Stability of the blow-up time and the blow-up set under perturbations

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Abstract

In this paper we prove a general result concerning continuity of the blow-up time and the blow-up set for an evolution problem under perturbations. This result is based on some convergence of the perturbations for times smaller than the blow-up time of the unperturbed problem together with uniform bounds on the blow-up rates of the perturbed problems.

We also present several examples. Among them we consider changing the spacial domain in which the heat equation with a power source takes place. We consider rather general perturbations of the domain and show the continuity of the blow-up time. Moreover, we deal with perturbations on the initial condition and on parameters in the equation. Finally, we also present some continuity results for the blow-up set.

1 Introduction

A remarkable and well known fact in nonlinear parabolic problems is that the solution may develop singularities in finite time, no matter how smooth the initial data are. In fact, for many differential equations or systems, the solutions become unbounded in finite time, a phenomena that is known as *blow-up*. In this work we are interested in studying how the blow-up behavior of an evolutionary problem is affected by perturbations of the problem. These perturbations may be of quite different nature. We may consider perturbations of the initial condition, of the coefficients of the equation, perturbations of the spacial domain, etc. We will present a very general continuity result together with some particular examples.

In an evolutionary problem, if the maximal solution is defined on a finite time interval, $[0, T)$ with $T < +\infty$, and

$$\lim_{t \nearrow T} \|u(\cdot, t)\|_{L^\infty} = +\infty,$$

we say that u blows up at time T. Typical examples where this happens are problems involving nonlinear reaction terms in the equation like the semilinear heat equation

$$u_t = \Delta u + |u|^{p-1}u, \qquad p > 1, \tag{1.1}$$

see [12, 22] and the references therein.

As we have mentioned above, our main interest here is to investigate the dependence of the blow-up time and the blow-up set (the set of points at which the solution becomes unbounded when approaching the blow-up time) with respect to perturbations of the problem. For the semilinear case (1.1), it is known that the blow-up time is continuous with respect to the initial data in L^∞ when $1 < p < p_S = (N+2)/(N-2)$, see [5, 16, 19, 21] if Ω is bounded (with Dirichlet boundary conditions) and [9] if $\Omega = \mathbb{R}^N$. Remark that the restriction on p being subcritical is not technical. Indeed if it does not hold, i.e., if $p \geq p_S$, then the blow-up time may be not even continuous as a function of the initial data, see [11]. This phenomenon is related to the possibility of having a nontrivial continuation after T, see [20]. Moreover, for subcritical p, in [16] it is proved that T is almost Lipschitz (up to a logarithmic factor). The one dimensional case was treated in [18] where it was shown that T is Lipschitz for some special initial data and some particular perturbations. Our aim in this work is to extend those continuity results and treat more general perturbations.

Assume that we are in the following general setting: we have $u = u(x,t)$ a particular solution of an evolutionary problem defined in Ω with finite blow-up time T and blow-up set $B(u)$, defined as

$$B(u) = \left\{ x \in \overline{\Omega} \, : \, \exists\, x_n \to x, \, t_n \nearrow T \text{ with } u(x_n, t_n) \to \infty \right\}.$$

Let also $\{u_h\}$ be a family of solutions associated to a family of evolution problems, that are perturbations of the original one, defined for $x \in \Omega_h$, with blow-up times T_h and blow-up sets $B(u_h)$. Observe that the evolution problems and the domains may be different for different values of h. We also extend u and u_h as zero outside their corresponding domains, Ω and Ω_h, if they are not the same, so we assume that all these functions are defined in a bigger set D.

Let us consider the following conditions:

(H1) For all $t_0 \in (0, T)$ there exists $h(t_0) > 0$ so that the solution u_h, for $0 < h < h(t_0)$, is defined at least up to time t_0 and

$$\liminf_{h \to 0} \|u_h\|_{L^\infty(\Omega_h \times (0,t_0))} \geq \|u\|_{L^\infty(\Omega \times (0,t_0))}.$$

Hypothesis **(H1)** holds, for instance, if we have

(H1*) For all $t_0 \in (0, T)$ there exists $h(t_0) > 0$ so that the solution u_h, for $0 < h < h(t_0)$, is defined at least up to time t_0 and

$$\lim_{h \to 0} \|u - u_h\|_{L^\infty(D \times (0,t_0))} = 0.$$

Both hypotheses, **(H1)** and **(H1*)** are related to some continuous dependence of the solutions before the blow-up time. From this property we will easily obtain that $\liminf_{h\to 0} T_h \geq T$. To obtain the other inequality we need to relate the size of the solution with the time that is left to reach the blow-up time. This relation can be expressed in the following hypothesis:

(H2) There exists a function $G : (0, \infty) \to (0, \infty)$ continuous and decreasing, verifying $G(0+) = +\infty$, such that for every $0 < t < T_h$,

$$\|u_h(\cdot, t)\|_{L^\infty(\Omega_h)} \leq G(T_h - t). \tag{1.2}$$

Observe that **(H2)** is equivalent to say that $T_h - t \leq G^{-1}(\|u_h(\cdot, t)\|_{L^\infty(\Omega_h)})$, which bounds the time left for explosion with the L^∞-norm of the solution. Another way to relate this two concepts can be accomplished by the use of an appropriate energy functional (related ideas are to be found in [21]). This is expressed in the following hypothesis,

(H2*) Assume there exist nonnegative functionals $V_h(u_h(\cdot, t))$ and $V(u(\cdot, t))$, such that $V(u(\cdot, t)) \to +\infty$ as $t \nearrow T$ and

$$\liminf_{h\to 0} V_h(u_h(\cdot, t)) \geq V(u(\cdot, t)), \quad \forall t < T, \tag{1.3}$$

and there also exists a function $G : (0, \infty) \to (0, \infty)$ continuous and decreasing, verifying $G(0+) = +\infty$, such that for every $0 < t < T_h$,

$$V_h(u_h(\cdot, t)) \leq G(T_h - t). \tag{1.4}$$

Observe that if we consider $V_h(u_h) = \|u_h\|_{L^\infty(\Omega_h)}$ we recover hypothesis **(H2)**.

For the behavior of the blow-up set we also need the following

(H3) There exists a function $H : (0, \infty) \to (0, \infty)$ continuous and decreasing, verifying $H(0+) = +\infty$, such that for all $x_h \in B(u_h)$ and $0 < t < T_h$,

$$|u_h(x_h, t)| \geq H(T_h - t). \tag{1.5}$$

With these hypotheses, our general continuity result reads as follows:

Theorem 1.1 *We have the following,*
i) If **(H1)** *and* **(H2)** *(or* **(H2*)***) hold then*

$$\lim_{h\to 0} T_h = T. \tag{1.6}$$

ii) If the convergence of the blow-up times given by (1.6) *holds and hypotheses* **(H1*)**, **(H3)** *also hold, then for every $\delta > 0$ there exists $h_0 > 0$ such that for $0 < h < h_0$, we have*

$$B(u_h) \subset B(u) + B_\delta(0) = \{x + y,\ x \in B(u),\ |y| < \delta\}, \tag{1.7}$$

that is, the blow-up set is uppersemicontinuous at $h = 0$.

We will apply this general result to deal with several types of perturbations in blow-up problems, like perturbations of the domain, on the initial condition or on the different terms of the equation, like the reaction or the diffusion.

To simplify the examples we will focus on solution to the most well known equation with blowing up phenomena, the semilinear heat equation,

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, & \Omega \times (0,T), \\ u = 0, & \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \Omega, \end{cases} \tag{1.8}$$

with a superlinear and subcritical exponent, that is, $1 < p < p_S$ (for $p \leq 1$ there is no blow-up and as we have mentioned above, for supercritical powers, continuity of T does not hold in general, see [11]).

Let us summarize our continuity results as follows (in the following statements u stands for a solution to (1.8), and as usual T is the blow-up time, $B(u)$ the blow-up set).

Theorem 1.2 (Perturbations of the initial condition). *Let u_h be the solution to* (1.8) *with initial condition $u_{0,h}$. Then, if $r > pN$ we have that*

$$\lim_{h\to 0} \|u_{0,h} - u_0\|_{L^r(\Omega)} = 0 \implies \lim_{h\to 0} T_h = T. \tag{1.9}$$

Theorem 1.3 (Perturbation of parameters in the equation). *Let us perturb the equation in* (1.8) *considering*

$$u_t = \Delta u + a_h(x)|u|^{p-1}u, \tag{1.10}$$

or

$$u_t = div(B_h(x)\nabla u) + |u|^{p-1}u, \tag{1.11}$$

in the same domain Ω and with the same initial condition u_0. If as $h \to 0$ it holds that $a_h \to 1$, in the case of (1.10)*, or $B_h(x) \to Id$, in the case of* (1.10)*, uniformly in $\overline{\Omega}$, then we have $T_h \to T$.*

Theorem 1.4 (Perturbations of the domain). *Let Ω, Ω_h be Lipschitz domains such that $\Omega, \Omega_h \subset B(0,R)$ for $R > 0$ large enough. Let u_h be the solution of* (1.8) *with Ω replaced by Ω_h, and with initial condition u_0^h satisfying $0 \leq u_0, u_0^h \leq M$ for some positive constant M and $\|u_0^h - u_0\|_{L^1(B(0,R))} \to 0$ as $h \to 0$. We consider the following general situation of domain perturbation:*

i) $\Omega_h \subset \{x \in \mathbb{R}^n : dist(x,\Omega) < h\}$

ii) if $\chi_h \in H_0^1(\Omega_h), \chi \in H_0^1(\Omega)$ are, respectively, the unique solution of $-\Delta\chi_h = 1$ in Ω_h and of $-\Delta\chi = 1$ in Ω, both with homogeneous Dirichlet boundary conditions, we have that $\|\chi_h - \chi\|_{L^2(B(0,R))} \to 0$ as $h \to 0$.

Then, we have $T_h \to T$ as $h \to 0$.

Remark 1.5 *Examples of domain perturbations satisfying i) and ii) above include: regular perturbations of a fixed domain; a domain where a small ball centered at $x_0 \in \Omega$ with radius h has been removed and others. See Subsection 3.4.*

Theorem 1.6 (Stability of the blow-up set). *Let Ω be an interval, $\Omega = (0,1)$, and let u_0 be a function with a unique maximum in $(0,1)$ (therefore, u blows up at a single point, $B(u) = \{x_0\}$). We have:*

1. *If u_h is the corresponding solution to the initial datum $u_{0,h}$, and $r > p$, then*

$$\lim_{h \to 0} \|u_{0,h} - u_0\|_{L^r(\Omega)} = 0 \implies B(u_h) = x_h \text{ (a single point), with } \lim_{h \to 0} x_h = x_0.$$

2. *If we perturb the domain considering $\Omega_h = (0, 1+h)$, $|h| < 1$, keeping the initial condition fixed, and u_h is the corresponding solution in Ω_h, then*

$$B(u_h) = x_h \text{ (a single point), with } \lim_{h \to 0} x_h = x_0.$$

Our general result could also be applied to deal with numerical approximations of blow-up problems. In fact, when one performs a numerical approximation of a parabolic equation one gets a discrete family u_h that approximates the continuous solution u. For numerical approximations of blow-up problems see, for example, [4, 15].

The rest of the paper is organized as follows: in Section 2 we prove Theorem 1.1 and discuss the necessity of the hypotheses. Next, in Section 3 we collect several examples in which continuity of blow-up takes place, i.e., we prove Theorems 1.2 to 1.6. Finally, in Section 4 we briefly comment on some possible extensions of this work.

2 General results on continuity of blow-up

In this section we prove Theorem 1.1. We divide the proof into two lemmas. Prior to this, let us make some comments on our hypotheses and state some properties that hold in our general setting.

1) Hypothesis **(H1*)** represents the continuity with respect to the parameter h of the flow of the evolutionary equation.

Quantitative versions of **(H1)** and **(H1*)** are respectively as follows,

(Q) There exist two functions $g(h) > 0$ and $f(h) > 0$, with $g, f \to 0$ as $h \to 0$ such that

$$\|u\|_{L^\infty(\Omega \times (0, T-g(h)))} - \|u_h\|_{L^\infty(\Omega_h \times (0, T-g(h)))} \leq f(h).$$

(Q*) There exist two functions $g(h) > 0$ and $f(h) > 0$, with $g, f \to 0$ as $h \to 0$ such that

$$\|u - u_h\|_{L^\infty(D \times (0, T-g(h)))} \leq f(h).$$

These functions will provide us with some explicit bounds on the difference of the blow-up times and the blow-up sets for u and u_h.

Properly speaking, **(Q)** is not a hypothesis different from **(H1)**, but a definition of the functions f and g. If **(H1)** holds, then there always exist two functions f and g satisfying **(Q)**. The same comment applies for **(Q*)** and **(H1*)**.

2) Hypothesis **(H2)** represents a uniform upper bound of the rate of explosion for all solutions u_h. **(H3)** is a uniform lower bound valid for every point in $B(u_h)$. We remark that, in general, **(H3)** is difficult to verify since it refers to *all* points in the blow-up set. In some cases, a lower bound for the maximum of the solution can be obtained from a comparison argument (see the final examples). Therefore it is most applicable in cases of single-point blow-up.

3) Now let us define the function

$$\psi(d) = \sup \{ u(x,t) \,:\, \mathrm{dist}(x, B(u)) \geq d,\ t \in [0,T) \} .$$

For this function ψ we have the following property,

(D) $\psi : (0,\infty) \to (0,\infty)$ is continuous, decreasing with $\psi(0+) = +\infty$, and such that for all $t \in (0,T)$

$$u(x,t) \leq \psi(\mathrm{dist}(x, B(u))).$$

This condition **(D)** is a bound for the solution u of the unperturbed problem far from its blow-up set.

We have the following result on the continuity of the blow-up time.

Lemma 2.1 *i) If* **(H1)** *and* **(H2)** *hold then*

$$\lim_{h\to 0} T_h = T.$$

Moreover, we have the estimate

$$-\tilde{g}(h) \leq T - T_h \leq g(h), \tag{2.1}$$

where $\tilde{g}(h) = G^{-1}(\|u(\cdot, T - g(h))\|_{L^\infty(\Omega)} - f(h))$*, and* f*,* g *and* G *are given in* **(H2)** *and* **(Q)**.
ii) If **(H1)** *and* **(H2*)** *hold then*

$$\lim_{h\to 0} T_h = T.$$

Proof. *i)* We first observe that from **(H1)** we have that $T_h \geq t_0$ for all $0 < h < h(t_0)$. This implies that $\liminf_{h\to 0} T_h \geq t_0$. Since this is obtained for all $t_0 < T$, we get

$$\liminf_{h\to 0} T_h \geq T.$$

On the other hand, if $\limsup_{h\to 0} T_h = \hat{T} > T$, then, there exists a sequence $h_n \to 0$ with $T_{h_n} \to \hat{T}$. Hence, if $0 < \tau < \hat{T} - T$, there exists n_0 such that for $n \geq n_0$ we have

$0 < \tau < T_{h_n} - t$ for each $0 < t < T$. From **(H2)** we have $\|u_{h_n}(\cdot,t)\|_{L^\infty(\Omega_{h_n})} \le G(T_{h_n} - t) \le G(\tau) < +\infty$, for all $0 < t < T$. This is in contradiction with **(H1)** and the fact that the solution u blows up at time T. This shows the continuity of the blow-up time.

Now, from **(Q)** we have that $u_h(\cdot,t)$ is well defined and finite at least for times $t \in (0, T - g(h))$. This means that $T_h \ge T - g(h)$, which implies that $T - T_h \le g(h)$.

On the other hand, if $T_h > T$ and if we denote by $t_0(h) = T - g(h) \in (0,T)$ we have $T_h - T \le T_h - t_0(h)$. From **(H2)** we get

$$T_h - t_0(h) \le G^{-1}(\|u_h(\cdot, t_0(h))\|_{L^\infty(\Omega_h)}).$$

But we also have $\|u_h(\cdot,t_0(h))\|_{L^\infty(\Omega_h)} \ge \|u(\cdot,t_0(h))\|_{L^\infty(\Omega)} - f(h)$. This implies that

$$T_h - t_0(h) \le G^{-1}(\|u(\cdot, t_0(h))\|_{L^\infty(\Omega)} - f(h))$$

from where the result follows.

ii) The proof in this case is very similar as the one provided in *i*). As in *i*), from **(H1)** we deduce that $\liminf_{h\to 0} T_h \ge T$. Moreover, if $\limsup_{h\to 0} T_h = \hat{T} > T$, then, there exists a sequence $h_n \to 0$ with $T_{h_n} \to \hat{T}$. Hence, if $0 < \tau < \hat{T} - T$, there exists n_0 such that for $n \ge n_0$ we have $0 < \tau < T_{h_n} - t$ for each $0 < t < T$. From **(H2*)** we have $V_{h_n}(u_{h_n}(\cdot,t)) \le G(T_{h_n} - t) \le G(\tau) < +\infty$, for all $0 < t < T$. This is in contradiction with (1.4) and the fact that $V(u(\cdot,t)) \to +\infty$ as $t \to T$. This shows the continuity of the blow-up time. □

Remark 2.2 *This lemma shows the first assertion of Theorem 1.1.*

Next, we present a simple example that shows that **(H2)** is necessary to get convergence of the blow-up times even if the deal with a simple ODE.

Example. Let

$$\begin{cases} u_t(t) = u^2(t), \\ u(0) = 1, \end{cases}$$

which has the explicit solution

$$u(t) = \frac{1}{1-t}, \tag{2.2}$$

with blow-up time $T = 1$. Let also u_h be the solution to the perturbed problem

$$\begin{cases} (u_h)_t(t) = f_h(u_h)(t), \\ u_h(0) = 1, \end{cases}$$

with

$$f_h(s) = \begin{cases} s^2 & s \le 1/h, \\ (s - 1/h)^{1+h} + (1/h)^2 & s > 1/h. \end{cases}$$

The sequence of reactions satisfy $f_h(s) \xrightarrow{h\to 0} f(s) = s^2$ in $s \in [0,\infty)$, not uniformly nor monotonically. Also, since $f_h(s) \sim s^{1+h}$ for s large, and $1+h > 1$, it is clear that u_h blows up at some finite time T_h. Let us estimate these blow-up times. To this end we observe that u_h coincides with u for times smaller than t_h, the first time at which $u(t_h) = 1/h$. This time t_h can be computed from the explicit formula (2.2) of the solution. We get $t_h = 1 - h$. Using the definition of f_h for $s > 1/h$ we get, after a simple integration

$$T_h - (1-h) = \int_{1/h}^{\infty} \frac{1}{(s-(1/h))^{1+h} + (1/h)^2}\, ds \equiv I(h).$$

Changing variables, we get

$$I(h) = h^{2h/(1+h)} \int_0^{\infty} \frac{1}{w^{1+h}+1}\, dw.$$

Now we observe that the last integral behaves like $1/h$ for h small, and therefore we conclude

$$\lim_{h\to 0} T_h = \infty \neq 1 = T.$$

In fact, the above blow-up times are given by the explicit formulae

$$T_h = \int_1^{\infty} \frac{ds}{f_h(s)}, \qquad T = \int_1^{\infty} \frac{ds}{f(s)}.$$

By Fatou's lemma we always have $\liminf_{h\to 0} T_h \geq T$. The condition to have convergence of the blow-up times, with perturbed reactions in the ODE setting, is obviously the convergence of the above integrals.

We conclude by noticing that in this example we have, from the fact that $u_h = u$ for every $0 < t < 1-h$, that **(H1*)** holds, clearly, **(H2)** does not hold.

We can also obtain the following result on the continuity of the blow-up set.

Lemma 2.3 *Assume that we have convergence of the blow-up times and that* **(H1*)** *and* **(H3)** *hold. Let f, g and ψ be given by* **(Q*)** *and* **(D)**. *Then*

$$dist(B(u_h), B(u)) = \sup_{x_h \in B(u_h)} dist(x_h, B(u)) \leq \theta(h)$$

where

$$\theta(h) = \begin{cases} \psi^{-1}(H(g(h)) - f(h)) & \text{if } T_h \leq T \\ \psi^{-1}(H(\tilde{g}(h) + g(h)) - f(h)) & \text{if } T_h > T \end{cases}$$

and $\tilde{g}(h)$ is defined in Lemma 2.1. In particular $\theta(h) \leq \psi^{-1}(H(\tilde{g}(h) + g(h)) - f(h)) \to 0$, which shows that the blow up set is uppersemicontinuous at $h = 0$.

Proof. Let $x_h \in B(u_h)$ and $t_0(h) = T - g(h)$. By **(H3)** and **(Q*)** we have

$$H(T_h - t_0) \leq |u_h(x_h, t_0)| \leq |u(x_h, t_0)| + \|u_h(\cdot, t_0) - u(\cdot, t_0)\|_{L^\infty(\Omega)} \leq |u(x_h, t_0)| + f(h).$$

Moreover, by **(D)** we have $u(x_h, t_0) \le \psi(\mathrm{dist}(x_h, B(u)))$, from where we get

$$H(T_h - t_0) \le \psi(\mathrm{dist}(x_h, B(u))) + f(h).$$

First, if $T_h \le T$, from Lemma 2.1 *ii*), we have that $T - T_h \le g(h)$, and therefore $0 \le T_h - t_0 \le g(h)$, which implies that $H(T_h - t_0) \ge H(g(h))$. In particular we get

$$\mathrm{dist}(x_h, B(u)) \le \psi^{-1}(H(g(h)) - f(h)).$$

On the contrary, if $T_h > T$, then

$$T_h - t_0 = T_h - T + g(h) \le G^{-1}(\|u(\cdot, T - g(h))\|_{L^\infty(\Omega)} - f(h)) + g(h) = \tilde{g}(h) + g(h),$$

where we have used again Lemma 2.1 *ii*). This implies that

$$H(T_h - t_0) \ge H(\tilde{g}(h) + g(h)),$$

and therefore

$$\mathrm{dist}(x_h, B(u)) \le \psi^{-1}(H(\tilde{g}(h) + g(h)) - f(h)).$$

This ends the proof of the lemma. □

Remark 2.4 *This lemma shows the second assertion of Theorem 1.1.*

The question of continuity of the blow-up set is in general a delicate matter. As we have already said, condition **(H3)** is quite difficult to be fulfilled. On the contrary, next example shows that condition **(H3)** is important for the convergence of the blow-up sets.

Example. Let us consider solutions to the porous medium equation with reaction,

$$u_t = \Delta u^m + u^{m-h}, \qquad x \in \mathbb{R}^N\,,\ t > 0$$

with a fixed $m > 1$ and u_0 nonnegative with compact support. If $h > 0$ is small ($m - h > 1$ is required to have blow-up) it is known, see for instance the book [22], that the solution blows up and the blow-up set is the whole space, while for $h = 0$ the blow-up set is a compact set. Therefore in this case we can not have

$$B(u_h) \subset B(u) + B(0, \delta), \tag{2.3}$$

for any $\delta > 0$.

Note that in this case **(H1*)** holds. On the other hand, as $u_h(x,t)$ is compactly supported for any $0 < t < T_h$, we can not have **(H3)**. Finally observe that (2.3) holds trivially in the case of perturbations with $h < 0$.

3 Examples of perturbations

In this section we consider several applications of the general results developed in the previous section and discuss the hypotheses involved in each case.

We will deal with continuity of the blow-up time under the following situations: perturbations of initial conditions, perturbations of parameters in the equation and perturbations of the domain.

Moreover, we finally present a result concerning stability of the blow-up set in one space dimension, when the problem is subject to perturbations of the initial condition or perturbations of the domain.

As we have mentioned in the introduction, we will present examples focusing on the problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, & \Omega \times (0,T), \\ u(x,t) = 0, & \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \Omega. \end{cases} \tag{3.1}$$

3.1 Preliminaries

Let us begin with some preliminary results.

Proposition 3.1 *Let u and v be the solutions to problem* (3.1) *with initial conditions u_0, $v_0 \in L^r(\Omega)$, respectively. Then if $r > pN$ we have*

$$\sup_{t\in[\tau_1,\tau_2]} \|u(\cdot,t) - v(\cdot,t)\|_{C^{2,\alpha}(\bar{\Omega})} \leq C(\tau_1,\tau_2,\delta)\|u_0 - v_0\|_{L^r(\Omega)}, \tag{3.2}$$

for every $0 < \tau_1 < \tau_2 < T(u_0)$, $\|u_0 - v_0\|_{L^r(\Omega)} \leq \delta$ and for some $\alpha > 0$.

Proof. The equation in (3.1) generates a continuous semiflow in $L^r(\Omega)$ if $r \geq (p-1)N/2$. As a matter of fact, using standard Sobolev embeddings we can see that if we denote by $f(u) = |u|^{p-1}u$ then

$$f : L^r(\Omega) \to L^{r/p}(\Omega) \hookrightarrow W^{-s,r}(\Omega)$$

with $s = (p-1)N/r$, and f is Lipschitz on bounded sets of $L^r(\Omega)$.

Using standard existence and uniqueness theories of solutions for this problem, we have that, for all $u_0 \in L^r(\Omega)$ there exists a unique solution defined in a maximal interval of existence

$$u \in C^0([0,T(u_0)), W^{2-s,r}(\Omega)) \cap C^1((0,T(u_0)), W^{2-s,r}(\Omega)).$$

Notice that $2 - s > 1$ provided $r > (p-1)N$.

Using now the variation of constants formula and the regularization of the linear semigroup, it can be shown the following: for all $\tau_2 < T(u_0)$, there exists some $\delta > 0$ such that, if $\|u_0 - v_0\|_{L^r(\Omega)} \leq \delta$, then the solution v starting at v_0 exists at least up to time τ_2

and, if $0 < \tau_1 < \tau_2 < T(u_0)$ is fixed, we have the existence of a constant $C(\tau_1, \tau_2, \delta)$ such that, for all $t \in [\tau_1, \tau_2]$ and all $v_0 \in L^r(\Omega)$ with $\|u_0 - v_0\|_{L^r(\Omega)} \leq \delta$, we have

$$\begin{aligned}
&\|u(\cdot,t) - v(\cdot,t)\|_{W^{2-s,r}(\Omega)} \leq C(\tau_1,\tau_2,\delta)\|u_0 - v_0\|_{L^r(\Omega)}\,, \\
&\|u_t(\cdot,t) - u_t(\cdot,t)\|_{W^{1,r}(\Omega)} \leq C(\tau_1,\tau_2,\delta)\|u_0 - v_0\|_{L^r(\Omega)}\,.
\end{aligned}$$

Choosing $r > pN$ we get that $W^{2-s,r}(\Omega) \hookrightarrow C^{1,\alpha}(\bar{\Omega})$ and $W^{1,r}(\Omega) \hookrightarrow C^{\alpha}(\bar{\Omega})$ for some $\alpha > 0$. Hence, we get

$$\|u(\cdot,t) - v(\cdot,t)\|_{C^{1,\alpha}(\bar{\Omega})} \leq C(\tau_1,\tau_2,\delta)\|u_0 - v_0\|_{L^r(\Omega)} \tag{3.3}$$

and

$$\|u_t(\cdot,t) - v_t(\cdot,t)\|_{C^{\alpha}(\bar{\Omega})} \leq C(\tau_1,\tau_2,\delta)\|u_0 - v_0\|_{L^r(\Omega)} \tag{3.4}$$

for all $t \in [\tau_1, \tau_2]$ and all $v_0 \in L^r(\Omega)$ with $\|u_0 - v_0\|_{L^r(\Omega)} \leq \delta$.

Note that the condition $r > pN$ allows to get also the previous requirements on r fulfilled. On the other hand, these regularity results are not optimal and they can be improved with a bootstrap argument.

Let us consider now $t \in [\tau_1, \tau_2]$ fixed and denote by $U(x) = u(x,t)$ and $V(x) = v(x,t)$. We have in this way that U and V are solutions of the elliptic problems

$$\begin{cases} -\Delta U = G(x), & x \in \Omega, \\ U = 0, & x \in \partial\Omega, \end{cases} \tag{3.5}$$

and

$$\begin{cases} -\Delta V = F(x), & x \in \Omega, \\ V = 0, & x \in \partial\Omega, \end{cases} \tag{3.6}$$

where $G(x) = |u(x,t)|^{p-1}u(x,t) - u_t(x,t)$ and $F(x) = |v(x,t)|^{p-1}v(x,t) - v_t(x,t)$.

Using (3.3) and (3.4) we get that $\|F - G\|_{C^{\alpha}(\bar{\Omega})} \leq C\|u_0 - v_0\|_{L^r(\Omega)}$ for some constant $C = C(\tau_1, \tau_2, \delta)$. Applying now Schauder estimates to the function $U(x) - V(x)$ we get the desired estimate. □

An immediate consequence of the above result is the following corollary.

Corollary 3.2 *If for some time t_0 the solution $u(x,t_0)$ has a unique maximum at $x = x_0$ and $\Delta u(x_0,t_0) < 0$, then for every v_0 with $\|u_0 - v_0\|_{L^r(\Omega)} \leq \delta$ it holds that $v(x,t_0)$ has a unique maximum at some point $\widetilde{x}_0$ which is close to x_0 when δ is small enough.*

For instance, this result can be used to obtain single point blow-up in one dimension for the approximations (in L^r-norm) of a solution with a unique maximum (that has single point blow-up thanks to the results in [10], see also [7]).

Following ideas from [21], we now relate the time left for a solution to get blow-up with the size of the solution at each time. This relation involves the energy functional associated to the problem,

$$\Phi(u(\cdot,t)) = \frac{1}{2}\int_{\Omega} |\nabla u|^2\, dx - \frac{1}{p+1}\int_{\Omega} |u|^{p+1}\, dx\,, \tag{3.7}$$

since we know that this energy is nonincreasing along the evolution orbits and tends to $-\infty$ as $t \to T$. We refer to [14] and [21] for a proof of this fact, but we include some details here for the sake of completeness.

Lemma 3.3 *Let u be a blowing up solution of problem* (3.1)*. Then, $\Phi(u(\cdot,t)) \to -\infty$ as $t \to T$, and moreover we have the estimate*

$$T - t \le C\Big(- \Phi(u(\cdot,t))\Big)^{-\frac{p-1}{p+1}} \qquad \textit{for } t_0 < t < T, \tag{3.8}$$

where t_0 is close to T and $C = C(p, |\Omega|)$. Equivalently, if $V(u(\cdot,t)) = -\Phi(u(\cdot,t))$ and $G(t) = (C/t)^{\frac{p+1}{p-1}}$ we have

$$V(u(\cdot,t)) \le G(T-t) \qquad \textit{for } t_0 < t < T. \tag{3.9}$$

Proof. We first take a sequence $t_n \to T$ such that

$$\|u\|_{L^\infty(\Omega\times[0,t_n])} \le \|u(\cdot,t_n)\|_{L^\infty(\Omega)}\,,$$

and define $\lambda_n = \|u(\cdot,t_n)\|_{L^\infty(\Omega)}$. In this way the sequence λ_n is nondecreasing. Also, there exists a sequence $x_n \in \Omega$ such that

$$\frac{\lambda_n}{2} \le u(x_n,t_n) \le \lambda_n\,. \tag{3.10}$$

Now we define the rescaled function

$$\phi_n(y,s) = \frac{1}{\lambda_n} u(x_n + a_n y, t_n + b_n s)\,,$$

whit $a_n = \lambda_n^{(1-p)/2}$ and $b_n = \lambda_n^{1-p}$. Since $p > 1$, we have that both a_n and b_n go to zero as $n \to \infty$. The function ϕ_n so defined satisfies the equation

$$(\phi_n)_s = \Delta\phi_n + \phi_n^p\,, \qquad \text{for } (y,s) \in \Omega_n \times (-\frac{t_n}{b_n}, 0)\,,$$

where $\Omega_n = \{y \in \mathbb{R}^N : x_n + a_n y \in \Omega\}$. Observe that Ω_n expand to cover the whole $\mathbb{R}^N$ as $n \to \infty$. Moreover, $0 \le \phi_n \le 1$. Then, by standard regularity theory, we have that $\phi_n(y,s) \to \phi(y,s)$ uniformly in compact sets of $\mathbb{R}^N \times (-\infty,0)$, where the function ϕ satisfies the equation

$$\phi_s = \Delta\phi + \phi^p\,, \qquad \text{in } \mathbb{R}^N \times (-\infty, 0)\,. \tag{3.11}$$

On the other hand,

$$\begin{aligned}
\int_{s_1}^{s_2}\int_{\Omega_n} |(\phi_n)_t|^2\,dr\,ds &= \frac{b^2}{\lambda_n}\int_{s_1}^{s_2}\int_{\Omega_n} |u_t(x_n + ar, t_n + bs)|^2\,dr\,ds \\
&\le \lambda^{\frac{(N-2)p-(N+2)}{2}} \int_0^{t_n}\int_\Omega |u_t(x,t)|^2\,dx\,dt \\
&= \lambda^{\frac{(N-2)p-(N+2)}{2}} \left(\Phi(u(\cdot,0)) - \Phi(u(\cdot,t_n))\right).
\end{aligned}$$

Notice that the parameter p is subcritical, which implies that $(N-2)p-(N+2)<0$. This implies that, if $\Phi(u(\cdot,t))$ is bounded, then

$$\int_{s_1}^{s_2}\int_{\Omega_n} |(\phi_n)_t|^2\,dr\,ds \to 0$$

for every $0<s_1<s_2<\infty$ and, therefore, the limit function ϕ does not depend on s. Moreover, ϕ is nonnegative and nontrivial. Indeed, if $x_n\to\bar{x}$, we have that $\phi(\bar{x})\geq 1/2$ by (3.10). We conclude with a contradiction since it is well know that for $1<p<p_S$ the only non-negative stationary solution of equation (3.11) is given by the trivial solution, see [13]. This contradiction proves that the energy functional $\Phi(u(\cdot,t))$ must blow up at time $t=T$.

In order to obtain the required estimate we observe that, since $t\to\Phi(u(\cdot,t))$ is non-increasing,

$$\frac{d\Phi(u(\cdot,t))}{dt}=-\int_\Omega |u_t(x,t)|^2\,dx\leq 0\,,$$

we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx = & -\int_\Omega |\nabla u|^2\,dx+\int_\Omega u^{p+1}\,dx=-2\Phi(u(\cdot,t))+\frac{p-1}{p+1}\int_\Omega u^{p+1}\,dx\\ \geq & \ 2|\Phi(u(\cdot,t_0))|+C_1\Big(\int_\Omega u^2\,dx\Big)^{\frac{p+1}{2}},\end{aligned}$$

where $t_0<T$ is close enough to T so that $\Phi(u(\cdot,t_0))<0$ and $C_1=\frac{p-1}{p+1}|\Omega|^{\frac{1-p}{p+1}}$.

Therefore, denoting by $s(t)=\int_\Omega u^2(x,t)\,dx$ and integrating between t_0 and T we obtain

$$\begin{aligned}T-t_0 &\leq \int_{s(t_0)}^{s(T)}\frac{d\sigma}{4|\Phi(u(\cdot,t_0))|+2C_1\sigma^{\frac{p+1}{2}}}\\ &\leq \int_0^\infty \frac{d\sigma}{4|\Phi(u(\cdot,t_0))|+2C_1\sigma^{\frac{p+1}{2}}}=C|\Phi(u(\cdot,t_0))|^{-\frac{p-1}{p+1}}\end{aligned}$$

where the constant $C=C(p,|\Omega|)$. □

3.2 Perturbations of the initial conditions

Example 1. L^r perturbations of the initial data.

We deal with solution to (3.1) and we perturb the initial condition $u_0(x)$ by considering a family of functions $u_{0,h}(x)$.

As we have mentioned in the introduction, continuity and almost Lipschitz dependence of the blow-up time with respect to the initial condition in L^∞-norm has already been established, see [15], [16], [18]. See also [6] for a proof of Lipschitz continuity for the porous medium equation in the whole line, $\Omega=\mathbb{R}$.

In [21] the author proves the continuity of the blow-up time with respect the initial data in the space $H_0^1(\Omega)$.

We want to consider here perturbations of the initial values in the space L^r. We consider a sequence of functions $u_{0,h}$ such that

$$\|u_{0,h} - u_0\|_{L^r(\Omega)} \to 0 \qquad \text{as } h \to 0.$$

In order to use Proposition 3.1 we impose the condition $r > pN$. Hence, if $t_0 < T$ (as usual T is the blow-up time of the solution u starting at u_0), we know that, for h small, $\|u(\cdot,t_0) - u_h(\cdot,t_0)\|_{C^{2,\alpha}(\bar\Omega)} \le C\|u_0 - u_{0,h}\|_{L^r(\Omega)}$, which allows us to use the continuity results of the blow-up times with this stronger norm that we have mentioned above.

Nevertheless, with the results that we developed in the previous sections, we can provide a simple proof of the continuity of the blow-up times. Observe that, as a byproduct of Proposition 3.1, we have the convergence

$$u_h(\cdot,t_0) \to u(\cdot,t_0) \qquad \text{strongly in } H^1(\Omega) = W^{1,2}(\Omega), \tag{3.12}$$

for every $0 < t_0 < T$ fixed. This in particular implies that **(H1)** holds. Moreover, this H^1 convergence and the fact that p is subcritical implies that, putting $V_h(u_h(\cdot,t)) = -\Phi(u_h(\cdot,t))$, we have from Lemma 3.3 that **(H2*)** holds. Hence Lemma 2.1 implies the convergence of the blow-up times.

3.3 Perturbations of parameters in the equation

Example 2. Perturbations in the reaction or in the diffusivity coefficients.

Again we look at solutions to problem (3.1). In this case we perturb that problem by introducing some coefficients, thus considering the equation

$$u_t = \Delta u + a_h(x)|u|^{p-1}u,$$

or even the equation

$$u_t = \operatorname{div}(B_h(x)\nabla u) + |u|^{p-1}u,$$

in the same domain Ω and with the same initial condition u_0. The coefficient a_h is a real function defined in Ω, while B_h is a function with values in the space of square $N \times N$ matrices. In both cases of perturbation we can use the same arguments as before as long as

$$u_h(\cdot,t_0) \to u(\cdot,t_0) \qquad \text{strongly in } H^1(\Omega), \tag{3.13}$$

for every $0 < t_0 < T$ fixed. And this is guaranteed if $a_h \to 1$, or $B_h \to I$ (the identity matrix), uniformly in $\overline{\Omega}$. This result generalizes the constant coefficient cases $a_h = 1 + h$ and $B_h = (1+h)I$ studied previously in [15].

Note that we can perturb the problem in several other ways, getting the same conclusion as long as we can obtain (3.13). For example, as in [15], we can deal with perturbations of the exponent, considering $u_t = \Delta u + |u|^{p_h - 1}u$ with $p_h \to p$.

3.4 Perturbation of the domain

In order to address the problem of domain perturbation, we first need a result on the behavior of solutions under perturbations of the domain in the presence of globally Lipschitz nonlinearities. For the sake of notation, let us denote by $u(x,t,\varphi,\mathcal{O},f)$ the solution of

$$\begin{cases} u_t = \Delta u + f(u), & \mathcal{O}\times(0,t_0), \\ u(x,t)=0, & \partial\mathcal{O}\times(0,t_0), \\ u(x,0)=\varphi(x), & \mathcal{O}. \end{cases} \tag{3.14}$$

where f is certain nonlinearity, $\mathcal{O}$ is an open set of $\mathbb{R}^N$ and φ is a function defined in $\mathcal{O}$. Observe that both, the initial condition φ and the solution u are defined in $\mathcal{O}$ but we can extend both of them by zero outside $\mathcal{O}$ so that we may consider them defined in $\mathbb{R}^N$.

Proposition 3.4 *Let Ω be a bounded Lipschitz domain and let*

$$\hat{\Omega}^h = \{x\in\mathbb{R}^N \,:\, dist(x,\Omega)<h\}$$

for $0<h\le h_0$ for some positive, small h_0. We consider a general nonlinearity $f(u)$ which is smooth and globally Lipschitz. Then, if $u_0\in L^\infty(\Omega)$ and $u_0^h\in L^\infty(\hat{\Omega}^h)$ with $\|u_0\|_{L^\infty(\Omega)}, \|u_0^h\|_{L^\infty(\hat{\Omega}^h)}\le M$ for all $0<h\le h_0$ and

$$\|u_0^h-u_0\|_{L^1(\Omega)}\to 0 \qquad as\ h\to 0,$$

then, for each $0<\tau_0<t_0$ we have

$$\sup_{t\in(\tau_0,t_0)}\|u(\cdot,t,u_0^h,\hat{\Omega}^h,f)-u(\cdot,t,u_0,\Omega,f)\|_{L^\infty(\hat{\Omega}^h)}\to 0 \qquad as\ h\to 0. \tag{3.15}$$

Proof. Observe that since the nonlinearity is globally Lipschitz, the solutions are globally defined in time for any initial condition and for any h. Moreover, $\hat{\Omega}^h$ is a nice smooth perturbation of the fixed domain Ω, from where the convergence stated in the proposition follows easily. □

In the following result we consider more general perturbations of the domain and we obtain convergence of solutions in the energy space H^1. The kind of perturbations we will consider satisfy the following,

(P) $\Omega_h\subset B(0,R)$ for $R>0$ fixed and if $\chi_h\in H_0^1(\Omega_h), \chi\in H_0^1(\Omega)$ are the unique solutions of $-\Delta\chi_h=1$ in Ω_h and of $-\Delta\chi=1$ in Ω, respectively, with homogeneous Dirichlet boundary conditions, we have $\|\chi_h-\chi\|_{L^2(B(0,R))}\to 0$ as $h\to 0$.

Observe that we regard the functions χ_h and χ defined in $B(0,R)$ by extending them by zero outside Ω_h and Ω respectively.

Proposition 3.5 *Let $\Omega \subset B(0,R)$ be a Lipschitz domain and let $\{\Omega_h\}_{0<h\le h_0}$, be a family of domains satisfying property* **(P)**. *We consider a general nonlinearity $f(u)$ which is smooth and globally Lipschitz. Then, if $u_0 \in L^2(\Omega)$ and $u_0^h \in L^2(\Omega_h)$ with*

$$\|u_0^h - u_0\|_{L^2(B(0,R))} \to 0, \quad as\ h \to 0,$$

then, for each $0 < \tau_0 < t_0$ we have

$$\sup_{t\in(\tau_0,t_0)} \left\| u(\cdot,t,u_0^h,\Omega_h,f) - u(\cdot,t,u_0,\Omega,f)\right\|_{H^1(B(0,R))} \to 0, \quad as\ h \to 0. \tag{3.16}$$

Proof. Condition **(P)** implies the spectral convergence of the Laplace operators in Ω_h, denoted by $-\Delta_{\Omega_h}$, to the Laplace operator in Ω, $-\Delta_\Omega$, see [1], [8]. By this we mean that if **(P)** is satisfied and if we denote by $\{\lambda_n^h\}_{n=1}^\infty$ the sequence of eigenvalues of $-\Delta_{\Omega_h}$, ordered and counting multiplicity, and by $\{\varphi_n^h\}_{n=1}^\infty$ a corresponding sequence of orthonormalized eigenfunctions and similarly for $\{\lambda_n\}_{n=1}^\infty$ and $\{\varphi_n\}_{n=1}^\infty$ for the operator $-\Delta_\Omega$, we have that $\lambda_n^h \to \lambda_n$ as $h \to 0$ for all $n = 1, 2, ...$, and the spectral projections converge as operators from $L^2(B(0,R))$ to $H_0^1(B(0,R))$. That is, if $\lambda_n < \lambda_{n+1}$ then

$$\sup_{\|\chi\|_{L^2(B(0,R))}\le 1} \left\{ \left\| \sum_{i=1}^n \langle \chi, \varphi_n^h\rangle \varphi_n^h - \sum_{i=1}^n \langle \chi, \varphi_n\rangle \varphi_n \right\|_{H^1(B(0,R))} \right\} \to 0. \tag{3.17}$$

We have denoted by $\langle\cdot,\cdot\rangle$ the scalar product in L^2. Using the expression of the linear semigroup $e^{\Delta_{\Omega_h} t}$ in terms of the eigenvalues and eigenfunctions of $-\Delta_{\Omega_h}$, that is

$$e^{\Delta_{\Omega_h} t}\chi = \sum_{i=1}^\infty e^{-\lambda_n^h t} \langle \chi, \varphi_n^h\rangle \varphi_n^h$$

and with the convergence of the eigenvalues and the spectral projections given by (3.17) we get the convergence of the linear semigroups, that is,

$$\|e^{\Delta_{\Omega_h} t}\chi - e^{\Delta_\Omega t}\chi\|_{H^1(B(0,R))} \le C(T)\theta(h)t^{-\gamma}\|\chi\|_{L^2(B(0,R))}, \tag{3.18}$$

for all $0 < t \le T$, and some $0 < \gamma < 1$, where $\theta(h) \to 0$ as $h \to 0$. To show (3.18) from the convergence of the eigenvalues and (3.17) we refer to [2], for a general result, and to [3] for a similar result with Neumann boundary conditions.

With the expression of $u(\cdot,t,u_0^h,\Omega_h,f)$ and $u(\cdot,t,u_0,\Omega,f)$ given by the variation of constants formula, that read as follows,

$$u(\cdot,t,u_0^h,\Omega_h,f) = e^{\Delta_{\Omega_h} t}u_0^h + \int_0^t e^{\Delta_{\Omega_h}(t-s)} f(u(\cdot,s,u_0^h,\Omega_h,f))\,ds$$

$$u(\cdot,t,u_0,\Omega,f) = e^{\Delta_\Omega t}u_0 + \int_0^t e^{\Delta_\Omega(t-s)} f(u(\cdot,s,u_0,\Omega,f))\,ds$$

substracting both expressions, applying some elementary computations, using (3.18) and with the singular Gronwall's inequality, see [17, Lemma 7.1.1], we obtain

$$\|u(\cdot,t,u_0^h,\Omega_h,f)-u(\cdot,t,u_0,\Omega,f)\|_{H^1(B(0,R))}\leq C(T)\theta(h)t^{-\gamma}, \tag{3.19}$$

for all $0<t\leq T$, where $\theta(h)\to 0$ as $h\to 0$. This gives the desired result. □

To the particular nonlinearity we are considering, $F(u)=|u|^{p-1}u$, we associate the truncated function F_k, defined by

$$F_k(u)=\begin{cases} -k^p & u<-k,\\ |u|^{p-1}u & -k<u<k,\\ k^p & k<u.\end{cases}$$

Observe that F_k is globally Lischitz and $F_k(u)=F(u)$ as long as $|u|\leq k$.

Proposition 3.6 *Let $\Omega\subset B(0,R)$ and let $\hat{\Omega}^h$ be as in Proposition 3.4. Assume we have a family of domains $\{\Omega_h\}_{0<h\leq h_0}$ satisfying* **(P)** *and $\Omega_h\subset\hat{\Omega}^h$ for all $0<h\leq h_0$. Assume also that we have $0\leq u_0\in L^\infty(\Omega)$, $0\leq u_0^h\in L^\infty(\Omega_0^h)$ and that there exists $M>0$ such that $\|u_0\|_{L^\infty(\Omega)},\|u_0^h\|_{L^\infty(\Omega_h)}\leq M$ and $\|u_0-u_0^h\|_{L^2(B(0,R))}\to 0$ as $h\to 0$. Let T be the existence time of $u(\cdot,t,u_0,\Omega,F)$ and T_h the existence time of $u(\cdot,t,u_0^h,\Omega_h,F)$. Then if $t_0<T$ and we denote by*

$$k=2+\sup_{0\leq t\leq t_0}\|u(\cdot,t,u_0,\Omega,F)\|_{L^\infty(\Omega)},$$

then, there exists $0<h_1<h_0$ such that for $0<h<h_1$, we have

$$T_h>t_0, \tag{3.20}$$

$$u(\cdot,t,u_0,\Omega,F)=u(\cdot,t,u_0,\Omega,F_k),\ \text{for all } 0<t\leq t_0, \tag{3.21}$$

$$u(\cdot,t,u_0^h,\Omega_h,F)=u(\cdot,t,u_0^h,\Omega_h,F_k),\ \text{for all } 0<t\leq t_0. \tag{3.22}$$

Therefore for each $t_0<T$ we have

$$\|u(\cdot,t_0,u_0,\Omega,F)-u(\cdot,t_0,u_0^h,\Omega_h,F)\|_{H^1(\mathbb{R}^N)}\to 0 \tag{3.23}$$

which in particular implies that $T_h\to T$ as $h\to 0$.

Proof. From the definition of k it is clear that we have (3.22). Moreover, it is clear that $k\geq M+2$.

If we consider the solution $u(x,t,M,\hat{\Omega}^{h_0},F)$ then, this solution will exists for certain time and therefore we will have the existence of a time $\tau_0>0$ small such that

$$0\leq u(x,t,M,\hat{\Omega}^{h_0},F)\leq M+1$$

for all $0\leq t\leq\tau_0$. By comparison arguments with respect to the initial condition and with respect to the domain, we have

$$0\leq u(x,t,u_0^h,\Omega_h,F)\leq u(x,t,u_0^h,\hat{\Omega}^h,F)\leq u(x,t,M,\hat{\Omega}^{h_0},F)\leq M+1$$

for $0 < h < h_0$ and $0 < t \le \tau_0$. This implies that for $0 < h \le h_0$,

$$u(x,t,u_0^h,\Omega_h,F) = u(x,t,u_0^h,\Omega_h,F_k), \qquad 0 < t \le \tau_0 \tag{3.24}$$

and also

$$u(x,t,u_0^h,\hat{\Omega}^h,F) = u(x,t,u_0^h,\hat{\Omega}^h,F_k), \qquad 0 < t \le \tau_0.$$

Applying now Proposition 3.4, we obtain that there exists $0 < h_1 < h_0$ such that for $0 < h < h_1$ we have,

$$\|u(\cdot,t,u_0^h,\hat{\Omega}^h,F_k) - u(\cdot,t,u_0,\Omega,F_k)\|_{L^\infty(\hat{\Omega}^h)} \le 1,$$

for $\tau_0 \le t \le t_0$, which in particular implies that, for those values of h and t, we have

$$0 \le u(\cdot,t,u_0^h,\Omega_h,F_k) \le u(\cdot,t,u_0^h,\hat{\Omega}^h,F_k) \le \sup_{\tau_0 \le t \le t_0} \|u(\cdot,t,u_0,\Omega,F_k)\|_{L^\infty(\Omega)} + 1 < k.$$

Therefore,

$$u(\cdot,t,u_0^h,\Omega_h,F_k) = u(\cdot,t,u_0^h,\Omega_h,F), \quad \tau_0 \le t \le t_0.$$

Putting together this last inequality and (3.24) we get (3.22) and also (3.20). Statement (3.23) is obtained from Proposition 3.5. This H^1 convergence implies the convergence of the energy (3.7) associated to the equation and in particular, with Lemma 3.3, we easily get that **(H2*)** holds true. From here we obtain the convergence of the blow-up times. □

We will consider now several examples where we obtain the continuity of the blow-up times when the domain is perturbed. We start with some basic but important examples where a straight proof of the convergence of the blow-up times can be obtained. We will also consider other not so simple examples where we must check that condition **(P)** in order to be able to apply Proposition 3.6.

Example 3. Dilatations of a fixed domain.

Consider the family of dilatations of a fixed domain, for $\lambda > 0$, let

$$\Omega^{(\lambda)} = \lambda\Omega = \{\lambda x \,:\, x \in \Omega\},$$

and study the family of evolution problems with $\Omega_h = \Omega^{(\lambda)}$, $\lambda = 1 + h$, $0 < |h| < 1$. If we solve

$$\begin{cases} (u_h)_t = \Delta u_h + |u_h|^{p-1}u_h, & \Omega_h \times (0,T_h) \\ u_h(x,t) = 0, & \partial\Omega_h \times (0,T_h) \\ u_h(x,0) = u_0(x), & \Omega_h, \end{cases} \tag{3.25}$$

we get by a simple scaling argument, that the solution u_h verifies

$$u_h(x,t) = \lambda^{-2/(p-1)} v_\lambda(\lambda^{-1}x, \lambda^{-2}t)$$

where v_λ is the solution to the problem with initial condition $u_0(x)$ replaced by $v_\lambda(x,0) = v_{\lambda,0}(x) \equiv \lambda^{2/(p-1)}u_0(\lambda x)$. Therefore, we have

$$T^{(\lambda)} = \lambda^2 \widetilde{T}_\lambda,$$

where $\widetilde{T}_\lambda$ is the blow-up time of the solution of the problem in the fixed domain Ω, but with perturbed initial condition $v_{\lambda,0}$. The continuity of this blow-up time $\widetilde{T}_\lambda$ as has been shown above in Example 1. From this it follows immediately that

$$T^{(\lambda)} \to T, \qquad \text{as } \lambda \to 1.$$

Example 4. Regular perturbations of a star-shaped domain.

Now assume that Ω is star-shaped (without loss of generality with respect to $0 \in \Omega$), and let Ω_h be a sequence of domains for which we have

$$\overline{\lambda}(h) = \inf\left\{\lambda > 0 \,:\, \Omega_h \subset \Omega^{(\lambda)}\right\} \to 1, \qquad \text{as } h \to 0, \tag{3.26}$$

and

$$\underline{\lambda}(h) = \sup\left\{\lambda > 0 \,:\, \Omega_h \supset \Omega^{(\lambda)}\right\} \to 1, \qquad \text{as } h \to 0. \tag{3.27}$$

Then, with the notations of the previous example, we have

$$\Omega^{(\underline{\lambda}(h))} \subset \Omega_h \subset \Omega^{(\overline{\lambda}(h))}$$

for every $h > 0$. If we consider now the solution u_h to the problem (3.25) in Ω_h with $u_0 \geq 0$, we have

$$u^{(\underline{\lambda}(h))}(x,t) \leq u_h(x,t) \leq u^{(\overline{\lambda}(h))}(x,t).$$

Hence, as $h \to 0$ we have

$$\begin{array}{ccc} T^{(\overline{\lambda}(h))} & \leq T_h & \leq T^{(\underline{\lambda}(h))} \\ \downarrow & & \downarrow \\ T & & T. \end{array}$$

This shows the continuity of the blow-up time, $T_h \to T$, for regular perturbations of a star-shaped domain, that is perturbations that verify (3.26) and (3.27).

Example 5. General perturbed domains satisfying condition **(P)**.

There are several interesting situations in which condition **(P)** holds. For instance, if we consider $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a bounded smooth domain, $x_0 \in \Omega$ and we denote by $\Omega_h = \Omega \setminus \bar{B}(x_0, h)$, then **(P)** holds. The main reason for this is that $H_0^1(\Omega \setminus \{x_0\}) = H_0^1(\Omega)$ since a single point has zero H^1-capacity in $\mathbb{R}^N$ for $N \geq 2$. Recall that the H^1-capacity of a closed set $K \subset \mathbb{R}^N$ is defined as

$$Cap(K) = \inf\left\{\|\nabla\phi\|_{L^2(\mathbb{R}^N)} : \phi \in C_0^\infty(\mathbb{R}^N), \phi = 1 \text{ in a neighborhood of } K\right\}$$

As a matter of fact if $K \subset\subset \Omega$ is such that $Cap(K) = 0$, then we have $H_0^1(\Omega \setminus K) = H_0^1(\Omega)$ and this implies that if $V_h \subset \Omega$, is a decreasing sequence of closed sets with $\cap V_h = K$, then the family $\Omega_h = \Omega \setminus V_h$ satisfies condition **(P)**. We refer to [1] for a proof of this result.

For this family of domains we can apply Proposition 3.6 and obtain the convergence of the blow-up times.

3.5 Stability of blow-up sets

Here we deal with solutions to our problem (3.1) but here we restrict ourselves to one space dimension, that is,

$$\begin{cases} u_t = u_{xx} + u^p, & (0,1) \times (0,T) \\ u(0,t) = u(1,t) = 0, & (0,T) \\ u(x,0) = u_0(x), & (0,1), \end{cases} \tag{3.28}$$

with (as before) $p > 1$, and $u_0 \geq 0$.

Example 6. Stability of the single-point blow-up in an interval.

Assume that u_0 has a unique maximum in $(0,1)$. Then it is known that u blows up at a single point, see [10] and [7]. Let x_0 be this blow-up point.

We perturb the initial condition considering a family $u_{0,h}$ of functions such that

$$\|u_{0,h} - u_0\|_{L^r(0,1)} \to 0, \quad \text{as } h \to 0,$$

with $r > p$. Note that under this hypothesis we have that $T_h \to T$.

From the C^2 convergence (see Proposition 3.1 and its Corollary 3.2) we obtain that for $0 < \tau < T$ and h small, the function $u_h(\cdot, \tau)$ has a unique maximum at some point $x_h(\tau)$. A comparison argument with the solution of the ODE, $z' = z^p$,

$$z(t) = C_p(T_h - t)^{-\frac{1}{p-1}},$$

gives that

$$u_h(x_h(t), t) \geq C_p(T_h - t)^{-\frac{1}{p-1}}$$

(otherwise, u_h and z cannot blow-up at the same time T_h).

This implies that **(H3)** holds, from where the convergence of the blow-up sets follows. Indeed we have, $B(u_h) = x_h$, a single point, with

$$\lim_{h \to 0} x_h = x_0.$$

Example 7. Stability of the single-point blow-up in a family of intervals.

We can also perturb the interval considering the dilatations $\Omega^{(\lambda)} = (0, \lambda)$, $\lambda > 0$. Note that we have a perturbation of the domain like the ones considered in Example 2, therefore the convergence of the blow-up times is guaranteed.

If we solve the problem in $\Omega^{(\lambda)}$ we get, by the same scaling argument used in Example 2, that the solution $u^{(\lambda)}$ verifies

$$u^{(\lambda)}(x,t) = \lambda^{-2/(p-1)} v_\lambda(\lambda^{-1}x, \lambda^{-2}t)$$

where v_λ is the solution to problem in $(0,1)$ with initial condition $u_0(x)$ replaced by $v_\lambda(x,0) = v_{\lambda,0}(x) \equiv \lambda^{2/(p-1)} u_0(\lambda x)$. Therefore, by the (uniform) convergence of $v_{\lambda,0}$ to u_0 as $\lambda \to 1$, we get, using the continuity of the blow-up set with respect to the initial condition,

$$\lim_{\lambda \to 1} x_\lambda = x_0.$$

4 Conclusions

In this final section we comment briefly on the results obtained throughout the previous pages and on further extensions.

In this paper we have proved a general result concerning continuity of the blow-up time and blow-up set for an evolution problem under perturbations. This result is based on some convergence of the solutions of the perturbed problem for times smaller than the blow-up time of the solution of the unperturbed problem together with some uniform bounds on the blow-up rates of the solutions of the perturbed problems. Obtaining these uniform bounds is a delicate subject in general, specially the ones appearing in **(H3)**, since they involve an estimate for every point in the blow-up set. It will be desirable to obtain continuity of the blow-up set under weaker hypotheses, but, as shown by one of our examples, this seems a very delicate issue.

We have also accompanied the general results by a number of examples where more specific information can be obtained. It is clear that we can extend some of our results to more general equations, like the evolution given by the porous medium equation with a source $u_t = \Delta u^m + u^p$ or the one given by the $q-$Laplacian with a source, $u_t = \mathrm{div}(|\nabla u|^{q-2}\nabla u)+u^p$. Our results are specially well suited for the case $p > m$ (or $p > q-1$) where there is single point blow-up. Finally, we mention that our general result may also be applied to systems, but this extension requires further analysis.

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