remark 2. 3. In the course of the proof of this lemma we have obtained a *contraction in L^∞* result for ordered solutions.

Now we can prove Theorem 1.5 for the general case.

Proof of Theorem 1.5. Assume that (17) does not hold and that there is a solution u bounded up to time T. Let $v = \varphi(u)$. We get

$$\begin{cases} \dfrac{1}{\varphi'(u)} v_t = v_{xx}, \\ -v_x(0,t) = \psi(t). \end{cases}$$

As either all solutions blow up or none of them does, we can take any initial data without loss of generality. We take as initial data $v(x,0) = \delta > 0$. It is easy to check that $v_t \geq 0$. Thus, $v(x,t) \geq \delta$ for all t. Hence, using condition (15), we have that

$$\begin{cases} v_t \geq c v_{xx}, \\ -v_x(0,t) = \psi(t), \\ v(x,0) = \delta. \end{cases}$$

Thus, v is a bounded supersolution of the heat equation with a flux at the boundary given by $\psi(t)$, a contradiction. Therefore, all solutions blow-up.

Now we consider the case in which (17) holds. Let v be a solution to (20) with $v(x,0)$ such that $v_t = v_{xx} \geq 0$. As (17) holds, v turns out to be bounded and moreover if T is small, then $v(x,t)$ is close to $v(0,t)$. Let $u = \varphi^{-1}(v)$. Then

$$\begin{cases} \dfrac{1}{\varphi'(u)} v_t = \dfrac{1}{\varphi'(u)} v_{xx} \geq \dfrac{1}{C} v_{xx}, \\ -v_x(0,t) = \psi(t). \end{cases}$$

Thus

$$\begin{cases} u_t \geq \dfrac{1}{C} (\varphi(u))_{xx}, \\ -(\varphi(u))_x(0,t) = \psi(t), \\ u(x,0) = \varphi^{-1}(v(x,0)). \end{cases}$$

Therefore, u is a supersolution of problem (13) (the constant appearing in front of the diffusion term is not relevant). As u is bounded up to time T, any solution to (13) with initial data smaller than $\varphi^{-1}(v(x,0))$ is bounded, and hence any solution is bounded (cf. Lemma 2.2). Moreover, as $u(x,t)$ is close to $u_0(x)$ if T is small we conclude that the solution that has initial data $\varphi^{-1}(v(x,0))$ has the property that

$$\sup_{0<t<T} \|u(\cdot,t)\|_\infty \leq \|u_0(\cdot)\|_\infty + \varepsilon. \quad \blacksquare \tag{23}$$

Remark 2. 4. In the case where $\varphi(u) = u^m$ and $\psi(t) = c(T-t)^{-s}$ with $s > 1/2$, problem (13) has a blowing up solution of self-similar form

$$u(x,t) = (T-t)^{-\alpha} f(\xi) \qquad \xi = x/(T-t)^\beta,$$

where the similarity exponents α and β are given by

$$\alpha = \frac{2s-1}{m+1}, \qquad \beta = \frac{s(1-m)+m}{m+1}.$$

However, in the critical case $s = 1/2$ there are not blowing up solutions of (13) of self-similar form, even if we allow the more general kind of similarity

$$u(x,t) = h(t)f(\xi) \qquad \xi = x/g(t).$$

As a corollary we can prove Theorem 1.6.

Proof of Theorem 1.6. *a)* As we assume $u_0(0) > 0$ we have that u is a supersolution of (13) with $\psi(t) = c(T-t)^{-s}$ with $s \geq 1/2$. Then Theorem 1.5 gives the result.
b) Given $\varepsilon > 0$, let $z(x,t)$ be a solution of (13) with $\psi(t) = (C+1)^p(T-t)^{-s}$, $C > 0$, with an initial $z(x,0)$ and a blow-up time for the flux T small enough so that $z(x,t)$ has the property (23). Hence z is a supersolution of (19) that remains bounded up to time T. Now, let $0 \leq u_0(x) \leq z(x,0)$. As z is bounded up to time T and satisfies (23), u must be bounded up to time T and must satisfy (23). ∎

3. BLOW-UP RESULTS FOR THE SYSTEM

In this section we prove Theorem 1.1. The ideas are similar to [19]. However, we give a sketch of the proofs to make the paper self-contained.

We want to apply a comparison argument. To this end, we define what we call a subsolution and a supersolution for (1)–(2).

DEFINITION 3.1. The continuous pair $(\underline{u}, \underline{v})$ is a subsolution of (1)–(2) if it satisfies

$$\begin{cases} u_t \leq (u^m)_{xx} \\ v_t \leq (v^n)_{xx} \end{cases} \qquad (x,t) \in D_T, \tag{24}$$

$$\begin{cases} -(u^m)_x(0,t) \leq u^{p_{11}}(0,t)v^{p_{12}}(0,t) \\ -(v^n)_x(0,t) \leq u^{p_{21}}(0,t)v^{p_{22}}(0,t) \end{cases} \qquad t \in (0,T), \tag{25}$$

in a weak sense.

DEFINITION 3.2. We call $(\overline{u}, \overline{v})$ a supersolution of (1)–(2) if it satisfies (24)–(25) in a weak sense with the opposite inequalities.

Remark 3. 1. A pair of functions satisfying (24)–(25) pointwise in their supports and with nonpositive flux at their free boundaries is a subsolution of (1)–(2). An analogous statement holds for supersolutions, [2].

With these definitions of super and subsolutions we can state a comparison lemma.

LEMMA 3.1. *Let* $(\overline{u},\overline{v})$ *be a nonnegative supersolution and* $(\underline{u},\underline{v})$ *be a nonnegative subsolution of* (1)–(2). *If*

$$\underline{u}(x,0)\le\overline{u}(x,0),\qquad \underline{v}(x,0)\le\overline{v}(x,0),$$

with

$$\underline{u}(0,0)<\overline{u}(0,0)\qquad \textit{and}\qquad \underline{v}(0,0)<\overline{v}(0,0),$$

then

$$\underline{u}(x,t)\le\overline{u}(x,t)\qquad \textit{and}\qquad \underline{v}(x,t)\le\overline{v}(x,t),$$

as long as both pairs of functions exist.

Proof. It is standard; therefore we omit the details. We argue in the support of $\underline{u}$ and $\underline{v}$, since at other points the conclusion is trivial, as we are considering nonnegative functions. Assume that the result is false. Let t_0 be the maximum time such that

$$\underline{u}(x,t)\le\overline{u}(x,t)\qquad \text{and}\qquad \underline{v}(x,t)\le\overline{v}(x,t),$$

up to time t_0. This time t_0 must be positive by continuity and well known properties of the porous medium equation. At that time t_0 we must have that $\underline{u}(0,t_0)=\overline{u}(0,t_0)$ or $\underline{v}(0,t_0)=\overline{v}(0,t_0)$. Let us assume that $\underline{u}(0,t_0)=\overline{u}(0,t_0)$. Now the result follows by an application of Hopf's Lemma. Indeed, $\overline{u}-\underline{u}$ satisfies a uniformly parabolic inequality in a neighbourhood of $x=0$, attains a minimum at $(0,t_0)$, and the corresponding flux is greater or equal than zero, a contradiction. ∎

Now we proceed with the blow-up part of Theorem 1.1. We organize the proof in two lemmas.

LEMMA 3.2. *If* (8) *or* (9) *hold, then there exist solutions* (u,v) *of* (1)–(3) *that blow up in a finite time* T.

Proof. Assume that (8) holds. We can assume that $v(0,0) > 0$. Therefore there is a value t_0 such that $v(0,t) > c$ for every $t \in (0,t_0)$. It follows that u satisfies $-(u^m)_x(0,t) \ge cu^{p_{11}}(0,t)$. Thus u is a supersolution of

$$\begin{cases} u_t = (u^m)_{xx}, & x > 0,\ t \in (0,t_0), \\ -(u^m)_x(0,t) = cu^{p_{11}}(0,t), & t \in (0,t_0), \\ u(x,0) = u_0(x), & x > 0. \end{cases} \tag{26}$$

We observe that choosing u_0 large we can obtain T as small as we want. So, we only have to choose u_0 large enough in order to guarantee that u blows up at a time $T < t_0$. This can be done because solutions of (26) blow up in a finite time if $2p_{11} > m+1$, see [9], [19]. The case in which (9) holds is treated in the same way. ∎

If (10) holds, then we can construct a subsolution of self-similar form with finite time blow-up. We will conclude that we have solutions with finite time blow-up using the comparison lemma (Lemma 3.1). The self-similar exponents are

$$\alpha_1 = \frac{n+1-2p_{22}+2p_{12}}{(m+1-2p_{11})(n+1-2p_{22})-4p_{21}p_{12}}, \qquad \beta_1 = \frac{1+(m-1)\alpha_1}{2},$$

$$\alpha_2 = \frac{m+1-2p_{11}+2p_{21}}{(m+1-2p_{11})(n+1-2p_{22})-4p_{21}p_{12}}, \qquad \beta_2 = \frac{1+(n-1)\alpha_2}{2}.$$

LEMMA 3.3. *If* (10) *holds, then there exists a pair of compactly supported functions θ_1, θ_2, such that*

$$\begin{array}{ll} \underline{u}(x,t) = (T-t)^{\alpha_1}\theta_1(\xi) & \xi = x/(T-t)^{\beta_1}, \\ \underline{v}(x,t) = (T-t)^{\alpha_2}\theta_2(\zeta) & \zeta = x/(T-t)^{\beta_2}, \end{array}$$

is a subsolution of (1)–(2).

Proof. To satisfy (24) and (25) we need that

$$\begin{cases} (\theta_1^m)''(\xi) \ge -\alpha_1\theta_1(\xi) + \beta_1\xi\theta_1'(\xi), \\ (\theta_2^n)''(\zeta) \ge -\alpha_2\theta_2(\zeta) + \beta_2\zeta\theta_2'(\zeta), \end{cases}$$

$$\begin{cases} -(\theta_1^m)'(0) \le \theta_1^{p_{11}}(0)\theta_2^{p_{12}}(0), \\ -(\theta_2^n)'(0) \le \theta_1^{p_{21}}(0)\theta_2^{p_{22}}(0). \end{cases}$$

We choose,

$$\theta_1(\xi) = A_1(a_1-\xi)^{\frac{1}{m-1}}.$$

Inserting this in the first inequality we get

$$\frac{m}{(m-1)^2}A_1^{m-1} \ge -\alpha_1(a_1-\xi) - \frac{\beta_1}{m-1}\xi \qquad \text{for } 0 \le \xi \le a_1.$$

Hence, it is enough to impose

$$\frac{m}{(m-1)^2}A_1^{m-1} \ge -\alpha_1 a_1 + \frac{|\beta_1|}{m-1}a_1,$$

that is

$$C_1 A_1^{m-1} \ge a_1. \tag{27}$$

Analogously we choose

$$\theta_2(\zeta) = A_2(a_2-\zeta)_+^{\frac{1}{n-1}},$$

and from the second equation we get

$$C_2 A_2^{n-1} \ge a_2. \tag{28}$$

The boundary conditions impose

$$\begin{array}{l} \frac{m}{m-1}A_1^m a_1^{\frac{1}{m-1}} \ge A_1^{p_{11}} a_1^{\frac{p_{11}}{m-1}} A_2^{p_{12}} a_2^{\frac{p_{12}}{n-1}}, \\ \frac{n}{n-1}A_2^n a_2^{\frac{1}{n-1}} \ge A_1^{p_{21}} a_1^{\frac{p_{21}}{m-1}} A_2^{p_{22}} a_2^{\frac{p_{22}}{n-1}}. \end{array} \tag{29}$$

Let

$$b_1 = A_1 a_1^{\frac{1}{m-1}}, \qquad b_2 = A_2 a_2^{\frac{1}{n-1}}.$$

Then, conditions (29) become

$$\begin{array}{l} \frac{m}{m-1}A_1^{m-1} b_1 \ge b_1^{p_{11}} b_2^{p_{12}}, \\ \frac{n}{n-1}A_2^{n-1} b_2 \ge b_1^{p_{21}} b_2^{p_{22}}. \end{array} \tag{30}$$

We fix $b_1 = b_2 = 1$, and then take A_1, A_2 large enough (and thus a_1 and a_2 small) to satisfy (27), (28) and (30). ∎

Now we only have to observe the following, if (10) holds but (8) and (9) do not then the subsolution that we have constructed blows up because in this case α_1 and α_2 are negative. Hence every solution that has initial data above the subsolution must blow-up in finite time.

This ends the proof of the existence of blowing up solutions if (8), (9) or (10) hold.

In order to prove the global existence part we will use as supersolutions a family of self-similar solutions of exponential form which exist for some specific values of the exponents. To this end we will use the following result of Gilding and Peletier, [10], [13], [14]:

THEOREM 3.1. *Let $a, b, V \in \mathbb{R}$ and $U \geq 0$. For fixed a and b, let $\mathcal{S}_A$ denote the set of values (U, V) such that there exists a weak, nonnegative, compactly supported solution f of*

$$(f^m)''(\eta) + a\eta f'(\eta) = bf(\eta) \qquad \text{for } 0 < \eta < \infty, \tag{31}$$

$$f(0) = U, \tag{32}$$

$$(f^m)'(0) = V, \tag{33}$$

and let $\mathcal{S}_B$ denote the set of values (U, V) for which there exists a bounded, positive, classical solution f of (31)–(33).

(a) *If $b < 0$ and $2a + b < 0$ then $\mathcal{S}_A = \{(0, 0)\}$ and $\mathcal{S}_B = \emptyset$.*

(b) *If $b < 0$ and $2a + b = 0$ then $\mathcal{S}_A = \{(0, V) : \ 0 \leq V < \infty\}$ and $\mathcal{S}_B = \emptyset$.*

(c) *If $b \leq 0$ and $2a + b > 0$ then there exists a unique V_* such that $\mathcal{S}_A = \{(U, U^{(m+1)/2}V_*) : \ 0 \leq U < \infty\}$ and*

$$\mathcal{S}_B = \{(U, V) : \ 0 \leq U < \infty, \quad U^{(m+1)/2}V_* < V < \infty\},$$

where $V_* > 0$ if $a + b < 0$, $V_* = 0$ if $a + b = 0$ and $V_* < 0$ if $a + b > 0$.

(d) *If $b > 0$ and $a \geq 0$ then there exists a unique $V_* < 0$ such that $\mathcal{S}_A = \{(U, U^{(m+1)/2}V_*) : \ 0 \leq U < \infty\}$ and $\mathcal{S}_B = \emptyset$.*

(e) *If $b > 0$ and $a < 0$, or $b = 0$ and $a \leq 0$, then $\mathcal{S}_A = \{(0, 0)\}$ and there exists a unique V_* such that $\mathcal{S}_B = \{(U, U^{(m+1)/2}V_*) : \ 0 \leq U < \infty\}$, where $V_* < 0$ if $b > 0$ and $V_* = 0$ if $b = 0$.*

Moreover, for each $(U, V) \in \mathcal{S}_A \cup \mathcal{S}_B$ there exists at most one weak solution of (31)–(33).

Remark 3. 2. In the case where $a = (m-1)b/2 > 0$ we have $V_* = -1$. This is a consequence of the existence of a self-similar solution of exponential form for a scalar problem, see [9].

The self-similar solutions of exponential form are constructed in next lemma.

LEMMA 3.4. *Let*

$$\begin{aligned} u(x,t) &= e^{\alpha_1 t} f(\xi) \qquad & \xi &= x/e^{\beta_1 t}, \\ v(x,t) &= e^{\alpha_2 t} g(\zeta) \qquad & \zeta &= x/e^{\beta_2 t}. \end{aligned} \tag{34}$$

If

$$4p_{12}p_{21} = (m+1-2p_{11})\,(n+1-2p_{22})\,, \tag{35}$$

with

$$m+1-2p_{11} > 0 \qquad n+1-2p_{22} > 0, \tag{36}$$

then, for any $\alpha_1 > 0$ *there is a self-similar solution of problem* (1)–(2) *global in time of the form* (34)*, where*

$$\alpha_2 = \left(\frac{m+1-2p_{11}}{2p_{12}}\right)\alpha_1, \qquad \beta_1 = \left(\frac{m-1}{2}\right)\alpha_1, \qquad \beta_2 = \left(\frac{n-1}{2}\right)\alpha_2. \tag{37}$$

Moreover, the supports of f *and* g *are compact.*

Proof. We are considering solutions of the form (34). Equations (1) impose the following relations for the exponents,

$$\begin{aligned} \alpha_1 &= m\alpha_1 - 2\beta_1, \\ \alpha_2 &= n\alpha_2 - 2\beta_2. \end{aligned} \tag{38}$$

The boundary conditions (2) imply that

$$\begin{aligned} m\alpha_1 - \beta_1 &= p_{11}\alpha_1 + p_{12}\alpha_2, \\ n\alpha_2 - \beta_2 &= p_{21}\alpha_1 + p_{21}\alpha_2. \end{aligned} \tag{39}$$

System (38)–(39) is reduced to

$$\begin{pmatrix} m+1-2p_{11} & -2p_{12} \\ -2p_{21} & n+1-2p_{22} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{40}$$

There is a nontrivial solution (α_1, α_2) of (40) if and only if

$$4p_{12}p_{21} = (m+1-2p_{11})\,(n+1-2p_{22})\,.$$

In this case, $\alpha_2, \beta_1, \beta_2$ are related to α_1 by (37).

On the other hand, the profiles must satisfy

$$\begin{aligned} (f^m)''(\xi) + \beta_1 \xi f'(\xi) &= \alpha_1 f(\xi), \\ (g^n)''(\zeta) + \beta_2 \zeta g'(\zeta) &= \alpha_2 g(\zeta), \end{aligned} \tag{41}$$

plus the boundary conditions

$$\begin{aligned} -(f^m)'(0) &= f^{p_{11}}(0) g^{p_{12}}(0), \\ -(g^n)'(0) &= f^{p_{21}}(0) g^{p_{22}}(0). \end{aligned} \tag{42}$$

As we are assuming (36), if $\alpha_1 > 0$, then $\alpha_2, \beta_1, \beta_2 > 0$. Moreover, $\beta_1 = (m-1)\alpha_1/2, \beta_2 = (n-1)\alpha_2/2$. Hence, using the remark after Theorem 3.1 we have solutions of the equations (42) with $V_{*1} = V_{*2} = -1$. That is, there are two one-parameter families (with parameters U_1, U_2) satisfying

$$\begin{aligned} f(0) &= U_1, \ \ -(f^m)'(0) = U_1^{(m+1)/2}, \\ g(0) &= U_2, \ \ -(g^n)'(0) = U_2^{(n+1)/2}. \end{aligned}$$

The boundary conditions impose

$$\begin{cases} U_1^{(m+1)/2} = U_1^{p_{11}} U_2^{p_{12}}, \\ U_2^{(n+1)/2} = U_1^{p_{21}} U_2^{p_{22}}. \end{cases}$$

Taking logarithms, this is equivalent to

$$\begin{pmatrix} m+1-2p_{11} & -2p_{12} \\ -2p_{21} & n+1-2p_{22} \end{pmatrix} \begin{pmatrix} \ln U_1 \\ \ln U_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Choosing a solution of this last equation (we are using again (35)) we conclude the proof of the theorem. ∎

Now, with the help of the self-similar solutions that we have just constructed we will prove the global existence part of the theorem.

LEMMA 3.5. *If none of the conditions* (8)–(10) *hold, then every nonnegative solution of* (1)–(3) *is global.*

Proof. It is enough to construct global supersolutions with initial data as large as needed. We achieve this with the aid of the self-similar solutions of exponential form that we found in Lemma 3.4.

First we choose $\tilde{p}_{12} \geq p_{12}$ such that $4\tilde{p}_{12} p_{21} = (m+1-2p_{11})(n+1-2p_{22})$, and we assume that $m+1 > 2p_{11}$ and $n+1 > 2p_{22}$. Let

$$\overline{u}(x,t) = e^{\alpha_1(t+\tau)} f\left(x/e^{\beta_1(t+\tau)}\right), \qquad \overline{v}(x,t) = e^{\alpha_2(t+\tau)} g\left(x/e^{\beta_2(t+\tau)}\right),$$

where $\alpha_1 > 0$ is arbitrary and α_2, β_1 and β_2 are given by (37) with $p_{12} = \tilde{p}_{12}$. Now we observe that this pair $(\overline{u}, \overline{v})$ is a supersolution of (1)–(3) as long as $\overline{v}(0,t) \geq 1$. This can be done by choosing τ large enough. This also allows to assume that

$\overline{u}(x,0) \geq u_0(x)$ and $\overline{v}(x,0) \geq v_0(x)$. Then, by the comparison Lemma 3.1, we obtain that every solution is global.

Now assume that $m+1 = 2p_{11}$. Then, as $4p_{12}p_{21} \leq (m+1-2p_{11})(n+1-2p_{22}) = 0$ we have $p_{12} = 0$ or $p_{21} = 0$. If $p_{12} = 0$ the equation for u is not coupled with the equation for v. Furthermore, as $m+1 = 2p_{11}$ every solution, u, is global (see [9]). For v we have that it satisfies a porous medium equation with a boundary condition of the form

$$-(v^n)_x(0,t) = u^{p_{21}}(0,t)v^{p_{22}}(0,t).$$

Now we only have to observe that $u(0,t)$ is bounded in every interval, $[0,T]$, (u is global) and that $2p_{22} \leq n+1$ to conclude that v is global, see [9]. The case $p_{21} = 0$ is analogous.

If $n+1 = 2p_{22}$ we reason in the same way as before. ▌

4. NON-SIMULTANEOUS BLOW-UP

In this section we prove theorems 1.2, 1.3 and 1.4.

Proof of Theorem 1.2. We want to choose initial data such that u blows up but v remains bounded. To do this we first fix v_0 such that $v_0(0) > 0$ and $v_t \geq 0$. Then u satisfies

$$\begin{cases} u_t = (u^m)_{xx} & (x,t) \in D_T, \\ -(u^m)_x(0,t) = u^{p_{11}}(0,t)v^{p_{12}}(0,t) \geq Cu^{p_{11}}(0,t) & t \in (0,T), \\ u(x,0) = u_0(x) & x > 0. \end{cases} \tag{43}$$

As $2p_{11} > m+1$, there are initial data w_0 such that the solution w of problem of

$$\begin{cases} u_t = (u^m)_{xx} & (x,t) \in D_T, \\ -(u^m)_x(0,t) = Cu^{p_{11}}(0,t) & t \in (0,T), \\ u(x,0) = u_0(x) & x > 0, \end{cases} \tag{44}$$

is increasing in time and blows up in a finite time, $\tilde{T}$. Hence, solutions u to (43) with initial data $u_0 = w(\tilde{T} - \varepsilon)$ will have blow-up time not greater than ε. That is, there are initial data u_0 such that the corresponding solution u is increasing in time and blows up in a finite time, T, which is as small as we want. Moreover,

$$u(0,t) \leq C(T-t)^{-\gamma_1}, \qquad \gamma_1 = \frac{1}{2p_{11} - (m+1)} > 0. \tag{45}$$

To prove this we proceed as follows. First we observe that as $u_t \geq 0$ then $(u^m)_{xx} \geq 0$ and hence $M(t) = \|u(\cdot,t)\|_\infty = \max u(\cdot,t) = u(0,t)$. Following ideas from

[15], we set

$$\varphi_M(y,s) = \frac{1}{M(t)} u(ay, bs+t), \qquad y \geq 0,\ -t/b \leq s \leq 0.$$

The function φ_M satisfies

$$0 \leq \varphi_M \leq 1, \qquad \varphi_M(0,0) = 1, \qquad (\varphi_M)_s \geq 0. \tag{46}$$

Choosing

$$a = M^{m-p_{11}}, \qquad b = M^{m+1-2p_{11}},$$

we have that φ_M is a solution of

$$\begin{cases} (\varphi_M)_s = (\varphi_M^m)_{yy}, \\ -(\varphi_M^m)_y(0,s) = \varphi^{p_{11}}(0,s) v^{p_{12}}(0, bs+t) \geq C\varphi^{p_{11}}(0,s). \end{cases} \tag{47}$$

We observe that there exists a number s_* such that that φ_M is well defined for every $(y,s) \in A = \{y > 0,\ s_* < s < 0\}$. Indeed, as $2p_{11} > m+1$ we have that $b \to 0$ and hence $-t/b \to -\infty$.

We claim that there exists a positive constant C such that for every M large enough it holds

$$(\varphi_M)_s(0,0) \geq C > 0. \tag{48}$$

As we will see later on, the blow-up rate follows from this inequality. Assume that the claim is false. Then, there is a sequence such that $(\varphi_{M_j})_s(0,0) \to 0$. From the results for bounded solutions of porous medium equations of [6] Section 6, we have that every sequence, $\{\varphi_{M_j}\}$ is equicontinuous. Therefore, passing to a subsequence if necessary, we have that

$$\varphi_{M_j} \to \varphi$$

uniformly on compact sets of $\{y \geq 0,\ s_* \leq s \leq 0\}$. This function φ is continuous and satisfies $\varphi(0,0) = 1$. Hence, there exists a neighbourhood of $(0,0)$, $U \subset A$, such that $\varphi > 1/2$ in U. As we have uniform convergence over $\overline{U}$ (we can assume that $\overline{U}$ is compact) for j large enough we have that $1/4 \leq \varphi_{M_j} \leq 1$. Therefore, φ_{M_j} is a solution of a uniformly parabolic equation in $\overline{U}$. Thus, from the fact that it is uniformly bounded we obtain, using well known Schauder estimates, [17], that

$$\|\varphi_{M_j}\|_{C^{2+\alpha,1+\alpha/2}} \leq C \qquad \text{in } \overline{U}.$$

Hence $\varphi_s(0,0) = 0$ and, since $\varphi_s \geq 0$ and $-(\varphi_s)_y \geq 0$, we have that φ does not depend on s. Thus φ is a bounded positive solution of the equation $(\varphi^m)_{yy} = 0$

with $-(\varphi^m)_y(0) \geq C$, $\varphi(0) = 1$. Such a solution does not exist, and therefore we have a contradiction.

Now we will obtain the blow-up rate from the claim. Writing the claim in terms of M,

$$M^{m-2p_{11}} M' \geq C,$$

which, after integration from t to T, and the change of variables $\tau = M(t)$ provides

$$\int_{M(t)}^{+\infty} \tau^{m-2p_{11}} \, d\tau \geq C(T-t).$$

Hence

$$M(t) = u(0,t) \leq C(T-t)^{\frac{1}{m+1-2p_{11}}}.$$

Thus, v satisfies

$$\begin{cases} v_t = (v^n)_{xx} & (x,t) \in D_T, \\ -(v^n)_x(0,t) \leq C(T-t)^{\frac{p_{21}}{m+1-2p_{11}}} v^{p_{22}} & t \in (0,T). \end{cases}$$

Hence v is a subsolution of (19) with initial data v_0. As we have chosen u_0 large, T is small and the solution of that problem with v_0 as initial data does not blow up. We conclude that v remains bounded up to time T. ∎

Proof of Theorem 1.3. Assume that (u,v) has finite time blow-up T and that v remains bounded up to time T. Therefore, u must blow-up at time T. As $v(x,t) \leq C$ for all $0 < t < T$, then u is a subsolution of (44). As u has finite time blow-up T, we must have $2p_{11} > m+1$, since solutions of (44) have blow-up only in this case (see [9], [19]).

Next we prove that $2p_{21} < 2p_{11} - (m+1)$. We start by showing that,

$$u(0,t) \geq c(T-t)^{-\gamma_1}, \qquad \gamma_1 = \frac{1}{2p_{11} - (m+1)} > 0. \tag{49}$$

To prove (49) we define φ_M as in the proof of Theorem 1.2 and obtain that it is a solution of

$$\begin{cases} (\varphi_M)_s = (\varphi_M^m)_{yy}, \\ -(\varphi_M^m)_y(0,s) \leq C\varphi^{p_{11}}(0,s). \end{cases} \tag{50}$$

The function is well defined for every $(y,s) \in A = \{y > 0,\ s_* < s < 0\}$, if M is large enough.

We claim that there exists a positive constant such that for every M large enough it holds

$$(\varphi_M)_s(0,0) \leq C. \tag{51}$$

Assume that the claim were false. Arguing again as in the proof of Theorem 1.2 we get that φ_{M_j} is a solution of a uniformly parabolic equation in a fixed neighbourhood $\overline{U}$ of the origin. Since it is uniformly bounded, we get that

$$\|\varphi_{M_j}\|_{C^{2+\alpha,1+\alpha/2}} \leq C \qquad \text{in } \overline{U}.$$

This implies (51).

To obtain the blow-up rate we write (51) in terms of M to get

$$M^{m-2p_{11}}(t)M'(t) \leq C.$$

Performing an integration from t to T we finally arrive to

$$M(t) = u(0,t) \geq C(T-t)^{-\frac{1}{2p_{11}-(m+1)}},$$

that is, (49).

Now we turn our attention to v, which is assumed to be bounded and satisfies

$$\begin{cases} v_t = (v^n)_{xx}, & (x,t) \in D_T, \\ -(v^n)_x(0,t) = u^{p_{21}}v^{p_{22}}(0,t) \geq C(T-t)^{-\gamma_1 p_{21}}v^{p_{22}}(0,t) & t \in (0,T), \\ v(x,0) = v_0(x) & x > 0. \end{cases} \tag{52}$$

By Theorem 1.6, we have $\gamma_1 p_{21} < 1/2$, which is equivalent to $2p_{21} < 2p_{11} - (m+1)$. ∎

Now we prove our continuity result, Theorem 1.4.

Proof of Theorem 1.4. Let (u_1, v_1) be a solution such that u_1 blows up at time T and v_1 remains bounded up to time T, say $v_1 \leq C$.

As u_1 blows up at time T it becomes large at time $T - \varepsilon$. We can find a neighbourhood of $(u_1(x,0), v_1(x,0))$ in L^∞ such that, if (u_2, v_2) is an increasing solution with initial data in such neighbourhood it satisfies that u_1 becomes large at time $T-\varepsilon$ and $v_2 \leq C+1$ up to $T-\varepsilon$ (this follows by continuity with respect to the initial conditions since up to $T-\varepsilon$ we are dealing with bounded solutions).

Let us restrict ourselves to increasing solutions and assume that both (11) and (12) hold. From the proof of the previous theorems we know that, given $v_0 \leq C+1$, if u_0 is large then u blows up and v remains bounded. From this fact, taking as initial data $(u_2(\cdot, T-\varepsilon), v_2(\cdot, T-\varepsilon))$, we can conclude that u_2 blows up and v_2 remains bounded. ∎

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