A NONLINEAR EIGENVALUE PROBLEM WITH INDEFINITE WEIGHTS RELATED TO THE SOBOLEV TRACE EMBEDDING

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ABSTRACT. In this paper we study the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^p_V(\partial\Omega)$, where V is an indefinite weight. This embedding leads to a nonlinear eigenvalue problem where the eigenvalue appears at the (nonlinear) boundary condition. We prove that there exists a sequence of variational eigenvalues $\lambda_k \nearrow +\infty$ and then show that the first eigenvalue is isolated, simple and monotone with respect to the weight. Then we prove a nonexistence result related to the first eigenvalue and we end this article with the study of the second eigenvalue proving that it coincides with the second variational eigenvalue.

1. Introduction.

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $V : \partial\Omega \to \mathbb{R}$ an indefinite weight. In this paper we consider the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^p_V(\partial\Omega)$, where

$$L^p_V(\partial\Omega) = \{u : \partial\Omega \to \mathbb{R};\ \int_{\partial\Omega} |u|^p\, V(x)\, d\sigma < +\infty\}.$$

We only require mild integrability hypotheses on the weight $V(x)$. More precisely, we assume

$$V^+ \not\equiv 0 \text{ on } \partial\Omega \quad \text{and} \quad V \in L^s(\partial\Omega), \tag{1.1}$$

where $s > (N-1)/(p-1)$ if $1 < p \le N$ and $s \ge 1$ if $p > N$.

Under these hypotheses on the weight V, this embedding is compact and therefore there exists a constant S_p such that the following inequality holds,

$$S_p^{1/p} \|u\|_{L^p_V(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}, \quad \text{where} \quad \|u\|^p_{L^p_V(\partial\Omega)} = \int_{\partial\Omega} |u|^p\, V(x)\, d\sigma.$$

By the compactness of the embedding, we can prove (see Theorem 1.1) that there exists functions, usually called *extremals*, where the constant S_p is attained. In fact, the extremals are solutions of

$$\begin{cases} \Delta_p u = |u|^{p-2}u, & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda V(x)|u|^{p-2}u, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and $\frac{\partial}{\partial\nu}$ is the outer unit normal derivative.

Observe that we are dealing with a nonlinear eigenvalue problem. In the case $p = 2$, this eigenvalue problem becomes linear and it is known as the *Steklov* problem, [4]. Our main concern here is the study of eigenvalues for problem (1.2).

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The study of the eigenvalue problem $-\Delta_p u = \lambda |u|^{p-2}u$ complemented with Dirichlet boundary conditions have received considerable attention in recent years. See for example [1, 2, 3, 5, 6, 9]. However, problem (1.2) is less covered in the literature. With $V \equiv 1$, problem (1.2) has been studied in [8] and in [10]. In those papers it is proved that there exists an unbounded sequence of eigenvalues and that the first eigenvalue is isolated and simple. First, we extend these results to our case, allowing a general indefinite weight. We prove,

Theorem 1.1. *Let $V(x)$ satisfy* (1.1)*, then there exists a sequence of eigenvalues λ_k of* (1.2) *such that $\lambda_k \to +\infty$ as $k \to +\infty$.*

The proof of this Theorem relies on the Ljusternik-Schnirelman critical point theory on C^1 manifolds using the *genus*, γ. We find the following variational characterization of a sequence of eigenvalues

$$\frac{1}{\lambda_k} = \sup_{C \in C_k} \min_{u \in C} \frac{\|u\|^p_{L^p_V(\partial\Omega)}}{\|u\|^p_{W^{1,p}(\Omega)}},$$

where $C_k = \{C \subset W^{1,p}(\Omega);\ C \text{ is compact, symmetric and } \gamma(C) \geq k\}$.

Regarding the first eigenvalue λ_1, following ideas from [5] and [10], we prove

Theorem 1.2. *The first eigenvalue of* (1.2) *is simple and isolated. Moreover, any associated eigenfunction does not change sign in Ω.*

The eigenfunctions associated to λ_1 are in fact the extremals for the embedding $W^{1,p}(\Omega) \hookrightarrow L^p_V(\partial\Omega)$. Hence our result says that the extremal is unique up to a multiplicative constant.

We observe that any eigenfunction associated to an eigenvalue $\lambda \neq \lambda_1$ changes sign in $\partial\Omega$. Also the number of nodal domains is finite. See section 3.

Moreover, we prove that the first eigenvalue is monotone with respect to the weight.

Theorem 1.3. *Let V_1, V_2 be two weight functions satisfying hypotheses* (1.1)*. If $V_1 \leq V_2$ then $\lambda_1(V_1) \geq \lambda_1(V_2)$.*

Related to λ_1 we have a nonexistence result. In fact, if we consider the equation

$$\begin{cases} \Delta_p u = |u|^{p-2}u - f(x), & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda_1 V(x)|u|^{p-2}u + g(x), & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where λ_1 is the principal eigenvalue and $f, g \geq 0$ are bounded and locally smooth, we have

Theorem 1.4. *The problem* (1.3) *has a solution if and only if $f \equiv 0$ on Ω and $g \equiv 0$ on $\partial\Omega$. In this case, $u = ku_1$ where u_1 is an eigenfunction associated to λ_1.*

Since λ_1 is isolated in the spectrum and there exists eigenvalues different from λ_1, it make sense to define the second eigenvalue of (1.2) as

$$\bar{\lambda}_2 := \inf\{\lambda \in \mathbb{R} \ :\ \lambda \text{ is an eigenvalue and } \lambda > \lambda_1\}.$$

Concerning the second eigenvalue, we have the following result

Theorem 1.5. *The eigenvalue λ_2 found in Theorem 1.1 coincides with $\bar{\lambda}_2$. In particular, $\bar{\lambda}_2$ is an eigenvalue for (1.2). Moreover, it holds the following variational characterization of $\bar{\lambda}_2 = \lambda_2$,*

$$\bar{\lambda}_2 = \lambda_2 = \inf_{u \in A} \left\{ \int_\Omega |\nabla u|^p + |u|^p \, dx \right\},$$

where $A = \{u \in W^{1,p}(\Omega);\ \|u\|_{L^p_V(\partial\Omega)} = 1$ and $|\partial\Omega^\pm| \geq c\}$, if $s > 1$ or $1 < p \leq N$ and $A = \{u \in W^{1,p}(\Omega);\ \|u\|_{L^p_V(\partial\Omega)} = 1$ and $\int_{\partial\Omega^\pm} V(x)\, d\sigma \geq c\}$, if $s = 1$ with $p > N$. Here $\partial\Omega^+ = \partial\Omega \cap \{u > 0\}$ and $\partial\Omega^- = \partial\Omega \cap \{u < 0\}$.

We want to remark that this last result was not known to hold in the case $V \equiv 1$.

The rest of the paper is organized as follows. In §2, we deal with the existence of a sequence of eigenvalues and prove Theorem 1.1. Next, in §3, we study the first eigenvalue and prove Theorems 1.2, 1.3 and the nonexistence result, Theorem 1.4. Finally, in §4 we prove Theorem 1.5.

2. Existence of $\{\lambda_k\}$

The proof is a rather straightforward adaptation of Theorem 1.3 in [8] where problem (1.2) with $V \equiv 1$ is considered, so we only make a sketch in order to make the paper self contained.

We introduce a topological tool, the *genus*. Given a Banach Space X, we consider the class

$$\Sigma = \{A \subset X : \ A \text{ is closed, } A = -A\}.$$

Over this class we define the genus, $\gamma : \Sigma \to \mathbb{N} \cup \{\infty\}$, as

$$\gamma(A) = \min\{k \in \mathbb{N} : \ \text{ there exists } \varphi \in C(A, \mathbb{R}^k - \{0\}),\ \varphi(x) = -\varphi(-x)\}.$$

For the properties of the genus and some applications we refer to [11].

Let us consider $M = \{u \in W^{1,p}(\Omega) \ : \ \|u\|^p_{W^{1,p}(\Omega)} = p\}$ and

$$\varphi(u) = \frac{1}{p} \int_{\partial\Omega} |u|^p \, V(x)\, d\sigma.$$

We are looking for critical points of φ restricted to the manifold M using a minimax technique. First we observe that φ satisfies the Palais-Smale condition on M.

Lemma 2.1. *Let $\beta > 0$ and $(u_j) \subset M$ be a Palais-Smale sequence on M above level β. Then there exists a subsequence that converges strongly in $W^{1,p}(\Omega)$.*

Proof. See [8]. □

Now we seek for critical values of φ.

Theorem 2.1. *Let $C_k = \{C \subset M \ : \ C$ is compact, symmetric and $\gamma(C) \leq k\}$ and let*

$$\beta_k = \sup_{C \in C_k} \min_{u \in C} \varphi(u). \tag{2.1}$$

Then $\beta_k > 0$ and there exists $u_k \in M$ such that $\varphi(u_k) = \beta_k$ and u_k is a weak solution of (1.2) with $\lambda_k = 1/\beta_k$. Moreover $\lim_k \beta_k = 0$, hence $\lim_k \lambda_k = +\infty$.

Proof. First, let us see that $\beta_k > 0$. It is immediate that $\gamma(M) = +\infty$, hence β_k is well defined in the sense that for every k, $C_k \neq \emptyset$. As we can choose a set $C \in C_k$ with the property $\int_{\partial\Omega} |u|^p V(x)\, d\sigma \neq 0$ if $u \in C$, we conclude that $\beta_k = \sup_{C\in C_k} \min_{u\in C} \varphi(u) > 0$. Now, for a fixed k let us prove the existence of the solution u_k. By a standard deformation argument we can assume that there exists a sequence $(u_j) \in M$ such that $\varphi(u_j) \to \beta_k$ and $Du_j \to 0$, see [8] for the details. Now, from Lemma 2.1 we can extract a converging subsequence $u_j \to u_k$ that gives us the desired solution that must verify, by the continuity of φ, $\varphi(u_k) = \beta_k$.

Let us see that $\lim_k \beta_k = 0$. Let E_j be a sequence of subspaces of $W^{1,p}(\Omega)$, such that $E_i \subset E_{i+1}$, $\overline{\cup E_i} = W^{1,p}(\Omega)$ and $\dim(E_i) = i$. Let E_i^c a topological complementary of E_i. Let

$$\tilde{\beta}_k = \sup_{C\in C_k} \min_{u\in C\cap E_{k-1}^c} \varphi(u).$$

$\tilde{\beta}_k$ is well defined and $\tilde{\beta}_k \geq \beta_k > 0$. Let us prove that $\lim_k \tilde{\beta}_k = 0$. Assume, by contradiction, that there exists a constant $\kappa > 0$ such that $\tilde{\beta}_k > \kappa > 0$ for all k. Then for every k there exists C_k such that

$$\tilde{\beta}_k > \min_{u\in C_k\cap E_{k-1}^c} \varphi(u) > \kappa.$$

Hence there exists $u_k \in C_k \cap E_{k-1}^c$ with $\tilde{\beta}_k > \varphi(u_k) > \kappa$. As M is bounded, we can assume, taking a subsequence if necessary, that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $u_k \to u$ strongly in $L^p(\partial\Omega)$. Hence $\varphi(u) \geq \kappa > 0$ but this is a contradiction with the fact that $u \equiv 0$ because $u_k \in E_{k-1}^c$. $\square$

3. Simplicity, isolation and monotonicity of λ_1

In this section we prove Theorems 1.2, 1.3 and 1.4. First we deal with Theorem 1.2 and we divide the proof in a series of lemmas in order to clarify the exposition. Then we deal with Theorem 1.3 and we end this section with the proof of the nonexistence result, Theorem 1.4.

Observe that solutions of (1.2), by a well known fact, belong to $C^{1,\alpha}_{\rm loc}(\Omega)$ (see [13], [7], etc.) but, as far as we know, with V under the hypotheses (1.1) this regularity is not known to hold up to the boundary.

First we prove that eigenfunctions associated to λ_1 must have definite sign.

Lemma 3.1. *Eigenfunctions associated to λ_1 are either positive or negative in Ω. Moreover if $u \in C^{1,\alpha}(\overline{\Omega})$ then u has definite sign in $\overline{\Omega}$.*

Proof. Let u be an eigenfunction associated to λ_1. Since $\|u\|_{W^{1,p}(\Omega)} = \|\,|u|\,\|_{W^{1,p}(\Omega)}$ and $\|u\|_{L^p_V(\partial\Omega)} = \|\,|u|\,\|_{L^p_V(\partial\Omega)}$, from the variational characterization of λ_1 given by (2.1), it follows that $|u|$ is also an eigenfunction associated to λ_1. By the strong maximum principle, see [14], or using Harnack inequality, see [12], it follows that $|u| > 0$ in Ω, therefore either $u > 0$ or $u < 0$ in Ω and so $u \geq 0$ or $u \leq 0$ in $\overline{\Omega}$.

If $u \in C^{1,\alpha}(\overline{\Omega})$, assume that there exists $x_0 \in \partial\Omega$ such that $|u(x_0)| = 0$. By Hopf's Lemma, [14], we have that $\frac{\partial |u|}{\partial \nu}(x_0) < 0$, but the boundary condition impose $\frac{\partial |u|}{\partial \nu}(x_0) = 0$, a contradiction. So $|u| > 0$ in $\overline{\Omega}$ and the result follows. $\square$

For the proof of the simplicity of λ_1 we use the following "Piccone's identity" proved in [1].

Lemma 3.2. *([1], Theorem 1.1) Let $v > 0$, $u \geq 0$ be two continuous functions in Ω differentiable a.e. Denote*

$$L(u,v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \nabla u,$$
$$R(u,v) = |\nabla u|^p - |\nabla v|^{p-2}\nabla\left(\frac{u^p}{v^{p-1}}\right)\nabla v.$$

Then **(i)** *$L(u,v) = R(u,v)$,* **(ii)** *$L(u,v) \geq 0$ a.e. and* **(iii)** *$L(u,v) = 0$ a.e. in Ω if and only if $u = kv$ for some $k \in \mathbb{R}$.*

Now, we define a *nodal domain* $\mathcal{N}$ of a function u as the closure of a connected component of $\Omega \setminus \{u = 0\}$. In the next result we give an estimate on the measure of $\mathcal{N} \cap \partial\Omega$ for an eigenfunction u.

Proposition 3.1. *Any eigenfunction u associated to a positive eigenvalue $0 < \lambda \neq \lambda_1$ changes sign on the boundary. Moreover, if $\mathcal{N}$ is a nodal domain of u then*

$$|\mathcal{N} \cap \partial\Omega| \geq (S_{p^*}^{-1}\lambda\|V\|_{L^s(\partial\Omega)})^{-\gamma}, \tag{3.2}$$

where $\gamma = \frac{s(N-1)}{sp-N}$ if $1 < p \leq N$ and $\gamma = 2s'$ if $p > N$, $s > 1$. If $p > N$ and $s = 1$ we get

$$\int_{\mathcal{N}\cap\partial\Omega} |V(x)|\, d\sigma \geq \frac{S_\infty}{\lambda}. \tag{3.3}$$

In particular, if $\partial\Omega^+ = \{x \in \partial\Omega \ :\ u(x) > 0\}$ and $\partial\Omega^- = \{x \in \partial\Omega \ :\ u(x) < 0\}$ then

$$|\partial\Omega^+| \geq (S_{p^*}^{-1}\lambda\|V\|_{L^s(\partial\Omega)})^{-\gamma}, \qquad |\partial\Omega^-| \geq (S_{p^*}^{-1}\lambda\|V\|_{L^s(\partial\Omega)})^{-\gamma}, \tag{3.4}$$

or

$$\int_{\partial\Omega^+} |V(x)|\, d\sigma \geq \frac{S_\infty}{\lambda}, \qquad \int_{\partial\Omega^-} |V(x)|\, d\sigma \geq \frac{S_\infty}{\lambda}. \tag{3.5}$$

Here S_q is the best constant in the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ and $|A|$ denotes the $N-1$ dimensional measure of a subset $A \subset \partial\Omega$.

Proof. Assume by contradiction that $u \geq 0$ (if $u \leq 0$ the argument is analogous). Arguing as in Lemma 3.1, it follows that $u > 0$ in Ω. Let $\varphi > 0$ be an eigenfunction associated to λ_1 and $\varepsilon > 0$. We apply Piccone's identity to the pair $\varphi, u+\varepsilon$. We have

$$\begin{aligned} 0 &\leq \int_\Omega L(\varphi, u+\varepsilon)\, dx = \int_\Omega R(\varphi, u+\varepsilon)\, dx \\ &\leq \lambda_1 \int_{\partial\Omega} \varphi^p\, V(x)\, d\sigma - \int_\Omega \varphi^p\, dx - \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\left(\frac{\varphi^p}{(u+\varepsilon)^{p-1}}\right) dx. \end{aligned} \tag{3.6}$$

As $\frac{\varphi^p}{(u+\varepsilon)^{p-1}} \in W^{1,p}(\Omega)$, it is admissible in the weak formulation of u. Then from (3.6) it follows that

$$0 \leq \int_{\partial\Omega} (\lambda_1 - \lambda\frac{u^{p-1}}{(u+\varepsilon)^{p-1}})\varphi^p\, V(x)\, d\sigma.$$

Letting $\varepsilon \to 0$ we get

$$0 \leq \int_{\partial\Omega} (\lambda_1 - \lambda)\varphi^p\, V(x)\, d\sigma,$$

which is impossible, as $\lambda > \lambda_1$ and $\int_{\partial\Omega} \varphi^p \, V(x)\, d\sigma = \|\varphi\|^p_{W^{1,p}(\Omega)}/\lambda_1 > 0$. Therefore, u changes sign.

For the second part, in the case $1 < p < N$, we consider $w(x) = u(x)$ if $x \in \mathcal{N}$ and 0 otherwise, then $w \in W^{1,p}(\Omega)$ and if we use w in the weak formulation of u, we get

$$\begin{aligned}\int_{\mathcal{N}} |\nabla u|^p + |u|^p \, dx &= \lambda \int_{\mathcal{N}\cap\partial\Omega} |u|^p \, V(x)\, d\sigma \leq \lambda \|V\|_{L^s(\partial\Omega)} \|u\|^p_{L^{s'p}(\mathcal{N}\cap\partial\Omega)} \\ &\leq \lambda \|V\|_{L^s(\partial\Omega)} \|u\|^p_{L^{p^*}(\mathcal{N}\cap\partial\Omega)} |\mathcal{N}\cap\partial\Omega|^{\frac{p^*-s'p}{s'p^*}}\end{aligned}$$

by Hölder inequality, where $p^* = \frac{p(N-1)}{N-p}$ is the critical exponent in the Sobolev trace imbedding Theorem. Now, by the Sobolev trace embedding Theorem, there exists a constant $S_{p^*} = S_{p^*}(N,p,\Omega)$ such that

$$S_{p^*} \|u\|^p_{L^{p^*}(\mathcal{N}\cap\partial\Omega)} = S_{p^*} \|w\|^p_{L^{p^*}(\partial\Omega)} \leq \int_{\Omega} |\nabla w|^p + |w|^p \, dx = \int_{\mathcal{N}} |\nabla u|^p + |u|^p \, dx.$$

Hence,

$$S_{p^*} \leq \lambda \|V\|_{L^s(\partial\Omega)} |\mathcal{N}\cap\partial\Omega|^{\frac{p^*-s'(p-1)}{s'p^*}}$$

and the proposition follows. The cases where $p \geq N$ and $s > 1$ can be handled in a similar fashion.

For the case $p > N$, $s = 1$, we proceed as before to obtain

$$\|u\|^p_{W^{1,p}(\mathcal{N})} \leq \lambda \|u\|^p_{L^\infty(\mathcal{N}\cap\partial\Omega)} \int_{\mathcal{N}\cap\partial\Omega} |V(x)| \, d\sigma.$$

Therefore (3.3) follows. The proof is complete. $\square$

As an easy consequence of Proposition 3.1 we get the following

Corollary 3.1. *Let (λ, u) be an eigenpair of (1.2) with $\lambda > \lambda_1$. Then the number of nodal domains of u is finite.*

Next, we make use of Piccone's identity to prove the simplicity of λ_1.

Proposition 3.2. *λ_1 is simple.*

Proof. We argue similarly as in Proposition 3.1. Let u, v be two eigenfunctions associated to λ_1. We can assume that u and v are both positive in Ω. We apply Piccone's identity to the pair u, $v + \varepsilon$ and obtain

$$\begin{aligned}0 &\leq \int_{\Omega} L(u, v+\varepsilon)\, dx = \int_{\Omega} R(u, v+\varepsilon)\, dx \\ &= \lambda_1 \int_{\partial\Omega} u^p \, V(x)\, d\sigma - \int_{\Omega} u^p \, dx - \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \left(\frac{u^p}{(v+\varepsilon)^{p-1}} \right) dx.\end{aligned}$$

Since the function $\frac{u^p}{(v+\varepsilon)^{p-1}} \in W^{1,p}(\Omega)$, it is admissible in the weak formulation of v. It follows, arguing as in Proposition 3.1, that

$$0 \leq \int_{\Omega} L(u, v+\varepsilon)\, dx \leq \lambda_1 \int_{\partial\Omega} u^p \Big(1 - \frac{v^{p-1}}{(v+\varepsilon)^{p-1}}\Big) V(x)\, d\sigma.$$

Letting $\varepsilon \to 0$, we obtain

$$\int_{\Omega} L(u,v)\, dx = 0,$$

but then, $L(u,v) = 0$ and by Theorem 3.2, there exists $k \in \mathbb{R}$ such that $u = kv$. $\square$

To end the proof of Theorem 1.2, we need a Lemma from [8].

Lemma 3.3. *([8], Lemma 2.1) Let $\phi \in W^{1,p}(\Omega)'$, where $W^{1,p}(\Omega)'$ denotes the dual space of $W^{1,p}(\Omega)$. Then there exists a unique weak solution $u \in W^{1,p}(\Omega)$ of $-\Delta_p u + |u|^{p-2}u = \phi$. Moreover, the operator $A_p : \phi \mapsto u$ is continuous.*

Now we can prove,

Proposition 3.3. *λ_1 is isolated, that is, there exists $\delta > 0$ such that there is no other eigenvalue of (1.2) in the interval $(\lambda_1, \lambda_1 + \delta)$.*

Proof. The result is a consequence of estimate (3.4) or (3.5). Suppose that the statement of the Proposition is false. Then there exists a sequence of eigenvalues $\lambda_n \searrow \lambda_1$. Let u_n be an eigenfunction associated to λ_n and we can assume that $\int_{\partial\Omega} |u_n|^p V(x)\, d\sigma = 1$. Now, by (1.2), it is easy to see that u_n is bounded in $W^{1,p}(\Omega)$, so there exists a subsequence (that we still denote u_n) and a function $u \in W^{1,p}(\Omega)$ such that $u_n \to u$ in $L^p(\partial\Omega)$ and weakly in $W^{1,p}(\Omega)$. Moreover, if we define ϕ_n, ϕ as

$$\langle \phi_n, v\rangle = \lambda_n \int_{\partial\Omega} |u_n|^{p-2} u_n v\, V(x)\, d\sigma, \quad \langle \phi, v\rangle = \lambda_1 \int_{\partial\Omega} |u|^{p-2} uv\, V(x)\, d\sigma,$$

we get that, by Hölder inequality, $\phi_n, \phi \in W^{1,p}(\Omega)'$ and $\phi_n \to \phi$ in $W^{1,p}(\Omega)'$. By the continuity of the operator A_p given by Lemma 3.3, we get that the sequence u_n converges strongly in $W^{1,p}(\Omega)$, therefore, passing to the limit in the weak formulation of u_n we deduce that u is an eigenfunction of (1.2) associated to λ_1. We can assume, by Lemma 3.1, that $u > 0$ (the case $u < 0$ is analogous). Then, from the fact that $u_n \to u$ in $L^p(\partial\Omega)$, it follows that $u_n \to u$ in measure, so $|\partial\Omega_n^-| \to 0$, but this contradicts either estimate (3.4) or estimate (3.5). □

Next we prove Theorem 1.3, that shows the monotonicity of the first eigenvalue with respect to the weight.

Proof of Theorem 1.3. Let u_1 be an eigenfunction associated to the first eigenvalue of the weight V_1. Then

$$\begin{aligned} \frac{1}{\lambda_1(V_1)} = \frac{\int_{\partial\Omega} |u_1|^p V_1(x)\, d\sigma}{\|u_1\|^p_{W^{1,p}(\Omega)}} &\le \frac{\int_{\partial\Omega} |u_1|^p V_2(x)\, d\sigma}{\|u_1\|^p_{W^{1,p}(\Omega)}} \\ &\le \sup_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\int_{\partial\Omega} |u|^p V_2(x)\, d\sigma}{\|u\|^p_{W^{1,p}(\Omega)}} = \frac{1}{\lambda_1(V_2)}, \end{aligned}$$

and the proof is finished. □

We end this section with the proof of Theorem 1.4, a nonexistence result for problem (1.3).

Proof of Theorem 1.4. The 'if' part is immediate. Let u be a solution of (1.3), therefore

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi + |u|^{p-2} u\phi\, dx - \lambda_1 \int_{\partial\Omega} |u|^{p-2} u\phi\, V(x)\, d\sigma = \int_\Omega f\phi\, dx + \int_{\partial\Omega} g\phi\, d\sigma.$$

If we choose $\phi = u^-$, we have

$$\int_\Omega |\nabla u^-|^p + |u^-|^p\, dx - \lambda_1 \int_{\partial\Omega} |u^-|^p\, V(x)\, d\sigma = -\int_\Omega f u^-\, dx - \int_{\partial\Omega} g u^-\, d\sigma.$$

This implies that either $u^- = 0$ or

$$\int_\Omega |\nabla u^-|^p + |u^-|^p \, dx \le \lambda_1 \int_{\partial\Omega} |u^-|^p \, V(x) \, d\sigma.$$

In the latter case, we have that u^- is a multiple of the principal eigenfunction, so $u^- > 0$ in Ω and so $u = -u^-$ and the proof is complete.

Suppose now that $u^- = 0$ then $u = u^+$ and then by the maximum principle (see [12], [14]), $u > 0$ in Ω. Applying now Piccone's identity to the pair $u_1, u+\varepsilon$ we get

$$\begin{aligned} 0 \le & \int_\Omega |\nabla u_1|^p + |u_1|^p \, dx - \lambda_1 \int_{\partial\Omega} |u_1|^p \, V(x) \, d\sigma \\ & - \int_\Omega f \frac{u_1^p}{(u+\varepsilon)^{p-1}} \, dx - \int_{\partial\Omega} g \frac{u_1^p}{(u+\varepsilon)^{p-1}} \, d\sigma \end{aligned}$$

which is a contradiction unless $f \equiv 0$ in Ω, $g \equiv 0$ on $\partial\Omega$. □

4. Variational characterization of the second eigenvalue

This section is concerned with the study of the second eigenvalue. Let us recall that, as λ_1 is isolated, it make sense to define

$$\bar\lambda_2 := \inf\{\lambda \in \mathbb{R} \ : \ \lambda \text{ is an eigenvalue of (1.2) and } \lambda > \lambda_1\}.$$

It can also be checked, arguing exactly as in the first part of the proof of Proposition 3.3 that $\bar\lambda_2$ is an eigenvalue of (1.2), i.e. the infimum is achieved.

The goal of this section is to show that this eigenvalue $\bar\lambda_2$ coincides with the second variational eigenvalue λ_2 found in Theorem 1.1. For this purpose, first we give a variational characterization of $\bar\lambda_2$.

Proof of Theorem 1.5. We deal with the case $s > 1$ or $1 < p \le N$. The arguments for the case $s = 1$ with $p > N$ are analogous and we leave the details to the reader. Let us call

$$\mu = \inf\left\{ \int_\Omega |\nabla u|^p + |u|^p \, dx \ : \ \|u\|^p_{L^p_V(\partial\Omega)} = 1 \text{ and } |\partial\Omega^\pm| \ge (S^{-1}\bar\lambda_2 \|V\|_{L^s(\partial\Omega)})^{-\gamma} \right\}.$$

It is easy to check that $\mu \le \bar\lambda_2$, because if we take u_2 an eigenfunction of (1.2) associated with $\bar\lambda_2$ such that $\int_{\partial\Omega} |u|^p \, V(x) \, d\sigma = 1$, by Proposition 3.1, we have that u_2 is admissible in the variational characterization of μ and the claim follows. Hence, the proof will follows if we show that $\mu \ge \lambda_2$. It is easy to check that

$$\frac{1}{\mu} = \sup\left\{ \int_{\partial\Omega} |u|^p \, V(x) \, d\sigma \ : \ \|u\|^p_{W^{1,p}(\Omega)} = 1, \ |\partial\Omega^\pm| \ge (S^{-1}\bar\lambda_2 \|V\|_{L^s(\partial\Omega)})^{-\gamma} \right\}.$$

Now, arguing as in Proposition 3.3, we see that the supremum is realized by a function $w \in W^{1,p}(\Omega)$, that is, $\|w\|_{W^{1,p}(\Omega)} = 1$, $|\partial\Omega^\pm| \ge (S^{-1}\bar\lambda_2 \|V\|_{L^s(\partial\Omega)})^{-\gamma}$ and $\mu^{-1} = \|w\|^p_{L^p_V(\partial\Omega)}$. As w^+ and w^- are not identically zero, if we consider the set

$$C = \operatorname{span}\{w^+, w^-\} \cap \{u \in W^{1,p}(\Omega) \ : \ \|u\|_{W^{1,p}(\Omega)} = 1\},$$

then $\gamma(C) = 2$. Hence, we obtain

$$\frac{1}{\lambda_2} \ge \inf_{u \in C} \int_{\partial\Omega} |u|^p \, V(x) \, d\sigma, \tag{4.7}$$

but, as w^+ and w^- have disjoint support, it follows that the infimum in (4.7) can be computed by minimizing the two variable function

$$G(a,b) := |a|^p \int_{\partial\Omega} |w^+|^p\, V(x)\, d\sigma + |b|^p \int_{\partial\Omega} |w^-|^p\, V(x)\, d\sigma$$

with the restriction

$$H(a,b) := |a|^p \|w^+\|_{W^{1,p}(\Omega)} + |b|^p \|w^-\|_{W^{1,p}(\Omega)} - 1 = 0.$$

Straightforward computations show that

$$\frac{1}{\lambda_2} \geq \frac{\int_{\partial\Omega} |w^+|^p\, V(x)\, d\sigma}{\|w^+\|_{W^{1,p}(\Omega)}}. \tag{4.8}$$

On the other hand, it can be checked that $u = w + tw^+$ is admissible in the variational characterization of μ for $t > -1$, then

$$Q(t) := \frac{\int_\Omega |\nabla(w + tw^+)|^p + |w + tw^+|^p\, dx}{\int_{\partial\Omega} |w + tw^+|^p\, V(x)\, d\sigma}$$

attains a minimum at $t = 0$. Therefore

$$\begin{aligned} 0 = Q'(0) =& p\frac{\int_\Omega |\nabla w|^{p-2}\nabla w \nabla w^+ + |w|^{p-2} w w^+\, dx}{\int_{\partial\Omega} |w|^p\, V(x)\, d\sigma} \\ & - \frac{\|w\|^p_{W^{1,p}(\Omega)}}{(\int_{\partial\Omega} |w|^p\, V(x)\, d\sigma)^2} \int_{\partial\Omega} |w|^{p-2} w w^+\, V(x)\, d\sigma, \end{aligned}$$

from where it follows that

$$\int_\Omega |\nabla w|^{p-2}\nabla w \nabla w^+ + |w|^{p-2} w w^+\, dx = \mu \int_{\partial\Omega} |w|^{p-2} w w^+\, V(x)\, d\sigma.$$

Therefore,

$$\mu = \frac{\|w^+\|_{W^{1,p}(\Omega)^p}}{\int_{\partial\Omega} |w^+|^p\, V(x)\, d\sigma}. \tag{4.9}$$

This last equation (4.9) together with (4.8) imply the desired result. □

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References

[1] W. Allegretto and Y-X. Huang. *A Piccone's identity for the p−Laplacian and applications.* Nonlinear Anal. TMA, **32 (7)** (1996), 143–175.

[2] A. Anane. *Simplicité et isolation de la première valeur propre du p−Laplacien avec poids.* C. R. Acad. Sci. Paris, **305 (I)** (1987), 725–728.

[3] A. Anane and N. Tsouli. *On the second eigenvalue of the p−Laplacian.* In "Nonlinear Partial Differential Equations" (Fès, 1994), 1–9, Pitman Res. Notes Math. Ser., 343, Longman, Harlow, 1996.

[4] I. Babuska and J. Osborn. Eigenvalue Problems. Handbook of Numer. Anal., Vol. II (1991). North-Holland.

[5] M. Cuesta. *Eigenvalue problems for the p−Laplacian with indefinite weights.* Electron. J. Differential Equations, 33, **2001** (2001), 1–9.

[6] M. Cuesta, D.G. de Figueiredo and J.-P. Gossez. *The beginning of the Fučik spectrum for the p−Laplacian.* J. Differential Equations, **159** (1999), 212–238.

[7] E. Dibenedetto. *$C^{1+\alpha}$ local regularity of weak solutions of degenerate quasilinear elliptic equations.* Nonlinear Anal. TMA, **7** (1983), 827–850.

[8] J. Fernández Bonder and J.D. Rossi. *Existence results for the $p-$Laplacian with nonlinear boundary conditions.* To appear in J. Math. Anal. Appl

[9] J. Garcia-Azorero and I. Peral. *Existence and non-uniqueness for the $p-$Laplacian: nonlinear eigenvalues.* Comm. Partial Differential Equations., **12** (1987), 1389–1430.

[10] S. Martinez and J.D. Rossi. *Isolation and simplicity for the first eigenvalue of the $p-$Laplacian with a nonlinear boundary condition.* Preprint.

[11] P. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations.* CBMS Regional Conf. Ser. in Math., no. 65, Amer. Math. Soc., Prov., R.I. (1986).

[12] J. Serrin. *Local behavior of solutions of quasilinear equations.* Acta Math., **111** (1962), 247–302.

[13] P. Tolksdorf. *Regularity for a more general class of quasilinear elliptic equations.* J. Differential Equations, **51** (1984), 126–150.

[14] J.L. Vazquez. *A strong maximum principle for some quasilinear elliptic equations.* Appl. Math. Optim., **12 (3)** (1984), 191–202.

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