APPROXIMATION OF THE SOBOLEV TRACE CONSTANT

JULIO D. ROSSI

Abstract. In this paper we study the Sobolev trace immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ with $1 < q < p^* = \frac{p(N-1)}{N-p}$ if $p > N$. We present an approximation procedure for the Sobolev trace constant and the extremals, that is the best constant that verifies $S^{1/p}\|u\|_{L^q(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}$ and functions where the constant is attained.

1. Introduction.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. In this paper we deal with the Sobolev trace immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ with $1 < q < p^* = \frac{p(N-1)}{N-p}$ if $p < N$. This immersion is a continuous, compact operator and therefore there exists a constant S such that

$$S^{1/p}\|u\|_{L^q(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}.$$

This Sobolev trace constant S can be characterized as

$$S = \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p + \int_\Omega |u|^p, \qquad \int_{\partial\Omega} |u|^q = 1 \right\}. \tag{1.1}$$

Using the compactness of the embedding it is easy to prove that there exists extremals, that is functions where the constant is attained. The extremals are weak solutions in $W^{1,p}(\Omega)$ of the following problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and $\frac{\partial}{\partial\nu}$ is the outer normal derivative. See [4] for a detailled analysis of the behaviour of extremals and best Sobolev constants in expanding domains for the linear case, $p = 2$.

In the case $p = q$ we have a nonlinear eigenvalue problem and the extremals are eigenfunctions of the first eigenvalue. In the linear case, that is for $p = 2$, this eigenvalue problem is known as the *Steklov* problem, [2]. In [5] it is proved that there exists a sequence of eigenvalues λ_n of (1.2) such that $\lambda_n \to +\infty$ as $n \to +\infty$. Also it is known that the first eigenvalue λ_1 is isolated and simple with a positive eigenfunction (see [8]). For the same type of results for the $p-$Laplacian with Dirichlet boundary conditions see [1], [6] and [7].

Key words and phrases. numerical approximations, p-Laplacian, nonlinear boundary conditions, Sobolev trace constant.

2000 *Mathematics Subject Classification.* 65J15, 65N15, 65N25.

Supported by ANPCyT PICT No. 5009. J. D. Rossi is a member of CONICET.

Our interest here is to approximate S. We remark that we are dealing with a nonlinear problem, (1.2), in the Banach space $W^{1,p}(\Omega)$. Let us describe a general approximation procedure. The idea is to replace the space $W^{1,p}(\Omega)$ with a subspace V_h in the minimization problem (1.1). To this end, let V_h be an increasing sequence of closed subspaces of $W^{1,p}(\Omega)$, such that

$$\begin{aligned} &\left\{ v \in V_h : \int_{\partial\Omega} |v|^q = 1 \right\} \neq \emptyset \\ &\qquad \text{and} \\ &\lim_{h\to 0} \inf_{v\in V_h} \|u - v\|_{W^{1,p}(\Omega)} = 0, \quad \forall \|u\|_{W^{1,p}(\Omega)} = 1. \end{aligned} \tag{1.3}$$

With this sequence of subspaces V_h we define our approximation of S by

$$S_h = \inf_{u_h \in V_h} \left\{ \int_\Omega |\nabla u_h|^p + \int_\Omega |u_h|^p, \qquad \int_{\partial\Omega} |u_h|^q = 1 \right\}, \tag{1.4}$$

We prove that under hypothesis (1.3) S_h approximates S,

Theorem 1.1. *Let u be an extremal for* (1.1). *Then, there exists a constant C independent of h such that,*

$$|S - S_h| \leq C \inf_{v\in V_h} \|u - v\|_{W^{1,p}(\Omega)},$$

for every h small enough.

Regarding the extremals we have,

Theorem 1.2. *Let u_h be a function in V_h where the infimum* (1.4) *is achived. Then from any sequence $h \to 0$ we can extract a subsequence $h_j \to 0$ that converges strongly to an extremal in $W^{1,p}(\Omega)$. That is, there exists an extremal of* (1.1), *w, with*

$$\lim_{h_j \to 0} \|u_{h_j} - w\|_{W^{1,p}(\Omega)} = 0.$$

We observe that the only requirement on the subspaces V_h is (1.3). This allows us, for example, to choose V_h as the usual finite elements spaces.

2. Proofs of the Theorems

Along this section we write C for a constant that does not depend on h and may vary from one line to another.

Proof of Theorem 1.1: As $V_h \subset W^{1,p}(\Omega)$ we have that

$$S \leq S_h. \tag{2.1}$$

Let us choose $v \in V_h$ such that $\|u - v\|_{W^{1,p}(\Omega)} \leq \inf_{V_h} \|u - w\|_{W^{1,p}(\Omega)} + \varepsilon$. We have that

$$S_h^{1/p} = \|u_h\|_{W^{1,p}(\Omega)} \leq \frac{\|v\|_{W^{1,p}(\Omega)}}{\|v\|_{L^q(\partial\Omega)}} \leq \frac{\|v - u\|_{W^{1,p}(\Omega)} + \|u\|_{W^{1,p}(\Omega)}}{\|v\|_{L^q(\partial\Omega)}}$$

$$= \left(\frac{\|v - u\|_{W^{1,p}(\Omega)} + S^{1/p}}{\|v\|_{L^q(\partial\Omega)}} \right).$$

Now we use that

$$\big| \|v\|_{L^q(\partial\Omega)} - 1 \big| \le \big| \|v\|_{L^q(\partial\Omega)} - \|u\|_{L^q(\partial\Omega)} \big| \le \|v-u\|_{L^q(\partial\Omega)} \le C\|v-u\|_{W^{1,p}(\Omega)}$$

and hypothesis (1.3) to obtain that for every h small enough,

$$S_h \le \left(\frac{\|v-u\|_{W^{1,p}(\Omega)} + S^{1/p}}{1 - C\|v-u\|_{W^{1,p}(\Omega)}} \right)^p \le S + C\|v-u_1\|_{W^{1,p}(\Omega)}. \tag{2.2}$$

From (2.1) and (2.2) the result follows.

Proof of Theorem 1.2: Theorem 1.1 and hypothesis (1.3) gives that

$$\lim_{h\to 0} \|u_h\|^p_{W^{1,p}(\Omega)} = \lim_{h\to 0} S_h = S.$$

Hence there exists a constant C such that for every h small enough,

$$\|u_h\|_{W^{1,p}(\Omega)} \le C.$$

Therefore we can extract a subsequence, that we denote by u_{h_j}, such that

$$\begin{aligned} u_{h_j} &\rightharpoonup w && \text{weakly in } W^{1,p}(\Omega), \\ u_{h_j} &\to w && \text{strongly in } L^p(\Omega), \\ u_{h_j} &\to w && \text{strongly in } L^q(\partial\Omega). \end{aligned} \tag{2.3}$$

Hence, from the $L^q(\partial\Omega)$ convergence we have,

$$1 = \lim_{h_j\to 0} \int_{\partial\Omega} |u_{h_j}|^q = \int_{\partial\Omega} |w|^q.$$

Therefore w is an admissible function in the minimization problem (1.1). Now we observe that,

$$\begin{aligned} \|u\|^p_{W^{1,p}(\Omega)} &\le \|w\|^p_{W^{1,p}(\Omega)} \le \liminf_{h_j\to 0} \|u_{h_j}\|^p_{W^{1,p}(\Omega)} \\ &\le \lim_{h_j\to 0} \|u_{h_j}\|^p_{W^{1,p}(\Omega)} = \lim_{h_j\to 0} S_h = S = \|u\|^p_{W^{1,p}(\Omega)}, \end{aligned}$$

and therefore,

$$\lim_{h_j\to 0} \|u_{h_j}\|_{W^{1,p}(\Omega)} = \|w\|_{W^{1,p}(\Omega)} = S^{1/p}. \tag{2.4}$$

The space $W^{1,p}(\Omega)$ being uniformly convex, the weak convergence, (2.3), and the convergence of the norms, (2.4), imply the convergence in norm. Therefore $u_{h_j} \to w$ in $W^{1,p}(\Omega)$. This limit w verifies $\|w\|^p_{W^{1,p}(\Omega)} = S$ and $\|w\|_{L^q(\partial\Omega)} = 1$. Hence it is an extremal and we have that

$$\lim_{h_j\to 0} \|u_{1,h} - w\|_{W^{1,p}(\Omega)} = 0,$$

as we wanted to prove.

Acknowledgements. We want to thank R. Duran and A. Lombardi for several suggestions and interesting discussions.

References

[1] A. Anane, *Simplicité et isolation de la premiere valeur propre du $p-$laplacien avec poids.* C. R. Acad. Sci. Paris, 305 (I), (1987), 725-728.

[2] I. Babuska and J. Osborn, Eigenvalue Problems, Handbook of Numer. Anal., Vol. II (1991). North-Holland.

[3] P. Ciarlet. The finite element method for elliptic problems. North-holland. 1980.

[4] M. del Pino and C. Flores, *Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains.* To appear in Comm. Partial Diff. Eq.

[5] J. Fernández Bonder and J.D. Rossi, *Existence results for the $p-$Laplacian with nonlinear boundary conditions.* To appear in J. Math. Anal. Appl.

[6] J. Garcia-Azorero and I. Peral, *Existence and non-uniqueness for the $p-$Laplacian: nonlinear eigenvalues.* Comm. Partial Differential Equations. Vol. 12 (1987), 1389-1430.

[7] P. Lindqvist, *On the equation* $\Delta_p u + \lambda |u|^{p-2} u = 0$. Procc. A.M.S., 109-1, (1990), 157–164.

[8] S. Martinez and J. D. Rossi. *Isolation and simplicity for the first eigenvalue of the p-Laplacian with a nonlinear boundary condition.* Preprint.

J. D. Rossi
Departamento de Matemática, FCEyN
UBA (1428) Buenos Aires, Argentina.
e-mail: `jrossi@dm.uba.ar`