The balance between nonlinear reaction and absorption at the boundary

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We study nonnegative solutions of a nonlinear diffusion equation with nonlinear boundary conditions. At a portion of the boundary we have reaction, while at another portion we have absorption. Namely, we deal with the following problem

$$\begin{cases} u_t = (u^m)_{xx} & (x,t) \in (0,L) \times (0,T), \\ -(u^m)_x(0,t) = u^p(0,t) & t \in (0,T), \\ (u^m)_x(L,t) = -u^q(L,t) & t \in (0,T), \\ u(x,0) = u_0(x) & x \in (0,L), \end{cases}$$

where m, p and q are positive parameters. We describe in terms of p, q, m and L when all solutions exist globally in time, when all solutions blow up in a finite time, and when there are both blowing up and global solutions. For blowing up solutions we find the blow-up rate and the blow-up set. We show that the blow-up rate is *discontinuous* as a function of the parameters. This is the first known example of this phenomenon in a parabolic problem with blow-up.

1. INTRODUCTION AND MAIN RESULTS

We deal with the following problem

$$\begin{cases} u_t = (u^m)_{xx} & (x,t) \in (0,L) \times (0,T), \\ -(u^m)_x(0,t) = u^p(0,t) & t \in (0,T), \\ (u^m)_x(L,t) = -u^q(L,t) & t \in (0,T), \\ u(x,0) = u_0(x) & x \in (0,L), \end{cases} \tag{1}$$

where $m, p, q, L > 0$ are parameters. Local existence in time follows by standard arguments, see [8] for a similar problem. The time T is the maximal existence time for the solution, which may be finite or infinite. If $T < \infty$, then u becomes unbounded in finite time and we say that it *blows up*. If $T = \infty$ we say that the solution is *global*. We assume that u_0 is continuous and nonnegative. Hence the solution that we are considering is nonnegative for all its time interval of existence, see [6], [8]. If $m = 1$, we have the well-known heat equation. For $m > 1$ the equation is called the porous medium equation, while for $m < 1$ it is known as the fast-diffusion equation. For every $m > 0$ problem (1) can be thought of as a model for nonlinear heat propagation. In this case u stands for the temperature and $-(u^m)_x$ represents the heat flux. Thus our nonlinear boundary conditions represent heat radiation or absorption through the boundary of the domain. A justification of the appearance of such boundary conditions in a combustion problem appears in [17].

In this article we are interested in the blow-up phenomenon, a subject that has deserved a great deal of attention in recent years, see for example the book [20] and the surveys [15] and [5]. For specific references about blow-up in problems with nonlinear boundary conditions see the survey [7]. In our problem we have a nonlinear source term at the boundary point $x = 0$ and a nonlinear absorption term at the other boundary point, $x = L$. The reaction term favors blow-up, while the absorption term tries to preclude it. The actual behaviour of solutions, blow-up or global existence, is determined by which term dominates. This depends on m, p, q and L as is shown in our first theorem. This theorem is restricted to $m > 1$. This is the most interesting case, as it is in this range of parameters where we have more diversity of behaviours. We state our results for $m \leq 1$ later, at the end of the introduction.

THEOREM 1.1. *Let $m > 1$.*
A) *If the parameters lie in any of the sets*

$$\begin{aligned}
\mathcal{B}_1 &= \{p > m\}, \\
\mathcal{B}_2 &= \{q \leq p = m\} \cup \{q > p = m,\ L \geq 2m/(m-1)\}, \\
\mathcal{B}_3 &= \{1 < p < m,\ p > q\}, \\
\mathcal{B}_4 &= \{(m+1)/2 < p < m,\ p = q\},
\end{aligned}$$

then there exist solutions that blow up in finite time.
B) *If the parameters lie in any of the sets*

$$\begin{aligned}
\mathcal{G}_1 &= \{p = q,\ 1 < p \leq (m+1)/2\}, \\
\mathcal{G}_2 &= \{p < q,\ 1 < p < m\} \cup \{q > p = m,\ L < 2m/(m-1)\}, \\
\mathcal{G}_3 &= \{p \leq 1\},
\end{aligned}$$

then every positive solution is global in time.

The content of this theorem is summarized in Figure 1.

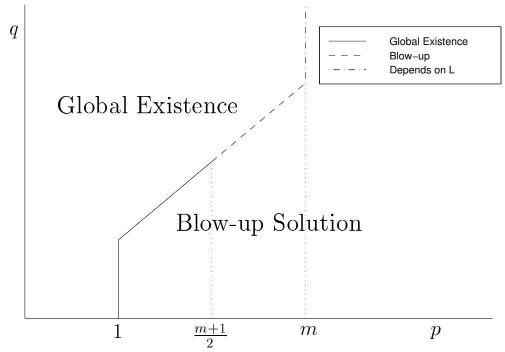

FIG. 1. Blow-up / Global existence.

In order to better understand Theorem 1.1 we will compare (1) with a related problem. Namely, we consider the porous medium equation ($m > 1$) in a bounded interval with nonlinear reaction at the origin and zero flux at the other end,

$$\begin{cases} u_t = (u^m)_{xx} & (x,t) \in (0,L) \times (0,T), \\ -(u^m)_x(0,t) = u^p(0,t) & t \in (0,T), \\ (u^m)_x(L,t) = 0 & t \in (0,T), \\ u(x,0) = u_0(x) & x \in (0,L). \end{cases} \tag{2}$$

This problem is studied in [8], where it is proved that solutions blow up if and only if $p > 1$ (similar results for systems are proved in [1]). In problem (1) the end $x = L$ is not insulated, but has an absorption term instead. Hence, our problem has less chances for blow-up than problem (2). Therefore we cannot expect blow-up for it if $p \le 1$. What if $p > 1$? Then the boundary absorption term may play a role. To understand this role we consider the *blow-up set*, $B(u)$, of a solution u, which is defined as

$$B(u) = \{x \in [0,L];\ \exists\, x_n \to x,\ t_n \nearrow T,\ \text{with}\ u(x_n,t_n) \to +\infty\}.$$

The blow-up set of solutions to problem (2) with $m > 1$ is known to be the whole interval $[0,L]$ if either $p < m$ or $p = m$ with $L \le 2m/(m-1)$; the interval $[0, 2m/(m-1)]$ if $p = m$ with $L > 2m/(m-1)$ and a single point $x = 0$ if $p > m$. For problem (1) there is some absorption at $x = L$. Therefore we expect that the blow-up set will not be bigger than that of problem (2). Thus it will consist at most of the origin if $p > m$ (see Theorem 1.5 below). Hence, absorption at $x = L$ cannot be very big, as u is bounded there; reaction wins the game. If $p < m$ the blow-up set may be the whole interval. If this is the case, absorption at $x = L$ – as well as reaction at $x = 0$ – becomes enormous. To have blow-up or not depends on which term dominates; hence the condition $p > q$. If $p = m$, things depend on the relation between L and $2m/(m-1)$. If $L \geq 2m/(m-1)$, then blow-up will be regional and the absorption term will play no role. On the other hand, if $L < 2m/(m-1)$, blow-up will not be truly regional, but will happen in the whole interval $[0, L]$ instead. Then the alternative blow-up/global existence will depend again on the condition $p > q$.

We still have to consider the segment $p = q$, $1 < p < m$. In this range of parameters both boundary conditions are of the same order and the balance becomes more delicate. If $1 < p \leq (m+1)/2$ there is enough cancellation to abort blow-up. On the contrary, if $(m+1)/2 < p < m$ we still have blow-up, though cancellation will provoke a diminution of the *blow-up rate*, *i.e.*, of the speed at which blow-up takes place (see Theorem 1.4 below).

The critical exponent for global existence, $p = (m+1)/2$, for the borderline case $p = q$, reminds us of another related problem, namely

$$\begin{cases} u_t = (u^m)_{xx} & (x,t) \in (0,+\infty) \times (0,T), \\ -(u^m)_x(0,t) = u^p(0,t) & t \in (0,T), \\ u(x,0) = u_0(x) & x \in (0,+\infty). \end{cases} \tag{3}$$

This problem in the half-line has been studied in [9] and [21], and will be very useful in our analysis. It is known that there are blowing up solutions to (3) if and only if $(m+1)/2 < p$. Moreover, the blow-up set of u, $B(u)$, is the whole half line if $(m+1)/2 < p < m$; the bounded interval $[0, 2m/(m-1)]$ if $p = m$ and a single point $x = 0$ if $p > m$ (see also [18] for similar results for a system). We remark that the blow-up set in the *regional* case $p = m$ of problem (3) coincides with the critical length $2m/(m-1)$ for the case $q > p = m$ for our problem.

Now we look for *Fujita type curves*, that is, for fixed m and L, curves in the pq-plane separating the regions of parameters where all nontrivial solutions blow up from that regions where there are both blowing up solutions and solutions that are global in time.

Stationary solutions of (1) are global solutions. Hence, the candidates for regions where all solutions blow up are the regions where there are no stationary solutions. The identification of the sets of parameters for which there exist stationary solutions of (1) is the content of our next theorem. Setting $w = u^m$, we look for nonnegative nontrivial solutions $w = w(x)$ of

$$\begin{cases} 0 = w_{xx} & x \in (0,L), \\ -w_x(0) = w^{p/m}(0), \\ w_x(L) = -w^{q/m}(L). \end{cases} \tag{4}$$

THEOREM 1.2. *Let* $m > 0$.
A) *If the parameters lie in the set*

$$\mathcal{E}_1 = \{p = m \neq q,\ L < 1\} \cup \{p > q,\ p > m\} \cup \{p < q,\ p < m\}$$

there exists a unique positive solution of (4).
B) *In the set*

$$\mathcal{E}_2 = \{p = q\} \cup \{p = m \neq q,\ L \geq 1\}$$

there is no solution of (4).
C) *In the set*

$$\mathcal{E}_3 = \{q < p < m\} \cup \{m < p < q\}$$

the existence and uniqueness of solutions of (4) *depends on* L*. Namely, there exists a critical value* $L^* = L^*(m,p,q) > 0$*, which is well defined in* $\mathcal{E}_3$*, given by,*

$$L^*(m,p,q) = \frac{m(q-p)}{q(p-m)} \left(\frac{q(p-m)}{p(q-m)} \right)^{\frac{p(q-m)}{m(q-p)}},$$

such that:
a) for $L > L^*$ *there is no solution of* (4)*;*
b) for $L = L^*$ *there is a unique positive solution of* (4)*;*
c) for $L < L^*$ *there exist exactly two different solutions of* (4).

Remark 1. 1. The following properties of L^* are easy to check. The limits are always understood to be taken from the interior of $\mathcal{E}_3$.
(i) $\lim_{p\to m} L^*(m,p,q) = 1$;
(ii) $\lim_{p\to q} L^*(m,p,q) = \lim_{q\to p} L^*(m,p,q) = 0$;
(iii) $\lim_{q\to 0} L^*(m,p,q) = 1$;
(iv) $\frac{\partial L^*}{\partial p} < 0$ and $\frac{\partial L^*}{\partial q} > 0$ if $m < p < q$;
(v) $\frac{\partial L^*}{\partial p} > 0$ and $\frac{\partial L^*}{\partial q} < 0$ if $q < p < m$;
(vi) $L^* < 1$ in $\mathcal{E}_3$.

Hence, if $L < 1$ the equation $L = L^*(m,p,q)$ defines q as a continuous function of p, $q = Q(p)$, for all $p > 1$. Indeed, $Q(p)$ is the only value that verifies $L = L^*(m,p,Q(p))$. On the other hand, if $L \geq 1$ there is no q such that $L = L^*(m,p,q)$.

To continue our study of the Fujita type curves, we prove that if the parameters are such that there exist blowing up solutions, but there are no stationary solutions, then every nontrivial solution blows up in finite time.

THEOREM 1.3. *Let* $m > 0$*. Let* m*,* p*,* q *and* L *be such that there are blowing up solutions and there is no stationary solution. Then all nontrivial solutions of* (1) *blow up.*

Now we can give a complete characterization of the Fujita type curves.

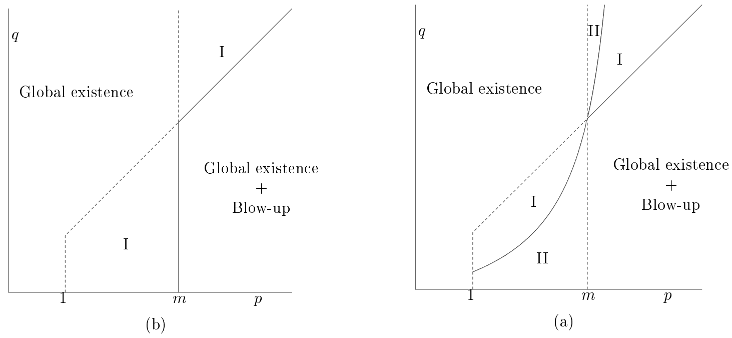

FIG. 2. Fujita type curves. In region A all nontrivial solutions blow up and in region B there exist both global and blowing up solutions.

COROLLARY 1.1. *Let $m > 1$. The Fujita type curves are given by:*
(i) $\{p = m,\ 0 \le q \le m\}$ *and* $\{p = q,\ p \ge m\}$ *if* $L \ge 1$*;*
(ii) $\{q = Q(p),\ p \ge 1\}$ *and* $\{p = q,\ p \ge m\}$ *if* $L < 1$.

An example is shown in Figure 2, where it can be seen that the picture becomes more complicated for $L < 1$.

Next we turn our interest to the asymptotic behaviour of blowing up solutions to problem (1). We first study the blow-up rate. We need an extra monotonicity assumption on u,

(H) $u(x,t)$ blows up in finite time T and verifies $u_t > 0$ for t near T.

Notice that if u_0 is smooth, compatible and verifies $(u_0^m)_{xx} \ge 0$ then $u(x,t)$ verifies (H).

Next theorem shows the dependence of the blow-up rate on the parameters.

THEOREM 1.4. *Let u be a solution of* (1) *with $m > 1$ satisfying* (H).
A) *In $\mathcal{B}_3 = \{1 < p < m,\ p > q\}$ the blow-up rate is given by*

$$\|u(\cdot,t)\|_\infty \sim (T-t)^{-\frac{1}{p-1}}.$$

B) *In $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_4 = \{(m+1)/2 < p < m,\ p = q\} \cup \{q \le p = m\} \cup \{q > p = m,\ L \ge 2m/(m-1)\} \cup \{p > m\}$ the blow-up rate is given by*

$$\|u(\cdot,t)\|_\infty \sim (T-t)^{-\frac{1}{2p-m-1}}.$$

This theorem shows that the blow-up rate is discontinuous in the set $\{p = q,\ (m+1)/2 < p < m\}$. As far as we know, this is the first example of this phenomenon in a parabolic problem with blow-up.

Remark 1. 2. In $\mathcal{B}_3$ the blow-up rate coincides with that of solutions of problem (2). In $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_4$ it is the same as that of solutions of problem (3).

Now we study the blow-up set for $u(x,t)$. We can characterize the blow-up set in terms of the parameters as follows,

THEOREM 1.5. *Let u be a solution of* (1) *with $m > 1$ satisfying* (H).
A) *In $\mathcal{B}_3 \cup \mathcal{B}_4 = \{1 < p < m,\ p > q\} \cup \{p = q,\ (m+1)/2 < p < m\}$ the blow-up set is*

$$B(u) = [0, L].$$

B) *In $\mathcal{B}_2 = \{p = m,\ p > q\} \cup \{q \geq p = m,\ L \geq 2m/(m-1)\}$ the blow-up set is given by*

$$B(u) = \begin{cases} [0, L] & \text{if } L < 2m/(m-1), \\ [0, 2m/(m-1)] & \text{if } L \geq 2m/(m-1). \end{cases}$$

C) *In $\mathcal{B}_1 = \{p > m\}$ the blow-up set is given by*

$$B(u) = \{0\}.$$

Therefore in $\mathcal{B}_3 \cup \mathcal{B}_4$ we have global blow-up, in $\mathcal{B}_2$ we have regional or global blow-up depending on L and in $\mathcal{B}_1$ we have single point blow-up.

Finally we state our results for $0 < m \leq 1$. Observe that theorems 1.2 and 1.3 already include this case. In this range of values of m we have single point blow up, and the situation is analogous to the case $p > m$ for $m > 1$. The proofs are similar to those for the porous medium equation ($m > 1$). Hence we will only point out the differences in appropriate places.

THEOREM 1.6. *Let $0 < m \leq 1$.*
(i) *There exist blowing up solutions for problem* (1) *if and only if $p > (m+1)/2$.*
(ii) *The Fujita type curves are given by $\{p = q,\ p \geq (m+1)/2\}$ if $L \geq 1$ and by $\{p = q,\ p \geq (m+1)/2\}$ and $\{q = Q(p),\ p \geq (m+1)/2\}$ if $L < 1$.*
(iii) *Let $u(x,t)$ be a solution that satisfies* (H). *Then the blow-up rate is given by*

$$\|u(\cdot,t)\|_\infty \sim (T-t)^{-\frac{1}{2p-m-1}},$$

and the blow-up set is

$$B(u) = \{0\}.$$

ORGANIZATION OF THE PAPER. In Section 2 we describe the sets of parameters for which there are stationary solutions, Theorem 1.2. In Section 3 we prove the blow-up results, Theorem 1.1, and we find the Fujita type curves, Theorem

1.3 and Corollary 1.1. In Section 4 we find the blow-up rates, Theorem 1.4. In Section 5 we look for the blow-up sets, Theorem 1.5. Finally, in the Appendix we prove an auxiliary result needed in Section 5.

2. STATIONARY SOLUTIONS

In this section we look for nontrivial nonnegative solutions of (4), that is, nontrivial nonnegative stationary solutions of (1).

Proof of Theorem 1.2. Let $w(x)$ be a solution of (4), then $w(x) = ax + b$. Since we are looking for nonnegative solutions, we must have $b \geq 0$, $aL + b \geq 0$. The boundary conditions yield $-a = b^{p/m}$, $a = -(aL + b)^{q/m}$. The first equation implies that $a = 0$ if and only if $b = 0$. In this case $w(x) \equiv 0$. Hence, to find nontrivial solutions of (4) we have to find positive parameters $b > 0$ satisfying $P(b) = 0$, with $P(b) = b - b^{p/q} - Lb^{p/m}$. As we are not interested in the root $b = 0$ of $P(b)$, we factor out b. Finally we are led to study the roots of

$$G(b) = 1 - b^{\frac{p}{q}-1} - Lb^{\frac{p}{m}-1}.$$

There are several cases. If $p = q$, then $G(b) = -Lb^{(p/m)-1} < 0$ and there is no positive root. If $p = m$, then $G(b) = 1 - L - b^{(m/q)-1}$; hence there is no root if $L \geq 1$ and there is just one root if $L < 1$.

In order to handle the remaining cases we compute $G'(b)$,

$$G'(b) = -\left(\frac{p}{q} - 1\right) b^{\frac{p}{q}-2} - L\left(\frac{p}{m} - 1\right) b^{\frac{p}{m}-2}.$$

Thus, if $p > q$ and $p > m$ we have $G'(b) < 0$, $G(0) = 1$, $G(\infty) = -\infty$, and hence there exists exactly one positive root of G. If $p < q$ and $p < m$, then $G(0) = -\infty$, $G'(b) > 0$, and $G(\infty) = 1$. Again we have just one positive root.

If $q < p < m$ or $m < p < q$, then $G(0) = -\infty$ and $G(\infty) = -\infty$. On the other hand, there exists just one value $\bar{b}$, given by

$$\bar{b} = \left(\frac{Lq(m-p)}{m(p-q)}\right)^{\frac{qm}{p(m-q)}},$$

such that $G'(\bar{b}) = 0$. Thus, the existence of roots depends on the sign of $G(\bar{b})$: if $G(\bar{b}) > 0$ there are two roots, if $G(\bar{b}) < 0$ there are no roots, and if $G(\bar{b}) = 0$ there is just one root. The critical value L^* is obtained from the last condition. ∎

3. BLOW-UP VERSUS GLOBAL EXISTENCE

In this section we prove Theorem 1.1. The basic idea is to compare from below with blowing up subsolutions or from above with global in time supersolutions.

Therefore we start by defining sub and supersolutions and by stating a comparison lemma.

DEFINITION 3.1. $i)$ We call $\overline{u}$ a supersolution of (1) if it satisfies

$$\begin{cases} \overline{u}_t \geq (\overline{u}^m)_{xx} & (x,t) \in (0,L) \times (0,T), \\ -(\overline{u}^m)_x(0,t) \geq \overline{u}^p(0,t) & t \in (0,T), \\ (\overline{u}^m)_x(L,t) \geq -\overline{u}^q(L,t) & t \in (0,T), \\ \overline{u}(x,0) > u_0(x) & x \in (0,L). \end{cases} \tag{5}$$

$ii)$ We call $\underline{u}$ a subsolution of (1) if it satisfies (5) with the opposite inequalities.

LEMMA 3.1. *Let $\overline{u}$, u and $\underline{u}$ be, respectively, a positive supersolution, a solution and a positive subsolution to problem* (1). *Then*

$$\overline{u} > u > \underline{u} \qquad (x,t) \in (0,L) \times [0,T).$$

Proof. We argue by contradiction (see [18] for a similar argument). Suppose that there exists a time t_0 such that $\overline{u}(x_0,t_0) = u(x_0,t_0)$ for a certain $x_0 \in [0,L]$. As $\overline{u}(x,0) > u(x,0)$ we have that $t_0 > 0$, thanks to the continuity of $\overline{u}$ and u down to $t = 0$.

If $x_0 \in (0,L)$ then we have a solution and a supersolution of the porous medium equation such that the difference attains an interior minimum. This contradicts the strong maximum principle.

Next we show that x_0 cannot be 0 nor L. Indeed, the boundary conditions imply

$$\frac{\partial(\overline{u} - u)}{\partial \eta}(x_0,t_0) \geq 0,$$

where η is the outwards normal. On the other hand, the function $(\overline{u} - u)(\cdot,t_0)$ is non-negative, so it has a minimum point at x_0. Moreover it satisfies

$$(\overline{u} - u)_t \geq (\overline{u}^m - u^m)_{xx} = m(\phi^{m-1}(\overline{u} - u))_{xx}$$

where $\overline{u} \geq \phi \geq u$. As $\overline{u}$ is strictly positive, $\phi(x_0,t_0)$ cannot vanish. Hence we can apply Hopf's Lemma to obtain

$$\frac{\partial(\overline{u} - u)}{\partial \eta}(x_0,t_0) < 0,$$

which is a contradiction.

Analogous arguments show that $u > \underline{u}$. ∎

Next we proceed with the proof of Theorem 1.1. As the arguments that we use depend strongly on the region of parameters under consideration, we organize the proof in several lemmas.

LEMMA 3.2. *In* $\mathcal{B}_1 = \{p > m > 1\}$ *there exist blowing up solutions.*

Proof. We use an auxiliary problem. For $\tilde{q} > q$ we set

$$\begin{cases} (u_k)_t = (u_k^m)_{xx} & (x,t) \in (0,L) \times (0,T_k), \\ -(u_k^m)_x(0,t) = u_k^p(0,t) & t \in (0,T_k), \\ (u_k^m)_x(L,t) = -ku_k^{\tilde{q}}(L,t) & t \in (0,T_k), \\ u_k(x,0) = u_{k0}(x) & x \in (0,L), \end{cases} \tag{6}$$

where u_{k0} will be chosen later.

If $p > m > 1$ it is known that problem (3) admits a self-similar solution

$$w(x,t) = (T-t)^{\alpha} F(\xi), \qquad \xi = \frac{x}{(T-t)^{\beta}},$$

with exponents $\alpha = -1/(2p-m-1) < 0$, $\beta = (p-m)/(2p-m-1) > 0$, see [10], [12] and [13]. This solution has support $(0,+\infty)$ for all time and blows up in finite time only at the point $x = 0$, see [18]. Moreover, for all $x > 0$ the function $w_x(x,t)$ is bounded for $t \in [0,T)$. Therefore, there exists a small positive constant k such that $(w^m)_x(L,t) \le -kw^{\tilde{q}}(L,t)$. Hence, if we take $u_{k0}(x) \ge w(x,0)$ then w is a subsolution of problem (6). So $u_k(x,t)$ blows up in finite time for every u_{k0} large. We can assume that the blow-up time T_k is as small as we want (just take T small). Hence we can assume, by taking u_{k0} large, that given $M > 0$, then $u_k(L,t) \ge M$ for all $t \in [0,T_k)$. To end the proof, we observe that $-ku_k^{\tilde{q}}(L,t) \le -u_k^q(L,t)$ holds if $kM^{\tilde{q}-q} \ge 1$. This implies that u_k is a subsolution of (1) with finite time blow-up. Hence every solution above this u_k has finite time blow-up. ∎

Remark 3. 1. The previous arguments extend to the case $0 < m < 1$ with $p > (m+1)/2$ if we can show the existence of a solution of (6) with finite time blow-up. To this end we use the results of [8]. In that paper it is proved that solutions of problem (1) in $[0,2L]$ with boundary condition at $x = 2L$ given by $(u^m)_x(2L,t) = 0$ blow up in finite time. Moreover, using as a barrier a singular solution of the form $U(x,t) = t^{1/(1-m)} f(x)$, where $f(0) = +\infty$, $f'(2L) = 0$ and $(f^m)'' + (1/(1-m))f = 0$ for $0 < x < 2L$, cf. [2], we can prove that the blow-up set is $B(u) = \{0\}$. Now take a decreasing in space solution. As it blows up only at $x = 0$ we have that $u(L,t)$ and $-(u^m)_x(L,t)$ are bounded and strictly positive. Hence v is a subsolution for problem (6). To show that this implies that (1) has solutions with finite time blow-up we just have to proceed as before.

The case $m = 1$ can be handled in an analogous way using that the problem in the half line has single point blow-up, [14], [19].

LEMMA 3.3. *In* $\mathcal{B}_2 = \{q \le p = m\} \cup \{q > p = m,\ L \ge 2m/(m-1)\}$ *there exist blowing up solutions.*

Proof. For $p = m$ there exists a self-similar solution w of (3), which blows up in finite time and has compact support, see [10], [12], [13] or [21]. It has the explicit form

$$w(x,t) = (T-t)^{-1/(m-1)} \left(\frac{m-1}{2m(m+1)}\right)^{\frac{1}{m-1}} \left(\frac{2m}{m-1} - x\right)_+^{\frac{2}{m-1}}.$$

If $L \ge 2m/(m-1)$, this solution is also a solution of our problem.

If $q \le p = m$ and $L < 2m/(m-1)$, a straightforward computation shows that $w(x,t)$ as above is a subsolution of our problem if we take

$$T \le \left(\frac{m-1}{2m(m+1)}\right) \left(\frac{2m}{m-1} - L\right)^2.$$

∎

LEMMA 3.4. *In* $\mathcal{B}_3 \cup \mathcal{B}_4 = \{1 < p < m,\ p > q\} \cup \{p = q,\ (m+1)/2 < p < m\}$ *there exist positive solutions with finite time blow-up.*

Proof. Using ideas from [22], we propose as the desired subsolution a function $w(x,t)$ of the form

$$w(x,t) = \varphi(\sigma), \qquad \sigma(x,t) = \alpha(x) + \beta(t), \tag{7}$$

where φ is a solution of the ODE

$$\varphi'(s) = \varphi^{p-m+1}(s). \tag{8}$$

First we deal with the case $1 < p < m$. So we have that

$$\varphi(\sigma) = (m-p)^{\frac{1}{m-p}} (\sigma + C)^{\frac{1}{m-p}}.$$

We will choose α and β such that w is a subsolution of (1), and β blows up at a finite time. The second condition ensures that w blows up in finite time.

We begin by the verification of the boundary condition at $x = 0$, that is, $-(w^m)_x(0,t) \le w^p(0,t)$. This is equivalent to

$$-m\varphi^{m-1}(\sigma(0,t))\varphi'(\sigma(0,t))\alpha'(0) \le \varphi^p(\sigma(0,t)).$$

Using that φ is a solution of (8) we arrive to the condition

$$-\alpha'(0) \le 1/m. \tag{9}$$

Next, let us check the inequality $w_t \le (w^m)_{xx}$. In terms of φ, α and β it reads,

$$\begin{aligned}\varphi'(\sigma)\beta'(t) \le\ & m(m-1)\varphi^{m-2}(\sigma)(\varphi')^2(\sigma)(\alpha'(x))^2+ \\ & m\varphi^{m-1}(\sigma)\varphi''(\sigma)(\alpha'(x))^2 + m\varphi^{m-1}(\sigma)\varphi'(\sigma)\alpha''(x).\end{aligned}$$

Using again that φ is a solution of (8), this is equivalent to

$$\begin{aligned}\beta'(t) \le\ & m(m-1)\varphi^{p-1}(\sigma)(\alpha'(x))^2+ \\ & m\varphi^{p-1}(\sigma)(\alpha'(x))^2 + m\varphi^{m-1}(\sigma)\alpha''(x).\end{aligned}$$

If we impose that

$$\alpha''(x) \ge c > 0, \tag{10}$$

the last term dominates as $\sigma \to \infty$. Thus we can take as β a solution of

$$\beta'(t) = c(\beta(t))^{\frac{m-1}{m-p}}$$

that blows up in finite time. The necessary and sufficient condition for the blow-up of β is then $(m-1)/(m-p) > 1$, which is equivalent to $p > 1$.

We still have to look at the boundary condition at $x = L$, $(w^m)_x(L,t) \le -w^q(L,t)$. In terms of φ and α this is equivalent to

$$m\varphi^{m-1}(\sigma(L,t))\varphi'(\sigma(L,t))\alpha'(L) \le -\varphi^q(\sigma(L,t)).$$

Using that φ is a solution of (8) we get

$$\alpha'(L) \le -\varphi^{q-p}(\sigma(L,t))/m. \tag{11}$$

To end this part of the proof we take a function α satisfying (9) and (10). Then we take $\varphi(0)$ such that condition (11) is fulfilled. This is always possible if $p > q$, as the function φ is increasing and unbounded.

In the case $p = q$ we choose $\alpha(x) = ax + b$ with $a = -1/m$ so that (9) and (11) are satisfied. In this case the condition for β reads

$$\beta'(t) \le m(m-1)\varphi^{p-1}(\sigma)(\alpha'(x))^2 + m\varphi^{p-1}(\sigma)(\alpha'(x))^2,$$

and we can choose β as a solution of

$$\beta'(t) = a^2(\beta(t))^{\frac{p-1}{m-p}}$$

that blows up in finite time. The necessary and sufficient condition for the blow-up of β is then $(p-1)/(m-p) > 1$, which is equivalent to $p > (m+1)/2$. ∎

Now we turn our attention to global existence results. In the next lemma we use the same technique as in Lemma 3.4.

LEMMA 3.5. *In* $\mathcal{G}_1 = \{p = q,\ 1 < p \le (m+1)/2\}$ *every solution is global.*

Proof. Now we construct supersolutions. Again we choose a function of the form (7) where φ is a solution of (8). Then, proceeding as before, we have that if $p = q$ and $1 < p \le (m+1)/2$ we can choose $\alpha(x) = ax + b$ and β in order to have a global supersolution with initial data as large as we want (we only have to observe that we can choose $\varphi(0)$ large). Indeed, we can take β as a solution of

$$\beta'(t) = C(\beta(t))^{\frac{p-1}{m-p}}$$

that exists globally. This is possible if and only if $(p-1)/(m-p) \le 1$, that is, if and only if $p \le (m+1)/2$. ∎

LEMMA 3.6. *In* $\mathcal{G}_2 = \{p < q,\ 1 < p < m\} \cup \{q > p = m,\ L < 2m/(m-1)\}$ *every solution is global.*

Proof. We look for a stationary supersolution of the form $u^m(x,t) = w(x) = ax + b$ with b large enough. From the boundary conditions we obtain that we need $-a \ge b^{p/m}$ and $a \ge -(aL+b)^{q/m}$. We set $-a = b^{p/m}$. Hence b must satisfy

$$b - b^{p/q} - Lb^{p/m} \ge 0. \tag{12}$$

If $p < q$ and $p < m$ then (12) is satisfied for b large. We remark that this supersolution is as large as we want. Hence, we can choose b such that $w^{1/m}(x) > u_0(x)$. Therefore $u(x,t)$ must be global.

If $p = m$ and $q > p$, the inequality (12) reads $(1-L)\,b - b^{p/q} \ge 0$. Thus, if $L < 1$ there exists a stationary supersolution as large as we want.

If $1 \le L < 2m/(m-1)$ we need a more subtle argument. First we note that in this range there are no stationary solutions. Hence, thanks to Theorem 1.3 (see proof below), if there is a nontrivial global solution then all solutions are global.

In order to find a global solution we choose a nontrivial initial data such that $u_t \ge 0$. Any smooth and compatible u_0 such that $(u_0^m)_{xx} \ge 0$ will do the job.

Now we prove that in this range of parameters any solution satisfying $u_t \ge 0$ is global. Indeed, if it blows up, then the blow-up set is $B(u) = [0, L]$ and, moreover, it blows up with the same rate at all points in the interval $[0, L]$,

$$u(x,t) \sim c(x)(T-t)^{-\alpha}, \qquad c(x) > 0,\ \alpha = 1/(2p-m-1),$$

see Section 5. Integrating the equation in $[0, L]$ and using the boundary conditions we get that

$$\frac{d}{dt}\int_0^L u(x,t)\,dt = -u^q(L,t)+u^p(0,t) \sim -c^q(L)(T-t)^{-\alpha q}+c^p(0)(T-t)^{-\alpha p}.$$

As $q > p$, the right hand side becomes negative for t close to T. But this contradicts the monotonicity of u. ∎

LEMMA 3.7. *In $\mathcal{G}_3 = \{p \le 1\}$ every solution is global.*

Proof. If $p \le 1$, every solution of (2) is global in time, [1], [8]. A comparison argument shows that the same is true for solutions of (1). ∎

Remark 3. 2. If $0 < m \le 1$ with $p \le (m+1)/2$, every solution of problem (2) is global, [8], [23]. Again a comparison argument shows that the same is true for solutions of (1).

To end this section we prove Theorem 1.3.

Proof of Theorem 1.3. It is enough to prove the result for solutions that are decreasing in x. Then a comparison argument will prove that it holds true for general solutions. In order to arrive to a contradiction, we assume that there exists a global, decreasing in x, nontrivial solution, $u(x,t)$, in a region of parameters where there is blow-up and no nontrivial stationary solution. Comparison with a suitable Barenblatt solution shows that u will eventually become positive in $[0, L]$.

As we are in a blow-up region and we have a comparison principle, if u becomes large enough in all $[0, L]$ it will blow up.

We claim that if u does not become large in all $[0, L]$ then $u(0,t)$ must be bounded. If, on the contrary, $u(0,t)$ is not bounded, then for any M large enough there is a time t_0 such that $u(0,t_0) \ge M$. Hence we can use an intersection comparison argument between u and a stationary solution w given by $w^m(x) = -M^{p/m}x + M^m$ to prove that $u(0,t) \ge M$ for all $t \ge t_0$. Indeed, if M is large enough we can assume that w and u_0 have only one intersection. As the number of intersections cannot increase, at time $t = t_0$ we must have $u(x,t_0) \ge w(x)$ and therefore $u(0,t) \ge M$ for all $t > t_0$. But this would imply (see below) that $u(L,t)$ is as big as we want, a contradiction.

To prove this last assertion, we note that if $u(L,t)$ is bounded, then $u(x,t)$ is a supersolution of

$$\begin{cases} u_t = (u^m)_{xx} & (x,t) \in (0,L) \times (0,+\infty), \\ u^m(0,t) = M^m & t \in (0,+\infty), \\ (u^m)_x(L,t) = -C & t \in (0,+\infty), \\ u(x,0) = u_0(x) & x \in (0,L). \end{cases}$$

A standard argument, using a Lyapunov functional, shows that solutions of the latter problem converge to the stationary solution $w^m(x,t) = -Cx + M^m$. Hence $u(L,t)$ becomes big, a contradiction.

Once we know that $u(x,t)$ is uniformly bounded, we can multiply the equation by $(u^m)_t$ and integrate by parts to arrive to

$$m\int_0^L u^{m-1}(u_t)^2\,dx = \int_0^L (u^m)_t u_t\,dx = \int_0^L (u^m)_t (u^m)_{xx}\,dx$$
$$= -\frac{d}{dt}\left(\frac{1}{2}\int_0^L (u^m)_x^2\,dx + \frac{m}{p+m}u^{p+m}(0,t) - \frac{m}{q+m}u^{q+m}(L,t)\right).$$

That is, we have a Lyapunov functional. As u is uniformly bounded, it must converge as $t\to\infty$ to a stationary state. However, there is only one, $w\equiv 0$. We just have to discard this possibility. In order to do this, we construct a small enough subsolution, preventing u going to $w\equiv 0$. The subsolution is $\underline{u}^m(x) = (ax+b)_+$, with $-a = b^{p/m}$. In $x = L$ we need $b - b^{p/q} - Lb^{p/m} \le 0$. This is true for all small $b > 0$ in the three cases in which there are no stationary solutions, namely, if (i) $p = q$, or (ii) $p = m$ with $L \ge 1$ or (iii) in the region $\mathcal{E}_3$ with $L > L^*$. The subsolution $\underline{u}$ is nontrivial in $[0, L]$ for all small $b > 0$. Indeed, it is positive for all $x < 1/b^{(p/m)-1}$. ∎

4. BLOW-UP RATE

In this section, we prove Theorem 1.4. We are assuming that the solution $u(x,t)$ of (1) satisfies hypothesis (H).

A consequence of hypothesis (H) is that $(u^m)_{xx} \ge 0$ and $u_x \le 0$. Therefore, we have the following properties

$$\begin{aligned} u(0,t) &= \max_{x\in[0,L]} u(x,t),\\ (u^m)_x(L,t) &= \max_{x\in[0,L]} (u^m)_x(x,t),\\ (u^m)_x(0,t) &= \min_{x\in[0,L]} (u^m)_x(x,t). \end{aligned} \tag{13}$$

LEMMA 4.1. *In* $\mathcal{B}_3 = \{1 < p < m,\ p > q\}$ *there exist constants* $C_1, C_2 > 0$ *such that*

$$C_1(T-t)^{-\frac{1}{p-1}} \le u(x,t) \le C_2(T-t)^{-\frac{1}{p-1}} \qquad \forall x\in[0,L]. \tag{14}$$

Proof. We start by proving that

$$\frac{1}{2}u(0,t) \le u(x,t) \le u(0,t) \qquad \forall x\in[0,L]. \tag{15}$$

The second inequality is an immediate consequence of (13). To prove the first one we apply (13) and the mean value theorem to get that for t near T,

$$u^m(x,t) - u^m(0,t) = x(u^m)_x(\xi,t) \geq x(u^m)_x(0,t) = -xu^p(0,t).$$

Thus, $u^m(x,t) \geq u^m(0,t)\,(1 - xu^{p-m}(0,t))$. Since $p - m < 0$ and $u(0,t)$ blows up with $u_t(0,t) \geq 0$, we obtain that $1 - Lu^{p-m}(0,t) \geq 1/2^m$ for t close to T. Hence (15).

Thanks to (15), in order to prove (14) it is enough to obtain analogous estimates for $u(0,t)$. We start by proving the upper estimate. We have

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L u(s,t)\,ds\right) &= \int_0^L u_t(s,t)\,ds = \int_0^L (u^m)_{xx}(s,t)\,ds \\ &= (u^m)_x(L,t) - (u^m)_x(0,t) = u^p(0,t) - u^q(L,t) \\ &\geq u^p(0,t)(1 - u^{q-p}(0,t)).\end{aligned}$$

By (H) we can assume that $u(0,t) \geq 2$ (this is valid at least for t near T). Since $q - p < 0$ we obtain

$$\frac{d}{dt}\left(\int_0^L u(s,t)\,ds\right) \geq \delta u^p(0,t) \geq \frac{\delta}{L^p}\left(\int_0^L u(s,t)\,ds\right)^p.$$

Integrating this inequality between t and T, we get

$$\int_0^L u(s,t)\,ds \leq C(T-t)^{-\frac{1}{p-1}}.$$

This inequality implies the upper estimate, as

$$\frac{1}{2}u(0,t) \leq u(L,t) \leq \frac{1}{L}\int_0^L u(s,t)\,ds.$$

To prove the lower estimate, we observe that

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L u(s,t)\,ds\right) &= u^p(0,t) - u^q(L,t) \leq u^p(0,t) \\ &\leq C\left(\int_0^L u(s,t)\,ds\right)^p.\end{aligned}$$

Integrating this inequality between t and T we get

$$\int_0^L u(s,t)\,ds \geq C(T-t)^{-\frac{1}{p-1}}.$$

This inequality implies the lower estimate, because

$$u(0,t) \geq \frac{1}{L}\int_0^L u(s,t)\,ds \geq C_1(T-t)^{\frac{1}{p-1}}.$$

∎

LEMMA 4.2. *In $\mathcal{B}_4 = \{p = q,\ (m+1)/2 < p < m\}$ there exist constants $C_1, C_2 > 0$ such that*

$$C_1(T-t)^{-\frac{1}{2p-m-1}} \leq u(x,t) \leq C_2(T-t)^{-\frac{1}{2p-m-1}} \qquad \forall x \in [0,L]. \tag{16}$$

Proof. It is easy to see that the arguments leading to (15) are also valid in this case. Hence,

$$\begin{aligned}
\frac{d}{dt}\left(\int_0^L u(s,t)\,ds\right) &= u^p(0,t) - u^p(L,t) \\
&= -Lp\,u^{p-1}(\xi,t)u_x(\xi,t) = -C\,u^{p-m}(\xi,t)(u^m)_x(\xi,t) \\
&\geq Cu^{2p-m}(0,t) \geq C\left(\int_0^L u(s,t)\,ds\right)^{2p-m}.
\end{aligned}$$

Integrating this inequality between t and T, we get

$$\int_0^L u(s,t)\,ds \leq C(T-t)^{-\frac{1}{2p-m-1}},$$

from where the upper estimate for $u(0,t)$ follows.

In order to prove the lower estimate, we observe that

$$\begin{aligned}
\frac{d}{dt}\left(\int_0^L u(s,t)\,ds\right) &= -Lp\,u^{p-1}(\xi,t)u_x(\xi,t) \leq Cu^{p-m}(L,t)u^p(0,t) \\
&\leq Cu^{2p-m}(L,t) \leq C\left(\int_0^L u(s,t)\,ds\right)^{2p-m}.
\end{aligned}$$

Integrating between t and T we get

$$\int_0^L u(s,t)\,ds \geq C(T-t)^{-\frac{1}{2p-m-1}},$$

and hence the lower estimate for $u(0,t)$. ∎

LEMMA 4.3. *In $\mathcal{B}_1 \cup \mathcal{B}_2 = \{p \geq m > 1\}$ there exist $C_1, C_2 > 0$ such that*

$$C_1(T-t)^{-\frac{1}{2p-m-1}} \leq u(0,t) \leq C_2(T-t)^{-\frac{1}{2p-m-1}}. \tag{17}$$

Proof. In this proof we follow the techniques given in [14]–[18] (see these papers for details).

We define $M(t) = \max u(\cdot, t) = u(0,t)$ and the function

$$\phi_M(r,s) = \frac{1}{M(t)} u(ar, bs+t),$$

where $a = M^{m-p}$, $b = M^{m+1-2p}$. If $p > m$ then a and b go to zero as $t \to T$; if $p = m$, then $a = 1$ and b goes to zero.

The function ϕ_M is a solution of the following problem:

$$\begin{cases} (\phi_M)_s = (\phi_M^m)_{rr} & (r,s) \in (0, \frac{L}{a}) \times (-\frac{t}{b}, 0), \\ -(\phi_M^m)_r(0,s) = \phi_M^p(0,s) & s \in (-\frac{t}{b}, 0), \\ (\phi_M^m)_r(\frac{L}{a}, s) = -M^{q-p}\phi_M^q(\frac{L}{a}, s) & s \in (-\frac{t}{b}, 0). \end{cases}$$

Moreover, using that $u_t \geq 0$, we get that $0 \leq \phi_M \leq 1$ and $\phi_M(0,0) = 1$. Since the functions ϕ_M are uniformly bounded solutions of the porous medium equation, we have that every sequence $\phi_{M_j}(r,s)$ is equicontinuous, *cf.* [6]. Then, $\phi_{M_j}(r,s) \to \Phi(r,s)$ as $M_j \to \infty$ uniformly on compact sets.

We claim that there exist two positive constants c and C such that

$$c \leq (\phi_M)_s(0,0) \leq C.$$

Now, if we rewrite these inequalities in terms of $M(t)$, we obtain

$$c \leq M^{m-2p}(t) M'(t) \leq C.$$

Integrating and taking into account that $M(t) = u(0,t)$, we get

$$C_1(T-t)^{-\frac{1}{2p-m-1}} \leq u(0,t) \leq C_2(T-t)^{-\frac{1}{2p-m-1}}.$$

In order to finish the proof we have to prove the claim. The upper bound follows by regularity theory, see [16], [18] for details. To obtain the lower estimate we suppose that there exists a sequence such that

$$(\phi_{M_j})_s(0,0) \to 0.$$

This implies that $\Phi_s(0,0) = 0$.

On the other hand, since $u_t \geq 0$, the function $w = \Phi_s$ is non-negative and satisfies

$$w_s = (m\Phi^{m-1}w)_{rr}.$$

Moreover, at $r = 0$,

$$-(m\Phi^{m-1}w)_r(0, s) = p\Phi^{p-1}w(0, s).$$

Observe that $\Phi(0, 0) > 0$ and that w has a minimum at $(0, 0)$. So by Hopf's Lemma we obtain that $w \equiv 0$. Then Φ is a stationary solution of the porous medium equation, $\Phi^m(r) = c_1 r + c_2$. But $\Phi(0) = 1$, and from the boundary condition at $x = 0$, $-(\Phi^m)_r(0) = \Phi^p(0) = 1$. Therefore, $\Phi^m(r) = 1 - r$. Hence, if $p > m$ we arrive to a contradiction with the fact that Φ is a nonnegative function in all $(0, +\infty)$. If $p = m$, Φ verifies at $x = L$

$$-1 = (\Phi^m)_r(L, s) = \begin{cases} 0 & p > q, \\ L - 1 & p = q, \\ -\infty & p < q, \end{cases}$$

which is a contradiction. Then the claim is proved and the Theorem follows. ▮

Remark 4. 1. The same proof applies if $0 < m \leq 1$ with $p > (m + 1)/2$.

5. BLOW-UP SETS

In this section we study the blow-up sets for blowing up solutions. As in the previous section we impose that u satisfies hypothesis (H).

LEMMA 5.1. *In* $\mathcal{B}_3 \cup \mathcal{B}_4 = \{1 < p < m,\ p > q\} \cup \{p = q,\ (m + 1)/2 < p < m\}$ *the blow-up set is* $B(u) = [0, L]$ *(global blow-up).*

Proof. It follows immediately from (14) and (16). ▮

LEMMA 5.2. *In* $\mathcal{B}_1 = \{p > m > 1\}$ *we have single point blow-up, i.e.,* $B(u) = \{0\}$.

Proof. We consider the problem

$$\begin{cases} v_t = (v^m)_{xx} & (x, t) \in (0, L) \times (0, T), \\ v(0, t) = C_2(T - t)^{-\frac{1}{2p-m-1}} & t \in (0, T), \\ (v^m)_x(L, t) = 0 & t \in (0, T), \\ v(x, 0) \geq u_0(x) & x \in [0, L], \end{cases}$$

where C_2 is the constant in the estimate (17). In [11] it is proved that $B(v) = \{0\}$. As u is a subsolution of this problem, the result follows. ▮

Remark 5. 1. If $0 < m < 1$ we can perform comparison with the supersolution given in [2], see Section 3. This proves that in this case the blow-up set is $B(u) = \{0\}$. For the case $m = 1$ we refer to [14], [19].

We still have to deal with the case $p = m$. In this case we characterize the blow-up profile and obtain the exact blow-up set as a consequence.

To obtain the asymptotic behaviour we rescale the solutions according to the blow-up rate (17),

$$v(x,\tau) = (T-t)^{\frac{1}{m-1}} u(x,t), \qquad \tau = -\ln(T-t).$$

We remark that, as $u(x,t) \le u(0,t) \le C_2(T-t)^{-1/(m-1)}$, then $v(x,\tau) \le C_2$. The function $v(x,\tau)$ satisfies the following problem,

$$\begin{cases} v_\tau = (v^m)_{xx} - \frac{1}{m-1} v & (x,\tau) \in (0,L) \times (-\ln T, +\infty), \\ -(v^m)_x(0,\tau) = v^m(0,\tau) & \tau \in (-\ln T, +\infty), \\ (v^m)_x(L,\tau) = -e^{\frac{q-m}{m-1}\tau} v^q(L,\tau) & \tau \in (-\ln T, +\infty), \\ v(x, -\ln T) = T^{\frac{1}{m-1}} u_0(x) & x \in (0,L). \end{cases} \tag{18}$$

In order to obtain information on the behaviour of u near $t = T$ we will study the behaviour of $v(x,\tau)$ as $\tau \to \infty$. As expected, stationary solutions related to (18) will play an outstanding role. In the Appendix, we study the corresponding stationary problems.

We will also need to know something about the decay of solutions to (18).

LEMMA 5.3. *Let v a solution of (18) that goes to zero in an interval of the form $[a-\delta, b+\delta]$. Then there exists a constant C such that*

$$v(x,\tau) \le C e^{-\frac{1}{m-1}\tau} \qquad x \in [a,b].$$

and this implies that $u(x,t) \le C$ for all $x \in [a,b]$.

Proof. We use ideas from [4], [21]. Since $v(x,\tau)$ goes to zero in $[a-\delta, b+\delta]$, we can use a solution of

$$\begin{cases} p_\tau = (p^m)_{xx} - \frac{1}{m-1} p & (x,\tau) \in [a-\delta, b+\delta] \times (0,+\infty), \\ p(a-\delta,\tau) = \varepsilon & \tau \in (0,+\infty), \\ p(b+\delta,\tau) = \varepsilon & \tau \in (0,+\infty), \\ v(x,0) = \varepsilon & x \in (a-\delta, b+\delta). \end{cases} \tag{19}$$

as a supersolution of our problem.

To finish the proof we only have to observe that solutions p of (19) go exponentially, with speed $e^{-\frac{1}{m-1}\tau}$, to a stationary solution that has the form

$$h(x) = \begin{cases} C(m)(a-x)^{\frac{2}{m-1}} & x < a, \\ 0 & a \le x \le b, \\ C(m)(x-b)^{\frac{2}{m-1}} & x > b. \end{cases}$$

If ε is small, $h(x)$ has a dead core including the interval $[a, b]$. The result follows. ∎

Now we have all the tools we need to complete our study of the blow-up sets.

LEMMA 5.4. *In the region* $\mathcal{B}_2 = \{q \le p = m\} \cup \{q > p = m,\ L \ge 2m/(m-1)\}$ *the blow-up set is given by*

$$B(u) = \begin{cases} [0, L] & \text{if } L < 2m/(m-1), \\ [0, 2m/(m-1)] & \text{if } L \ge 2m/(m-1). \end{cases}$$

Proof. First, we study the case $q < p = m$. Multiplying the equation in (18) by $(v^m)_\tau$ and integrating in x we obtain

$$\int_0^L (v^m)_\tau v_\tau \, dx = -v^q (v^m)_\tau (L, \tau) e^{-\frac{m-q}{m-1}\tau} - \frac{d}{dt} F(v)(\tau),$$

where

$$F(v)(\tau) = \int_0^L \frac{(v^m)_x^2}{2}(x, \tau) \, dx + \frac{1}{2} v^{2m}(0, \tau).$$

If $q < p = m$ we observe that the exponential term that appears at the boundary condition has a negative exponent and therefore is integrable in time. Hence it is easy to verify that $F(v)$ is a Lyapunov functional for problem (18). Therefore, we have that the $\omega-$limit set of $w(x, \tau)$ consists of nontrivial solutions of the following problem,

$$\begin{cases} (w^m)_{xx} - \frac{1}{m-1} w = 0 & x \in (0, L), \\ -(w^m)_x(0) = w^m(0), \\ (w^m)_x(L) = 0. \end{cases} \tag{20}$$

This problem has a unique solution, see Lemma A.1 in the Appendix. Hence we conclude that

$$v(x, \tau) \to w(x) \qquad \tau \to \infty.$$

In case $L < 2m/(m-1)$, as the solution of (20) is positive in the whole interval $[0, L]$, we obtain that $u(x, t)$ has global blow-up. If $L = 2m/(m-1)$

the limit w is explicit and is given by (A.3). Hence we have global blow-up. Finally, if $L > 2m/(m-1)$, $v(x,\tau)$ the limit is again given by (A.3). Hence, $[0, 2m/(m-1)) \subset B(u)$. To prove that in this case $B(u) = [0, 2m/(m-1)]$ (regional blow-up) we observe that $v(x,\tau)$ goes to zero exponentially with the precise rate $v(x,\tau) \le Ce^{-\frac{1}{m-1}\tau}$ in every compact set included in $(2m/(m-1), L]$, Lemma 5.3.

The case $q = p = m$ can be handled in an analogous way. The differences that appear are the following. First, in the problem in self-similar variables the exponential term at the boundary does not appear,

$$\begin{cases} v_\tau = (v^m)_{xx} - \frac{1}{m-1}v & (x,\tau) \in (0,L) \times (-\ln T, +\infty), \\ -(v^m)_x(0,\tau) = v^m(0,\tau) & \tau \in (-\ln T, +\infty), \\ (v^m)_x(L,\tau) = -v^m(L,\tau) & \tau \in (-\ln T, +\infty), \\ v(x, -\ln T) = T^{\frac{1}{m-1}} u_0(x) & x \in (0,L). \end{cases}$$

In this case we define the Lyapunov functional by

$$F(v)(\tau) = \int_0^L \frac{(v^m)_x^2}{2}(x,\tau)\,dx + \frac{1}{2}v^{2m}(0,\tau) - \frac{1}{2}v^{2m}(L,\tau),$$

and obtain that

$$\int_0^L (v^m)_\tau v_\tau \,dx = -\frac{d}{d\tau}F(v)(\tau).$$

The limit stationary problem now becomes

$$\begin{cases} (w^m)_{xx} - \frac{1}{m-1}w = 0 & x \in (0,L), \\ -(w^m)_x(0) = w^m(0), \\ (w^m)_x(L) = -w^m(L). \end{cases}$$

Using Lemma A.1 we know that if $L \ge 2m/(m-1)$ this problem has a unique compactly supported solution while if $L < 2m/(m-1)$ every solution is positive. Then we can conclude the argument exactly as in the case $q < p = m$, using Lemma 5.3.

In the case $q > p = m$ with $L \ge 2m/(m-1)$, we first prove that $(v^m)_x(L,\tau)$ is bounded. To do so we observe that hypothesis (H) implies that $(v^m)_x(\cdot,\tau)$ is a non-decreasing function. Moreover it is negative and bounded at $x = 0$. Then it is bounded for all $x \in [0, L]$.

The boundedness of $(v^m)_x(L,t)$ implies that $v^q(L,\tau)e^{\frac{q-m}{m-1}\tau} \le C$. Though the exponential term that appears in the boundary condition has a positive exponent, the bound $v^q(L,\tau) \le Ce^{-\frac{q-m}{m-1}\tau}$ allows us to proceed as in the case $q < p = m$ to obtain that $v(x,\tau) \to w(x)$ as $\tau \to \infty$. Then we can conclude that the blow-up set $B(u)$ is the interval $[0, 2m/(m-1)]$. ∎

APPENDIX

In this Appendix we study the stationary problems

$$\begin{cases} (w^m)_{xx} - \frac{1}{m-1}w = 0 & x \in (0,L), \\ -(w^m)_x(0) = w^m(0), \\ (w^m)_x(L) = 0, \end{cases} \tag{A.1}$$

$$\begin{cases} (w^m)_{xx} - \frac{1}{m-1}w = 0 & x \in (0,L), \\ -(w^m)_x(0) = w^m(0), \\ (w^m)_x(L) = w^m(L). \end{cases} \tag{A.2}$$

LEMMA A.1. *Let $m > 1$. The stationary problems* (A.1) *and* (A.2) *have a unique solution, which is compactly supported if $L > 2m/(m-1)$ and positive if $L \leq 2m/(m-1)$.*

Proof. In both problems the proof follows the same arguments. Therefore we only study problem (A.1).

If $L > 2m/(m-1)$ the problem has a unique compactly supported solution, [21],

$$w(x) = \left(\frac{m-1}{2m(m+1)}\right)^{\frac{1}{m-1}} \left(\frac{2m}{m-1} - x\right)_+^{\frac{2}{m-1}}. \tag{A.3}$$

If $L = 2m/(m-1)$, then (A.3) is also a solution. However, it is not compactly supported, but positive in $(0, 2m/(m-1))$. If $L < 2m/(m-1)$ there is no compactly supported solution.

To study the existence of other solutions of (A.1) we use ideas from [3]. We set $\tilde{w} = w^m$ and use a shooting argument beginning with $\tilde{w}(L) = k$, $\tilde{w}_x(L) = 0$. Integrating the equation we get

$$\frac{\tilde{w}_x^2}{2} - \frac{m}{m^2-1}\tilde{w}^{\frac{1+m}{m}} = -\frac{m}{m^2-1}k^{\frac{1+m}{m}}.$$

Evaluating at $x = 0$ and using the boundary condition, we observe that $\tilde{w}(0)$ must be a root of the following function

$$P(z) = \frac{z^2}{2} - \frac{m}{m^2-1}z^{\frac{1+m}{m}} + \frac{m}{m^2-1}k^{\frac{1+m}{m}}.$$

As,

$$P'(z) = z - \frac{1}{m-1}z^{\frac{1}{m}},$$

P attains a unique minimum at the point

$$z_0 = \left(\frac{1}{m-1}\right)^{\frac{m}{m-1}}.$$

In order to guarantee that $P(z)$ has a root we need that $P(z_0) \le 0$. This imposes the following restriction on k,

$$k \le k_c = \left(\frac{1}{m-1}\right)^{\frac{m}{m-1}} \left(1 - \frac{m+1}{2m}\right)^{\frac{m}{m+1}} \le 1.$$

If $k < k_c$ we have two roots $R_1(k)$, $R_2(k)$. As $P(k) \ge 0$ and $P'(k) \le 0$ we conclude that $k \le R_1(k) \le R_2(k)$. On the other hand, if we integrate again the equation we have, using that $\tilde{w}_x \le 0$,

$$\int_k^{\tilde{w}(x)} \frac{ds}{\sqrt{s^{\frac{1+m}{m}} - k^{\frac{1+m}{m}}}} = \sqrt{\frac{2m}{m^2-1}}(L-x).$$

At $x = 0$ this reads

$$\int_k^{\tilde{w}(0)} \frac{ds}{\sqrt{s^{\frac{1+m}{m}} - k^{\frac{1+m}{m}}}} = \sqrt{\frac{2m}{m^2-1}}L.$$

Hence for each root $R_i(k)$ we have a solution of (A.1) in an interval $[0, L_i(k)]$ where $L_i(k)$ is given by

$$L_i(k) = \sqrt{\frac{m^2-1}{2m}}\, k^{\frac{m-1}{2m}} \int_1^{\frac{R_i(k)}{k}} \frac{ds}{\sqrt{s^{\frac{1+m}{m}} - 1}}.$$

Now we observe that, as $R_1(k) \le R_2(k)$ we have that $L_1(k) \le L_2(k)$.

In order to prove the uniqueness of the solution of (A.1) in the interval $[0, L]$ we study the monotonicity of the functions $L_i(k)$.

First we observe that $L_1(k)$ is increasing with k because $R_1(k)/k$ increases with k. Indeed, a direct calculation shows that

$$\frac{d}{dk}\left(\frac{R_i(k)}{k}\right) = -\left(\frac{m-1}{2m}\right) \frac{R_i^2(k)}{k^2((m-1)R_i(k) - R_i^{\frac{1}{m}}(k))},$$

while we recall that $R_1(k) \le (m-1)^{-m/(m-1)}$. To see that $L_2(k)$ is decreasing we just observe, after differentiating and simplifying the resulting expression, that this is equivalent to

$$\psi(x) = -\frac{x}{\sqrt{x^{p+1}-1}\left(m-1-R_2^{\frac{1-m}{m}}\right)} + \int_1^x \frac{dv}{\sqrt{v^{p+1}-1}} \le 0$$

where we have set $x = R_2(k)/k$. In fact one can check that

$$\frac{d}{dk}L_2(k) = \sqrt{\frac{m-1}{2m(m+1)}}\, k^{-\frac{m+1}{2m}}\psi(k).$$

We note that $R_2(k) \geq (m-1)^{-m/(m-1)}$ and also $x(k)$ is a decreasing function of k, so $x(k) \geq x(k_c) = (2m/(m-1))^{m/(m+1)}$. In that range of values it is easy to check that $\psi'(k) < 0$ for $k > 0$ and also that $\psi(0) = 0$. Therefore $\psi(k) < 0$ and $L_2(k)$ is decreasing.

We want to get a bound for $L_2(k)$, $0 \leq k \leq k_c$. As

$$\lim_{k \to 0} R_2(k) = \left(\frac{2m}{m^2-1}\right)^{\frac{m}{m-1}},$$

a direct computation shows that

$$\lim_{k \to 0} L_2(k) = \frac{2m}{m-1}$$

Therefore L_2 attains a maximum at $k = 0$ that is $L_2(0) = 2m/(m-1)$, which is exactly the support of the explicit solution (A.3).

Since $R_1(k_c) = R_2(k_c)$ we have $L_1(k_c) = L_2(k_c)$. Also, as $R_1(k) \to 0$ when $k \to 0$, one can easily check that $L_1(0) = 0$.

We conclude that we have a unique solution of (A.1) given by (A.3) for $L \geq 2m/(m-1)$ and positive for $L < 2m/(m-1)$. ∎

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