LOCALIZATION OF BLOW-UP POINTS FOR A PARABOLIC SYSTEM WITH A NONLINEAR BOUNDARY CONDITION

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The authors localize the blow-up points of positive solutions of the system $u_t = \Delta u$, $v_t = \Delta v$ with conditions $\frac{\partial u}{\partial \eta} = f(v)$, $\frac{\partial v}{\partial \eta} = g(u)$ at the boundary of a bounded smooth domain Ω under some restrictions on f and g and the initial data $(\Delta u_0, \Delta v_0 > c > 0)$.

If Ω is a ball, the hypothesis on the initial data can be removed.

I. Introduction.

In this paper we study the localization of blow-up points of positive solutions of the following problem

$$\left\{ \begin{array}{ll} u_t = \Delta u & \text{in } \Omega \times (0,T) \\ v_t = \Delta v & \text{in } \Omega \times (0,T) \end{array} \right. \tag{1.1}$$

with boundary conditions

$$\left\{ \begin{array}{ll} \dfrac{\partial u}{\partial \eta} = f(v) & \text{in } \partial\Omega \times (0,T) \\ \dfrac{\partial v}{\partial \eta} = g(u) & \text{in } \partial\Omega \times (0,T) \end{array} \right. \tag{1.2}$$

and initial data

$$\left\{ \begin{array}{ll} u(x,0) = u_0(x) & \text{in } \Omega \\ v(x,0) = v_0(x) & \text{in } \Omega \end{array} \right. \tag{1.3}$$

Here Ω is a bounded smooth domain in ^{N}, η is the exterior normal vector and f, g are C^2. Existence and uniqueness of a classical solution up to some time T can be found in [1] (see also [8]). For problem (1.1)-(1.3) it is known that the existence

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of global solutions depends on the growth of certain functions derived from f and g. More concretely , Rossi and Wolanski ([8]) proved that if f and g are such that

$$\int^{\infty} \frac{ds}{(f \circ F^{-1} \circ G)(s)} < +\infty \tag{1.4}$$

then every positive solution of (1.1)-(1.3) blows-up in finite time, say T (here F and G are primitives of f and g, respectively, with $F(0) = G(0) = 0$).

Condition (1.4) is a natural one, because it is equivalent to the blow-up of solutions of the ordinary system $x' = f(y), y' = g(x)$.

By the result of [1] (see [8]), at that time T we have that,

$$\limsup_{t \nearrow T} \left\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \right\} = +\infty.$$

We can deduce that both functions, u and v, go to infinity at time T. In fact, suppose that u remains bounded in $\overline{\Omega} \times [0,T)$. Then v is a solution of

$$\begin{cases} v_t = \Delta v \\ \frac{\partial v}{\partial \eta} \leq K, \end{cases}$$

where K is a bound for $f(u)$, and it is well known that v remains bounded up to time T (see [9] or [4]) and hence T is not the blow-up time. So in our problem we have

$$\lim_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = +\infty \qquad \lim_{t \nearrow T} \|v(\cdot,t)\|_{L^\infty(\Omega)} = +\infty.$$

We deal here with the localization of the blow-up set. We prove that the blow-up only occurs on the boundary of the domain Ω.

The blow-up phenomena for a single heat equation with a nonlinear boundary condition was studied by Levine and Payne ([5]) and by Walter ([9]).

About the localization problem it is known for a single heat equation with boundary condition $\frac{\partial u}{\partial \eta} = f(u)$ that the blow-up set is contained in the boundary of the domain if f is convex and such that $\int^{+\infty} \frac{1}{f} < \infty$ (see [7]) (see also [6] for a localization result in a ball). In the special case where $f(u) = u^p$ Bei Hu and Hong-Ming Yin obtained in [3] the same localization result.

For the system (1.1)-(1.3) with $f(u) = u^p$ and $g(v) = v^q$, K. Deng localized the blow-up points in the special case of radial solutions in the boundary of a ball with p and q greater than 1 ([2]).

In this work we suppose that f and g satisfy: given $\lambda > 0$, there exists $k > 0$ such that

$$\frac{f(s+\lambda)}{f(s)+\lambda} \leq k, \qquad \frac{g(s+\lambda)}{g(s)+\lambda} \leq k. \tag{1.5}$$

For examples that verifies (1.5), take $f(x) = x^p$, $f(x) = e^x$ or $f(x) = x^p \ln(x)$ with $p \geq 1$.

Following the ideas of [7] we obtain a more general result, i.e. we may take f and g convex functions and we may consider Ω as a bounded smooth domain (without any symmetry).

We enounce our theorems :

Theorem 1. *Let Ω be a bounded smooth domain, (u,v) a solution of (1.1)-(1.3). Suppose f and g are convex, $F^{-1}\circ G$ is convex, (1.4) and (1.5) hold and the initial data obey $\Delta u_0, \Delta v_0 \geq c > 0$. Then, given a subdomain $\Omega' \subset\subset \Omega$, there exists a finite constant $K = K(u_0, v_0, \Omega') > 0$ such that*

$$\sup_{0<t<T} \left(\|u(x,t)\|_{L^\infty(\Omega')} + \|v(x,t)\|_{L^\infty(\Omega')} \right) < K, \tag{1.7}$$

where T is the blow-up time of (u,v).

Remark 1. We suspect that condition (1.5) is probably not needed, but we have not been able to remove it so far.

In the case where $\Omega = B(0,1)$ we can remove the hypothesis $\Delta u_0, \Delta v_0 \geq c > 0$ and we obtain the following result.

Theorem 2. *If $\Omega = B(0,1)$ and f, g and Ω' are as in Theorem 1 then conclusion (1.7) holds.*

Remark 2. The solution (u,v) considered in Theorem 2 need not be radial.

Remark 3. As a corollary of these Theorems we obtain the following: if $f(v) = v^p$ and $g(u) = u^q$, $p \geq 1$, $q \geq 1$ and $pq > 1$ (the blow-up condition ([2], [8])), and the initial data satisfy $\Delta u_0, \Delta v_0 \geq c > 0$, then the blow-up set is contained in the boundary of Ω (If Ω is a ball the last condition need not be imposed).

For simplicity, we prove Theorem 2 first (Section II), because in the proof of Theorem 1 we use some technical facts. We postpone the proof of Theorem 1 to section III.

II. Localization results for $\Omega = B(0,1)$.

First we state the following Lemmas, where (u,v) is a solution of (1.1)-(1.3), $\Omega = B(0,1)$ and f and g satisfies the hypothesis of Theorems 1 and 2 (we remark that u and v are not necessarily radial).

Lemma 2.1. *If $F^{-1}\circ G$ is convex, there exits a constant $C > 0$ such that*

$$Cv \geq F^{-1}\circ G(u)$$

In this Lemma we do not use the fact that Ω is a ball.

Proof. Let $\alpha = Cv$ and $\beta = F^{-1}\circ G(u)$, then α and β satisfies (we use here the fact that $F^{-1}\circ G$ is convex)

$$\alpha_t = \Delta\alpha, \qquad \beta_t \leq \Delta\beta,$$

and at $\partial\Omega$

$$\frac{\partial}{\partial\eta}(\alpha-\beta)=Cg(u)-\left(F^{-1}\circ G\right)'(u)\ f(v)$$

$$=Cg\circ G^{-1}\circ F(\beta)-\left(F^{-1}\circ G\right)'\left(G^{-1}\circ F(\beta)\right)f\left(\frac{\alpha}{C}\right).$$

We want to use a comparison argument. We choose a large C in order to obtain $Cv_0 > F^{-1}\circ G(u_0)$. Suppose that the conclusion is false and let $t_0 = \max\{t>0:(\alpha-\beta)(\tau)>0, 0<\tau<t\}$. At that time t_0 there exists $x_0\in\partial\Omega$ such that $\alpha(x_0,t_0)=\beta(x_0,t_0)$. There (at (x_0,t_0))

$$\frac{\partial}{\partial\eta}(\alpha-\beta)=Cg\circ G^{-1}\circ F(\alpha)-\left(F^{-1}\circ G\right)'\left(G^{-1}\circ F(\alpha)\right)f\left(\frac{\alpha}{C}\right)$$

$$=g\circ G^{-1}\circ F(\alpha)\left(C-\frac{1}{f(\alpha)}f\left(\frac{\alpha}{C}\right)\right)$$

Taking C large enough makes the last member positive, which is a contradiction.

Notation. We denote by S^{n-1} the unitary sphere in ^{n}, $|S^{n-1}|$ its $n-1$ dimensional measure and $d\sigma$ the $n-1$ measure in S^{n-1}.

Lemma 2.2. *Let*

$$W(r,t)=\frac{1}{\mid S^{n-1}\mid}\int_{S^{n-1}}u(r\xi,t)\ d\sigma(\xi),$$

$$Z(r,t)=\frac{1}{\mid S^{n-1}\mid}\int_{S^{n-1}}v(r\xi,t)\ d\sigma(\xi).$$

Then (W,Z) *satisfies*

$$\left\{\begin{array}{ll} W_t=W_{rr}+\dfrac{n-1}{r}W_r, & Z_t=Z_{rr}+\dfrac{n-1}{r}Z_r; \\ W_r\geq f(Z), \quad Z_r\geq g(W) & \textit{at} \quad r=1. \end{array}\right. \tag{2.1}$$

Proof. As in [7], we compute W_t using (1.1) :

$$W_t(r,t)=\frac{1}{\mid S^{n-1}\mid}\int_{S^{n-1}}\Delta u(r\xi,t)\ d\sigma(\xi). \tag{2.2}$$

On the other hand, we have

$$W_r(r,t)=\frac{1}{\mid S^{n-1}\mid r^{n-1}}\int_{S^{n-1}}\nabla u(r\xi,t)\xi r^{n-1}\ d\sigma(\xi),$$

and by Gauss Theorem we get

$$W_r(r,t)=\frac{1}{\mid S_r\mid}\int_{B_r}\Delta u(x,t)dx.$$

Passing to spherical coordinates gives

$$W_r(r,t) = \frac{1}{\mid S^{n-1} \mid r^{n-1}} \int_0^r (\int_{S^{n-1}} \Delta u(\rho\xi,t)\rho^{n-1}\, d\sigma(\xi))\, d\rho. \tag{2.3}$$

Now we compute W_{rr} :

$$W_{rr}(r,t) = \frac{1}{\mid S^{n-1} \mid r^{n-1}} \int_{S^{n-1}} \Delta u(r\xi,t)r^{n-1}\, d\sigma(\xi) \tag{2.4}$$

$$+\frac{1-n}{\mid S^{n-1} \mid r^{n}} \int_0^r (\int_{S^{n-1}} \Delta u(\rho\xi,t)\rho^{n-1}\, d\sigma(\xi))\, d\rho$$

From (2.2), (2.3) and (2.4) we get the first statement in (2.1).
From (2.3) with $r = 1$

$$W_r(1,t) = \frac{1}{\mid S^{n-1} \mid} \int_{S^{n-1}} \frac{\partial u}{\partial \eta}(\xi,t)\, d\sigma(\xi),$$

using (1.2)

$$W_r(1,t) = \frac{1}{\mid S^{n-1} \mid} \int_{S^{n-1}} f(u(\xi,t))\, d\sigma(\xi),$$

and now we can use the fact that f is convex and obtain (via Jensen's inequality),

$$W_r(1,t) \geq f(W(1,t)).$$

In the same way we can obtain the statements for Z.

Lemma 2.3. *Let $x \in B(0,1)$,*

$$w(r,t) = \frac{1}{\mid B_r \mid} \int_0^r (\int_{S^{n-1}} u(x+\rho\xi,t)\rho^{n-1}\, d\sigma(\xi))\, d\rho$$

$$z(r,t) = \frac{1}{\mid B_r \mid} \int_0^r (\int_{S^{n-1}} v(x+\rho\xi,t)\rho^{n-1}\, d\sigma(\xi))\, d\rho$$

Then z, w satisfy

$$\begin{cases} w_t = w_{rr} + \frac{(n+1)}{r} w_r & \text{for} \quad 0 < r < 1 - \|x\|, \\ z_t = z_{rr} + \frac{(n+1)}{r} z_r & \text{for} \quad 0 < r < 1 - \|x\|. \end{cases}$$

Proof. First we compute w_t:

$$w_t(r,t) = \frac{1}{\mid B \mid r^n} \int_0^r (\int_{S^{n-1}} \Delta u(x+\rho\xi,t)\rho^{n-1}\, d\sigma(\xi))\, d\rho.$$

Now we compute w_r:

$$w_r(r,t) = \frac{1}{\mid B \mid r} \int_{S^{n-1}} u(x + r\xi, t)\ d\sigma(\xi) - \tag{2.5}$$

$$-\frac{n}{\mid B \mid r^{n+1}} \int_0^r (\int_{S^{n-1}} u(x + \rho\xi, t)\rho^{n-1}\ d\sigma(\xi))\ d\rho,$$

and w_{rr} :

$$w_{rr}(r,t) = \frac{1}{\mid B \mid r} \int_{S^{n-1}} \nabla u(x + r\xi, t)\xi\ d\sigma(\xi) - \frac{1}{\mid B \mid r^2} \int_{S^{n-1}} u(x + r\xi, t)\ d\sigma(\xi) -$$

$$-\frac{n}{\mid B \mid r^2} \int_{S^{n-1}} u(x+r\xi, t)\ d\sigma(\xi) + \frac{n(n+1)}{\mid B \mid r^{n+2}} \int_0^r (\int_{S^{n-1}} u(x+\rho\xi, t)\rho^{n-1}\ d\sigma(\xi))\ d\rho.$$

Using the divergence theorem for the first term we obtain

$$w_{rr}(r,t) = \frac{1}{\mid B \mid r^n} \int_{B_r} \Delta u(x + y, t) dy -$$

$$-\frac{(n+1)}{r} \{ \frac{1}{\mid B \mid r} \int_{S^{n-1}} u(x + r\xi, t) d\sigma(\xi)$$

$$-\frac{n}{\mid B \mid r^{n+1}} \int_0^r (\int_{S^{n-1}} u(x + \rho\xi, t)\rho^{n-1} d\sigma(\xi)) d\rho \}.$$

From this we obtain the first equation, taking spherical coordinates and using (2.5).

Analogously for z.

Lemma 2.4. *If w, z are as in Lemma 2.3, then*

$$u(x,t) \leq \max \left\{ \|u_0\|_{L^\infty(\Omega)}, \sup_t \{w(r,t)\} \right\}$$

$$v(x,t) \leq \max \left\{ \|v_0\|_{L^\infty(\Omega)}, \sup_t \{z(r,t)\} \right\}$$

for every $r \in (0, 1 - \|x\|)$.

Proof. By lemma 2.3 we may consider $w(r,t)$ as a radial solution of the heat equation in $^{n+2}$ and then the maximum principle implies

$$w(0,t) \leq \max \left\{ \sup_{0<\rho<r} \{w(\rho, o)\}, \sup_t \{w(r,t)\} \right\}$$

$$\text{with } 0 < r < 1 - \|x\|.$$

Now, by the definition of $w(r,t)$, $w(0,t) = u(x,t)$ and $\sup_{0<\rho<r} w(\rho, 0) \leq \|u_0\|_{L^\infty(B)}$ and the conclusion follows. Analogously for z.

Lemma 2.5. *Let $\lambda > 0$ such that $(W_0)_r, (Z_0)_r > -\lambda$. Then W_r and $Z_r > -\lambda$, as long as they are defined.*

Proof. Let $\widetilde{W} = W_r + \lambda$. $\widetilde{W}$ satisfies

$$\begin{cases} \widetilde{W}_t = \widetilde{W}_{rr} + \frac{n-1}{r}\widetilde{W}_r - \frac{n-1}{r^2}\widetilde{W} + \frac{n-1}{r^2}\lambda \\ \widetilde{W}(0,t) = \lambda \qquad \widetilde{W}(1,t) \geq \lambda \\ \widetilde{W}(r,0) > 0 \end{cases} \tag{2.6}$$

By the minimum principle we obtain $\widetilde{W} > 0$.

Lemma 2.6. *There exists $\varepsilon > 0$ such that*

$$\begin{cases} a = r^{n-1}(W_r + \lambda) - \varepsilon r^{n+1} f(Z + \lambda r) \geq 0 \\ b = r^{n-1}(Z_r + \lambda) - \varepsilon r^{n+1} g(W + \lambda r) \geq 0 \end{cases}$$

Proof. Using Lemma 2.2 and taking ε small enough we can see that a satisfies

$$\begin{cases} a_t \geq a_{rr} - \frac{n-1}{r}a_r \\ a(r,0) \geq 0 \qquad a(0,t) = 0 \\ a(1,t) = f(Z) + \lambda - \varepsilon f(Z + \lambda), \end{cases}$$

we have $a(1,t) \geq 0$ if ε is small enough because of our hypothesis (1.5). Analogously for b using (1.5) again.

By these Lemmas we can prove Theorem 2.

Proof of Theorem 2. We begin by localize the blow-up set of u.

By Lemma 2.4 its enough to control $\sup_t\{w(r,t)\}$. It holds

$$w(r,t) \leq \frac{|S^{n-1}|}{|B|\, r^n} \int_0^{r+\|x\|} W(\rho,t)\rho^{n-1}\, d\rho,$$

which leads us to control $W(\rho,t)$ with $0 \leq \rho \leq r + \|x\| < 1$. W, Z satisfy

$$W_r + \lambda \geq \varepsilon r^2 f(Z + \lambda r). \tag{2.7}$$

Using Lemma 2.1 and Jensen inequality we obtain that $CZ \geq F^{-1} \circ G(W)$, then by (2.7)

$$W_r + \lambda \geq \varepsilon r^2 f(Z + \lambda r) \geq \varepsilon r^2 f \circ F^{-1} \circ G(W) \geq \varepsilon r^2 f \circ F^{-1} \circ G\left(\frac{W + \lambda r}{D}\right),$$

where $D > 1 + \lambda / \min u_0$. We can write the last inequality as

$$\frac{W_r + \lambda}{D} \geq \frac{\varepsilon r^2}{D} \varphi \left(\frac{W + \lambda r}{D} \right),$$

where $\varphi = f \circ F^{-1} \circ G$. Integrating from r to 1 gives

$$\int_{\frac{W+\lambda r}{D}}^{\infty} \frac{1}{\varphi(s)} ds \geq \frac{\varepsilon}{3D} \left(1 - r^3\right).$$

Using (1.4) we have that $\frac{W+\lambda r}{D}$ is bounded by a constant that depends on r, then $W(\rho, t)$ is bounded for every $0 < t < T$, $0 \leq \rho \leq r + \|x\| < 1$.

It rests to localize the blow-up set of v. It is sufficient to control $Z(\rho, t)$ for $0 < t < T$, $0 \leq \rho \leq r + \|x\| < 1$. Integrating (2.7) from r_1 to r_2 we obtain

$$\int_{r_1}^{r_2} (W_r + \lambda)\, dr \geq \varepsilon \int_{r_1}^{r_2} r^2 f(Z + \lambda r)\, dr,$$

hence

$$W(r_2, t) - W(r_1, t) + \lambda (r_2 - r_1) \geq \varepsilon r_1^2 f(Z(r_1, t) + \lambda r_1)(r_2 - r_1).$$

Thus the control over W implies that Z is bounded for $0 < t < T$, $0 \leq \rho \leq r + \|x\| < 1$.

III. Localization results for smooth Ω.

First we introduce some notation and a well known fact . Let Ω be a bounded domain with smooth boundary $\partial\Omega$. We denote $\Omega_r = \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) > r\}$ and $\partial\Omega_r = \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) = r\}$. We use the following construction. We define by

$$\Phi : \partial\Omega \times (0, R) \mapsto \Omega - \overline{\Omega}_R$$

$$\Phi(r, \xi) = \xi - r\eta(\xi),$$

where $\eta(\xi)$ is the exterior normal vector at $\xi \in \partial\Omega$.

We recall that Φ is a diffeomorphism if R is small enough. We also use the existence of a smooth function $\mu(r, \xi)$ such that

$$\int_{\partial\Omega_r} u \; d\sigma = \int_{\partial\Omega} u(\Phi(r, \xi))\mu(r, \xi) \; d\sigma(\xi)$$

and

$$\int_{\Omega - \overline{\Omega}_r} u(x) dx = \int_0^r \int_{\partial\Omega} u(\Phi(\rho, \xi)\mu(\rho, \xi) \; d\sigma(\xi) \; d\rho$$

This function μ is the Jacobian of Φ and is C^k if the boundary of Ω is C^{k+2}. If Ω is a ball $\mu(r, \xi) = r^{n-1}$ as in Section II.

As in Section II we need some previous Lemmas.

Lemma 3.1. *If (u,v) is a solution of (1.1) and $\Delta u_0, \Delta v_0 \geq c > 0$, then $\Delta u, \Delta v \geq c > 0$.*

Proof. Let $0 < b < c$. There exist $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$

$$u(x,\varepsilon) > u(x,0) + b\varepsilon,$$

$$v(x,\varepsilon) > v(x,0) + b\varepsilon.$$

Let $u_\varepsilon(x,t) = u(x,t+\varepsilon) - b\varepsilon$ and $v_\varepsilon(x,t) = v(x,t+\varepsilon) - b\varepsilon$. u_ε and v_ε are solutions of the heat equation in the domain $\Omega \times (0, T-\varepsilon)$ and satisfy

$$u_\varepsilon(x,0) > u(x,0), \qquad v_\varepsilon(x,0) > v(x,0),$$

$$\frac{\partial u_\varepsilon}{\partial \eta}(x,t) = f(v(x,t+\varepsilon)) = f(v_\varepsilon(x,t) + b\varepsilon) \geq f(v_\varepsilon(x,t)),$$

and

$$\frac{\partial v_\varepsilon}{\partial \eta}(x,t) \geq f(u_\varepsilon(x,t)).$$

Using a comparison principle we obtain

$$u_\varepsilon(x,t) > u(x,t), \qquad v_\varepsilon(x,t) > v(x,t).$$

This implies that $u_t \geq b$, $v_t \geq b$ in $\overline{\Omega} \times [0,T)$. As $b < c$ is arbitrary we can conclude that

$$\Delta u = u_t \geq c, \qquad \Delta v = v_t \geq c.$$

This proves the Lemma.

Lemma 3.2. *Let $u(x,t)$ be a positive solution of the heat equation, $r_0 = R/2$. Then $z(r,t) = \int_{\Omega_r} u(x,t)dx$ satisfies*

$$z_t(r,t) \geq z_{rr}(r,t) + A z_r(r,t),$$

$$\textit{where } A = \max_{[0,r_0]\times\partial\Omega} \frac{|\mu_r(r,\xi)|}{\mu(r,\xi)}.$$

Proof. We use ideas from [7]. First we compute z_t:

$$z_t(r,t) = \int_{\Omega_r} u_t(x,t)dx = \int_{\Omega_r} \Delta u(x,t)dx. \tag{3.1}$$

Now we observe that

$$z(r,t) = \int_{\Omega_r} u(x,t)dx = \int_\Omega u(x,t)dx - \int_{\Omega-\Omega_r} u(x,t)dx$$

$$= \int_{\Omega} u(x,t)dx - \int_0^r \int_{\partial\Omega} u(\Phi(\rho,\xi),t)\mu(\rho,\xi)\ d\sigma(\xi)\ d\rho.$$

Now we compute z_r:

$$z_r(r,t) = -\int_{\partial\Omega} u(\Phi(r,\xi),t)\mu(r,\xi)\ d\sigma(\xi) \tag{3.2}$$

and z_{rr}:

$$z_{rr}(r,t) = \int_{\partial\Omega} < \nabla u(\Phi(r,\xi),t);\eta(\xi) > \mu(r,\xi)\ d\sigma(\xi) - \int_{\partial\Omega} u(\Phi(r,\xi),t)\mu_r(r,\xi)\ d\sigma(\xi)$$

$$= \int_{\partial\Omega} \frac{\partial u}{\partial\eta}(\Phi(r,\xi),t)\mu(r,\xi)\ d\sigma(\xi) - \int_{\partial\Omega} u(\Phi(r,\xi),t)\mu(r,\xi)\{\frac{\mu_r}{\mu}\}\ d\sigma(\xi)$$

$$= \int_{\Omega_r} \Delta u(x,t)dx - \int_{\partial\Omega} u(\Phi(r,\xi),t)\mu(r,\xi)\{\frac{\mu_r}{\mu}\}\ d\sigma(\xi).$$

Using (3.1) gives

$$z_{rr}(r,t) = z_t(r,t) - \int_{\partial\Omega} u(\Phi(r,\xi),t)\mu(r,\xi)\{\frac{\mu_r}{\mu}\}\ d\sigma(\xi),$$

which implies by using (3.2) and the expression for A,

$$z_t(r,t) = z_{rr}(r,t) + \int_{\partial\Omega} u(\Phi(r,\xi),t)\mu(r,\xi)\{\frac{\mu_r}{\mu}\}\ d\sigma(\xi)$$

$$\geq z_{rr}(r,t) + Az_r(r,t).$$

Corollary 3.1. *Let $\theta = \frac{A}{2}$ and $\omega = \theta^2 - A\theta$ where A is defined in Lemma 3.2. Then*

$$\overline{z}(r,t) = \exp(\theta r - \omega t)\int_{\partial\Omega_r} \frac{\partial v}{\partial\eta} d\sigma \qquad \overline{w}(r,t) = \exp(\theta r - \omega t)\int_{\partial\Omega_r} \frac{\partial u}{\partial\eta}\ d\sigma$$

satisfy

$$\begin{cases} \overline{w}_t \geq \overline{w}_{rr} \\ \overline{z}_t \geq \overline{z}_{rr} \end{cases}$$

__Proof__. We observe that $\widetilde{w}(r,t) = \int_{\partial\Omega_r} \frac{\partial u}{\partial\eta}\ d\sigma = \int_{\Omega_r} \Delta u(x,t)dx$. And now we can apply Lemma 3.2 to Δu, which is a solution of the heat equation. We obtain

$$\widetilde{w}_t \geq \widetilde{w}_{rr} + A\widetilde{w}_r$$

and after some computation using our choice of θ and ω , we have

$$\overline{w}_t \geq \overline{w}_{rr}.$$

Analogously for $\overline{z}$.

Lemma 3.3. *If $0 < r < r_0$ and A is defined in Lemma 3.2, then*

$$\frac{\partial}{\partial r}(e^{-Ar}\int_{\partial\Omega_r} u \ d\sigma) \leq -e^{-Ar}\int_{\partial\Omega_r} \frac{\partial u}{\partial \eta} \ d\sigma.$$

Proof.

$$\frac{\partial}{\partial r}(e^{-Ar}\int_{\partial\Omega_r} u \ d\sigma) = -Ae^{-Ar}\int_{\partial\Omega_r} u \ d\sigma + e^{-Ar}\frac{\partial}{\partial r}(\int_{\partial\Omega_r} u \ d\sigma)$$

$$= -Ae^{-Ar}(\int_{\partial\Omega_r} u \ d\sigma) + e^{-Ar}\frac{\partial}{\partial r}(\int_{\partial\Omega} u(\Phi(r,\xi))\mu(r,\xi) \ d\sigma(\xi)$$

$$= e^{-Ar}\{-A\int_{\partial\Omega_r} u d\sigma + \int_{\partial\Omega} u(\Phi(r,\xi))\frac{\mu_r}{\mu}\mu d\sigma(\xi) -$$

$$\int_{\partial\Omega} < \nabla u(\Phi(r,\xi)); \eta(\xi) > \mu(r,\xi) d\sigma(\xi)\}$$

$$\leq -e^{-Ar}\int_{\partial\Omega_r} \frac{\partial u}{\partial \eta} \ d\sigma$$

Lemma 3.4. *Let $\widetilde{w}(r,t) = \int_{\partial\Omega_r} f(v) \ d\sigma$ and $\widetilde{z}(r,t) = \int_{\partial\Omega_r} g\,(u) \ d\sigma$. Then*

$$\widetilde{w}_t \leq \widetilde{w}_{rr} + B\widetilde{w}_r + C\widetilde{w}, \qquad \widetilde{z}_t \leq \widetilde{z}_{rr} + B\widetilde{z}_r + C\widetilde{z}.$$

$$0 \leq \int_{\partial\Omega_r} \Delta(f(v))dx \leq A\widetilde{w} - \widetilde{w}_r, \qquad 0 \leq \int_{\partial\Omega_r} \Delta(g\,(u))dx \leq A\widetilde{z} - \widetilde{z}_r.$$

Proof. As before we begin by $\widetilde{w}_r$:

$$\widetilde{w}_r(r,t) = \frac{\partial}{\partial r}\left\{\int_{\partial\Omega} f(v(\Phi(r,\xi),t))\mu(r,\xi) \ d\sigma(\xi)\right\}$$

$$= -\left\{\int_{\partial\Omega} < \nabla(f(v)); \eta(\xi) > (\Phi(r,\xi),t)\mu(r,\xi) \ d\sigma(\xi)\right\} +$$

$$\left\{\int_{\partial\Omega} f(v(\Phi(r,\xi),t))\frac{\mu_r}{\mu}(r,\xi)\mu(r,\xi) \ d\sigma(\xi)\right\}.$$

It follows that

$$\widetilde{w}_r(r,t) \leq -\int_{\partial\Omega_r} \frac{\partial(f(v))}{\partial \eta} \ d\sigma + A\int_{\partial\Omega_r} f(v) \ d\sigma$$

and using the divergence theorem we obtain

$$0 \leq \int_{\partial\Omega_r} \Delta(f(v))dx \leq A\widetilde{w} - \widetilde{w}_r$$

The first inequality is trivial because f is convex and $\Delta v \geq 0$.

In order to prove the first inequality we rewrite $\widetilde{w}_r$ as

$$\widetilde{w}_r(r,t) = -\int_{\Omega} \Delta(f(u))dx + \int_0^r \int_{\partial\Omega} \Delta(f(u))(\Phi(\rho,\xi),t)\mu(\rho,\xi)\ d\sigma(\xi)\ d\rho +$$

$$+ \int_{\partial\Omega} f(u(\Phi(r,\xi),t))\mu_r(r,\xi)\ d\sigma(\xi),$$

and we compute $\widetilde{w}_{rr}$ as

$$\widetilde{w}_{rr} = \int_{\partial\Omega} \Delta(f(u))(\Phi(r,\xi),t)\mu(r,\xi)\ d\sigma(\xi) -$$

$$\int_{\partial\Omega} < \nabla(f(u)); \eta(\xi) > (\Phi(r,\xi),t)\frac{\mu_r}{\mu}\mu(r,\xi)\ d\sigma(\xi) +$$

$$+ \int_{\partial\Omega} f(u(\Phi(r,\xi),t))\frac{\mu_{rr}}{\mu}\mu(r,\xi)\ d\sigma(\xi).$$

Now let $\overline{\mu}$ be a $C^2(\overline{\Omega})$ function that satisfies $\overline{\mu}(\Phi(r,\xi)) = \frac{\mu_r}{\mu}(r,\xi)$ for $(r,\xi) \in [0,r_0] \times \partial\Omega$. With this $\overline{\mu}$ we can compute, using that $\Delta f(u) \geq 0$,

$$\int_{\partial\Omega_r} \frac{\partial f(u)}{\partial\eta}\overline{\mu}\ d\sigma = \int_{\Omega_r} \Delta(f(u))\overline{\mu}dx - \int_{\Omega_r} f(u)\Delta\overline{\mu}dx + \int_{\partial\Omega_r} f(u)\frac{\partial\overline{\mu}}{\partial\eta}\ d\sigma$$

$$\Big| \int_{\partial\Omega_r} \frac{\partial f(u)}{\partial\eta}\overline{\mu}\ d\sigma \Big| \leq C_1 \int_{\Omega_r} \Delta(f(u)) + C_2 \int_{\partial\Omega_r} f(u)\ d\sigma$$

$$\Big| \int_{\partial\Omega_r} \frac{\partial f(u)}{\partial\eta}\overline{\mu}\ d\sigma \Big| \leq C_1(A\widetilde{w} - \widetilde{w}_r) + C_2\widetilde{w}.$$

It follows that,

$$\widetilde{w}_{rr}(r,t) \geq \int_{\partial\Omega_r} \Delta(f(u))\ d\sigma - B\widetilde{w}_r - C\widetilde{w} \geq \int_{\partial\Omega_r} f'(u)\Delta u\ d\sigma - B\widetilde{w}_r - C\widetilde{w}$$

$$\widetilde{w}_t(r,t) = \int_{\partial\Omega_r} f'(u)\Delta u\ d\sigma.$$

We finally obtain,

$$\widetilde{w}_t \leq \widetilde{w}_{rr} + B\widetilde{w}_r + C\widetilde{w}$$

Corollary 3.2. *Let* $\beta < -\max\{A, \frac{|B|}{2}\}$ *and* $\gamma = C + \beta^2 - B\beta$. *Then* $\underline{z}(r,t) = \exp(\beta r - \gamma t)\widetilde{z}(r,t)$ *and* $\underline{w}(r,t) = \exp(\beta r - \gamma t)\widetilde{w}(r,t)$ *satisfies*

$$\underline{z}_t \leq \underline{z}_{rr}, \qquad \underline{w}_t \leq \underline{w}_{rr},$$

$$\underline{w}_r < 0, \qquad \underline{z}_r < 0.$$

Proof. As

$$\underline{z}_r = \exp(\beta r - \gamma t)[\beta\widetilde{z} + \widetilde{z}_r] = \exp(\beta r - \gamma t)[(A+\beta)\widetilde{z} + (\widetilde{z}_r - A\widetilde{z})]$$

and each term is negative, we have the last inequality. Next we compute:

$$\underline{z}_t = \exp(\beta r - \gamma t)[\widetilde{z}_t - \gamma\widetilde{z}],$$

$$\underline{z}_r = \exp(\beta r - \gamma t)[\widetilde{z}_r + \beta\widetilde{z}],$$

$$\underline{z}_{rr} = \exp(\beta r - \gamma t)[\widetilde{z}_{rr} + 2\beta\widetilde{z}_r + \beta^2\widetilde{z}].$$

By Lemma 4.3, we have

$$\underline{z}_t \leq \underline{z}_{rr} + (B - 2\beta)\underline{z}_r + (-\gamma + C + \beta^2 - B\beta)\underline{z}.$$

Thus, by our choice of β and γ, we obtain the first inequality. Analogously for $\underline{w}$.

Proposition 3.1. *There exist a constant* $C > 0$ *and a function* $h \in C^2([0, r_0])$ *, nonnegative , decreasing and convex, such that*

$$h(r_0) = 0 \ , \ h(0) < L = \max\{1, \sup_{0\leq t\leq T} e^{(\gamma-\omega)t}\},$$

$$\int_{\partial\Omega_r} \frac{\partial u}{\partial\eta}\, d\sigma \geq C h(r) \int_{\partial\Omega_r} f(v)\, d\sigma$$

and

$$\int_{\partial\Omega_r} \frac{\partial v}{\partial\eta}\, d\sigma \geq C h(r) \int_{\partial\Omega_r} g(u)\, d\sigma$$

for $0 < r < r_0$.

Proof. We use a comparison argument. First we choose a C^2, nonnegative, decreasing and convex function h such that $\overline{z}(r,0) \geq h(r)\underline{z}(r,0)$ and $\overline{w}(r,0) \geq h(r)\underline{w}(r,0)$ with $h(r_0) = 0$ and $h(0) < L$. Next we observe that $h(r)\underline{z}(r,t)$ is a subsolution of the heat equation (we use Corollary 3.2 and the properties of h) and, by Corollary 3.1, $\overline{z}$ is a super solution. To conclude the comparison argument we observe that we have $\overline{z} \geq h(0)\underline{z}(0,t)$ at $r = 0$ because $h(0) < L$, and $\int_{\partial\Omega_{r_0}} \frac{\partial u}{\partial\eta}\, d\sigma \geq 0$ at

$r = r_0$. We conclude that $\overline{z} \geq h(r)\underline{z}$. We finish the proof of the Proposition recalling the definitions of $\overline{z}$ and $\underline{z}$. The same argument applies to obtain the second inequality.

Proof of Theorem 1. In this proof C and K are constants. From Lemma 3.3 we have

$$\frac{\partial}{\partial r}(e^{-Ar}\int_{\partial\Omega_r} u\ d\sigma) \leq -e^{-Ar}\int_{\partial\Omega_r}\frac{\partial u}{\partial\eta}\ d\sigma.$$

Then using Proposition 3.1

$$\frac{\partial}{\partial r}(e^{-Ar}\int_{\partial\Omega_r} u\ d\sigma) \leq -e^{-Ar}Ch(r)\int_{\partial\Omega_r} f(v)\ d\sigma$$

By Jensen inequality

$$\frac{\partial}{\partial r}(e^{-Ar}\int_{\partial\Omega_r} u\ d\sigma) \leq -Ch(r)f(e^{-Ar}\int_{\partial\Omega_r} v\ d\sigma/K).$$

Analogously

$$\frac{\partial}{\partial r}(e^{-Ar}\int_{\partial\Omega_r} v\ d\sigma) \leq -Ch(r)g(e^{-Ar}\int_{\partial\Omega_r} u\ d\sigma/K).$$

We observe that the conclusion of Lemma 2.1 holds here. Then we have

$$C_1\int_{\partial\Omega_r} v\ d\sigma \geq F^{-1}\circ G\left(C_2\int_{\partial\Omega_r} u\ d\sigma\right),$$

consequently

$$\frac{\partial}{\partial r}(e^{-Ar}\int_{\partial\Omega_r} u\ d\sigma/K) \leq -Ch(r)f\circ F^{-1}\circ G(e^{-Ar}\int_{\partial\Omega_r} u\ d\sigma/K).$$

An easy argument shows (integrating from 0 to r)

$$\int_{\partial\Omega_r} u\ d\sigma \leq Ke^{Ar}H^{-1}(C\int_0^r h(s)ds),$$

where $H(s) = \int_s^\infty 1/f\circ F^{-1}\circ G$.

Now we observe that

$$u(x,t) \leq \frac{1}{\mid B(x,\delta)\mid}\int_{B(x,\delta)} u \leq \frac{1}{\mid B(x,\delta)\mid}\int_{r_0}^{r_1}\int_{\partial\Omega_r} u\ d\sigma,$$

because $\Delta u > 0$.

The last term is bounded by an expression that depends only on $\mathrm{dist}(x,\partial\Omega)$, so we can control u in $\Omega_{r_0} - \Omega_{r_1}$ and then in Ω_{r_0} recalling that $\Delta u \geq 0$. Using a similar argument to one used in the proof of Theorem 2 we obtain that the blow-up set of v is localized in $\partial\Omega$.

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