

ESTIMATES FOR NONLINEAR HARMONIC MEASURES ON TREES

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ABSTRACT. In this paper we give some estimates for nonlinear harmonic measures on trees. In particular, we estimate in terms of the size of a set D the value at the origin of the solution to $u(x) = F((x, 0), \dots, (x, m-1))$ for every $x \in \mathbb{T}_m$, a directed tree with m branches with initial datum $f + \chi_D$. Here F is an averaging operator on \mathbb{R}^m , x is a vertex of a directed tree \mathbb{T}_m with regular m -branching and (x, i) denotes a successor of that vertex for $0 \leq i \leq m-1$. We also provide a characterization of the subsets of the tree for which the unique continuation property holds.

1. INTRODUCTION

Let us first recall some well known facts for the classical p -Laplacian. Let Ω be the unit ball in \mathbb{R}^N , $N > 1$. We say that u is p -harmonic (p -superharmonic/ p -subharmonic) in Ω ($1 < p < \infty$) if $u \in W^{1,p}(\Omega)$ and

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \psi \, dx = 0, \quad (\geq 0 / \leq 0)$$

whenever $\psi \in C_0^\infty(\Omega)$ ($\psi \geq 0$). Let E be a subset of $\partial\Omega$. Consider the following class

$$U_p(E) = \left\{ v : v \geq 0 \text{ and } p\text{-superharmonic s. t. } \liminf_{x \in \Omega, x \rightarrow y} v(y) \geq \chi_E(y) \forall y \in \partial\Omega \right\}.$$

The p -harmonic measure of the set E relative to the domain Ω is the function $\omega_p(\cdot, E)$ whose value at any $x \in \Omega$ is given by $\omega_p(x, E) = \inf\{v(x) : v \in U_p(E)\}$. We simply denote $\omega_p(E)$ when $x = 0$. For a deeper discussion about of p -harmonic measure, we refer the reader to [2, 3, 5, 6, 11, 12, 16].

In this context, the following problem for the p -Laplacian remains open, see [24].

Boundary Comparison Principle. For $\delta > 0$ consider I_δ a spherical cap with length $\delta/2$. Given $\epsilon > 0$, find $\delta = \delta(\epsilon, M, p) > 0$ such that

$$|u(0) - v(0)| < \epsilon$$

for all p -harmonic functions u and v in Ω that extend to $\bar{\Omega}$, are bounded $\|u\|_\infty \leq M$, $\|v\|_\infty \leq M$, and satisfy $u(y) = v(y)$ for all $y \in \partial\Omega \setminus I_\delta$.

Closely related to this problem is the following:

p -harmonic Measure Estimates. Does there exist $\alpha > 0$ such that

$$\omega_p(I_\delta) \sim \delta^\alpha \quad \text{as } \delta \rightarrow 0?$$

In [17], the authors study the second question in the case $p = \infty$. They showed that $\omega_\infty(I_\delta) \sim \delta^{\frac{1}{3}}$. Similar ideas can be used to obtain the result for $1 < p < \infty$, see [18].

In this work we provide answers to both problems for the F -harmonic function in a directed tree where F is an averaging operator on \mathbb{R}^m , see below for a precise definition.

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Regular trees are discrete models of the unit ball of \mathbb{R}^N and hence our results can be seen as a contribution in order to the study of the previously mentioned open problem.

We remark that for the linear case, $p = 2$, the solutions to these problems are well known and the starting point for their study is the mean value property for harmonic functions. One of the main interests of the present work is to show what kind of results can be proved when the mean value property under consideration is nonlinear.

Now, let us introduce briefly some definitions and notations needed in order to make precise the statements of our main results (but we refer the reader to Section 2 where more details can be found). Let $F: \mathbb{R}^m \rightarrow \mathbb{R}$ be a continuous function. We call F an averaging operator if it satisfies the following:

$$\begin{aligned} F(0, \dots, 0) &= 0 \text{ and } F(1, \dots, 1) = 1; \\ F(tx_1, \dots, tx_m) &= tF(x_1, \dots, x_m); \\ F(t + x_1, \dots, t + x_m) &= t + F(x_1, \dots, x_m), \end{aligned}$$

for all $t \in \mathbb{R}$;

$$F(x_1, \dots, x_m) < \max\{x_1, \dots, x_m\},$$

if not all x_j 's are equal; F is nondecreasing with respect to each variable; in addition, we will assume that F is permutation invariant, that is, $F(x_1, \dots, x_m) = F(x_{\tau(1)}, \dots, x_{\tau(m)})$ for each permutation τ of $\{1, \dots, m\}$ and that there exists $0 < \kappa < 1$ such that

$$(1.1) \quad F(x_1 + c, \dots, x_m) \leq F(x_1, \dots, x_m) + c\kappa$$

for all $(x_1, \dots, x_m) \in \mathbb{R}^m$ and for all $c > 0$.

As examples of averaging operators we mention the following: The first example is taken from [9]. For $1 < p < +\infty$, the operator $F^p(x_1, \dots, x_m) = t$ from \mathbb{R}^m to \mathbb{R} defined implicitly by

$$\sum_{j=1}^m (x_j - t)|x_j - t|^{p-2} = 0$$

is a permutation invariant averaging operator. Next, we consider, for $0 \leq \alpha, \beta \leq 1$ with $\alpha + \beta = 1$

$$\begin{aligned} F_0(x_1, \dots, x_m) &= \frac{\alpha}{2} \left(\max_{1 \leq j \leq m} \{x_j\} + \min_{1 \leq j \leq m} \{x_j\} \right) + \frac{\beta}{m} \sum_{j=1}^m x_j, \\ F_1(x_1, \dots, x_m) &= \alpha \operatorname{median}_{1 \leq j \leq m} \{x_j\} + \frac{\beta}{m} \sum_{j=1}^m x_j, \\ F_2(x_1, \dots, x_m) &= \alpha \operatorname{median}_{1 \leq j \leq m} \{x_j\} + \frac{\beta}{2} \left(\max_{1 \leq j \leq m} \{x_j\} + \min_{1 \leq j \leq m} \{x_j\} \right), \end{aligned}$$

where

$$\operatorname{median}_{1 \leq j \leq m} \{x_j\} := \begin{cases} y_{\frac{m+1}{2}} & \text{if } m \text{ is even,} \\ \frac{y_{\frac{m}{2}} + y_{(\frac{m}{2}+1)}}{2} & \text{if } m \text{ is odd,} \end{cases}$$

with $\{y_1, \dots, y_m\}$ a nondecreasing rearrangement of $\{x_1, \dots, x_m\}$. F_0, F_1 and F_2 are permutation invariant averaging operators. Note that F_0 and F_1 verify (1.1) but F_2 does not.

Associated with an averaging operator F we have an equation on a tree. In what follows x is a vertex of a directed tree \mathbb{T}_m with regular m -branching and (x, i) is a successor of that vertex for all $0 \leq i \leq m-1$ (see Section 2 for more details). A function $u: \mathbb{T}_m \rightarrow \mathbb{R}$ is called F -subharmonic function if the inequality $u(x) \leq F(u(x, 0), \dots, u(x, m-1))$ holds

for all $x \in \mathbb{T}_m$ and F -superharmonic if the opposite inequality holds for all $x \in \mathbb{T}_m$. We say that u is F -harmonic if u is both F -subharmonic and F -superharmonic, that is, u is a solution to the equation

$$u(x) = F(u(x, 0), \dots, u(x, m-1)).$$

Let $f : [0, 1] \rightarrow \mathbb{R}$ be a bounded function, $c > 0$ and $E \subset \partial\mathbb{T}_m$. We define $U_F(f, E, c)$ as the set of all F -superharmonic functions u such that

$$\liminf_{k \rightarrow +\infty} u(x_k) \geq f(\pi) + c\chi_E(\pi),$$

for all $\pi = (x_1, \dots, x_k, \dots) \in \partial\mathbb{T}_m$.

Finally, we need the notion of solution to the Dirichlet Problem, (DP) in the sequel. Given a bounded function $f : [0, 1] \rightarrow \mathbb{R}$, u is a solution to the Dirichlet Problem with boundary data f if it is F -harmonic and verifies

$$\lim_{k \rightarrow +\infty} u(x_k) = f(\pi), \quad \forall \pi = (x_1, \dots, x_k, \dots) \in \partial\mathbb{T}_m.$$

Now, we are ready to state the main result of this paper.

Theorem 1.1. *Let F be a permutation invariant averaging operator with the property (1.1), $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function, I be a subinterval of $[0, 1]$ with measure $|I|$ and $c > 0$. If u is the solution of (DP) with boundary data f , then*

$$0 \leq \inf \left\{ w(\emptyset) - u(\emptyset) : w \in U_F(f, I, c) \right\} \leq 2c(m|I|)^\gamma$$

for any $\gamma \leq -\log_m(\kappa)$.

Note that the obtained bound depends precisely both on the tree (m is the parameter that controls the branching of the tree), the interval where the perturbation of f takes place (through its measure), the size of the perturbation (given by c) and on the operator F (κ appears in (1.1)).

We immediately deduce, taking $f \equiv 0$ and $c = 1$, the following result.

Corollary 1.2 (p -harmonic Measure Estimates). *If I is a subinterval of $[0, 1]$, then*

$$0 \leq \omega_F(I) \leq 2(m|I|)^\gamma$$

for all $\gamma \leq -\log_m(\kappa)$.

Finally, we state the Boundary Comparison Principle.

Corollary 1.3 (Boundary Comparison Principle). *Let F be a permutation invariant averaging operator that satisfies the property (1.1). Given $\varepsilon > 0$ and $M > 0$, there exists $\delta = \delta(\varepsilon, M, m) > 0$ so that, if $f, g : [0, 1] \rightarrow \mathbb{R}$ are continuous functions, $f = g$ in $[0, 1] \setminus I$, and $\|f\|_\infty + \|g\|_\infty \leq M$, where I is a subinterval of $[0, 1]$ with $|I| = \delta$, then*

$$|u(\emptyset) - v(\emptyset)| < \varepsilon$$

where u and v are the solutions of (DP) with boundary data f y g respectively

Let us end the introduction with a brief comment on previous bibliography. For nonlinear mean values on a finite graph we refer to [15] and references therein. For equations on trees like the ones considered here, see [1, 10, 9] and [22, 23], where it is proved the existence and uniqueness of a solution using game theory. Here we use ideas from these references. Nonlinear mean value properties that characterize solutions to PDEs can be found, for example, in [14], [21], [7] and [8]. These mean value properties

reveal to be quite useful when designing numerical schemes that approximate solutions to the corresponding nonlinear PDEs, see [19, 20].

Organization of the paper. In Section 2 we collect some preliminary facts concerning trees, averaging operators, F -harmonic functions and F -harmonic measures; in Section 3 we prove existence and uniqueness for the Dirichlet problem and a comparison principle; in Section 4 we give a characterization of the sets for which the unique continuation hold; Finally in Section 5 we prove Theorem 1.1 and Corollary 1.3.

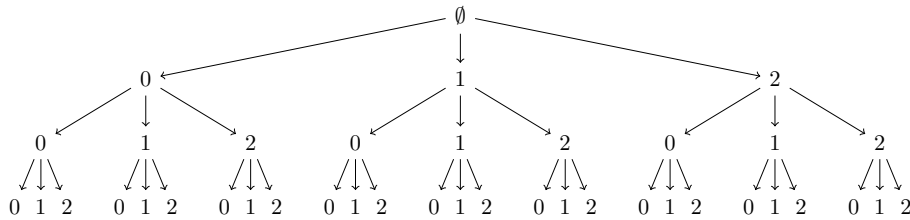
2. PRELIMINARIES

We begin with a review of the basic results that will be needed in subsequent sections. The known results are generally stated without proofs, but we provide references where the proofs can be found. Also, we introduce some of our notational conventions.

2.1. Directed Tree. Let $m \in \mathbb{N}_{>2}$. In this work we consider a directed tree \mathbb{T}_m with regular m -branching, that is, \mathbb{T}_m consists of the empty set \emptyset and all finite sequences (a_1, a_2, \dots, a_k) with $k \in \mathbb{N}$, whose coordinates a_i are chosen from $\{0, 1, \dots, m-1\}$. The elements in \mathbb{T}_m are called vertices. Each vertex x has m successors, obtained by adding another coordinate. We will denote by $\mathcal{S}(x)$ the set of successors of the vertex x . A vertex $x \in \mathbb{T}_m$ is called an n -level vertex ($n \in \mathbb{N}$) if $x = (a_1, a_2, \dots, a_n)$. The set of all n -level vertices is denoted by \mathbb{T}_m^n .

Example 2.1. Let $\kappa \in \mathbb{N}_{\geq 3}$. The $1/\kappa$ -Cantor set, that we denote by $C_{1/\kappa}$, is the set of all $x \in [0, 1]$ that have a base κ expansion without the digit 1, that is $x = \sum a_j \kappa^{-j}$ with $a_j \in \{0, 1, \dots, \kappa-1\}$ with $a_j \neq 1$. Thus $C_{1/\kappa}$ is obtained from $[0, 1]$ by removing the second κ -th part of the line segment $[0, 1]$, and then removing the second interval of length $1/\kappa$ from the remaining intervals, and so on. This set can be thought of as a directed tree with regular m -branching with $m = \kappa - 1$.

For example, if $\kappa = 3$, we identify $[0, 1]$ with \emptyset , the sequence $(\emptyset, 0)$ with the first interval right $[0, 1/3]$, the sequence $(\emptyset, 1)$ with the central interval $[1/3, 2/3]$ (that is removed), the sequence $(\emptyset, 2)$ with the left interval $[2/3, 1]$, the sequence $(\emptyset, 0, 0)$ with the interval $[0, 1/9]$ and so on.



A branch of \mathbb{T}_m is an infinite sequence of vertices, each followed by its immediate successor. The collection of all branches forms the boundary of \mathbb{T}_m , denoted by $\partial\mathbb{T}_m$.

We now define a metric on $\mathbb{T}_m \cup \partial\mathbb{T}_m$. The distance between two sequences (finite or infinite) $\pi = (a_1, \dots, a_k, \dots)$ and $\pi' = (a'_1, \dots, a'_k, \dots)$ is m^{-K+1} when K is the first index k such that $a_k \neq a'_k$; but when $\pi = (a_1, \dots, a_K)$ and $\pi' = (a_1, \dots, a_K, a'_{K+1}, \dots)$, the distance is m^{-K} . Hausdorff measure and Hausdorff dimension are defined using this metric. We have that \mathbb{T}_m and $\partial\mathbb{T}_m$ have diameter one and $\partial\mathbb{T}_m$ has Hausdorff dimension one. Now, we observe that the mapping $\psi : \partial\mathbb{T}_m \rightarrow [0, 1]$ defined as

$$\psi(\pi) := \sum_{k=1}^{+\infty} \frac{a_k}{m^k}$$

is surjective, where $\pi = (a_1, \dots, a_k, \dots) \in \partial\mathbb{T}_m$ and $a_k \in \{0, 1, \dots, m-1\}$ for all $k \in \mathbb{N}$. Whenever $x = (a_1, \dots, a_k)$ is a vertex, we set

$$\psi(x) := \psi(a_1, \dots, a_k, 0, \dots, 0, \dots).$$

We can also associate to a vertex $x = (a_1, \dots, a_k)$ an interval I_x of length $\frac{1}{m^k}$ as follows

$$I_x := \left[\psi(x), \psi(x) + \frac{1}{m^k} \right].$$

Observe that for all $x \in \mathbb{T}_m$, $I_x \cap \partial\mathbb{T}_m$ is the subset of $\partial\mathbb{T}_m$ consisting of all branches that start at x . With an abuse of notation, we will write $\pi = (x_1, \dots, x_k, \dots)$ instead of $\pi = (a_1, \dots, a_k, \dots)$ where $x_1 = a_1$ and $x_k = (a_1, \dots, a_k) \in \mathcal{S}(x_{k-1})$ for all $k \in \mathbb{N}_{\geq 2}$.

Finally we will denote by \mathbb{T}_m^x the set of the vertices $y \in \mathbb{T}_m$ such that $I_y \subset I_x$.

2.2. Averaging Operator. The following definition is taken from [1]. Let $F: \mathbb{R}^m \rightarrow \mathbb{R}$ be a continuous function. We call F an averaging operator if it satisfies the following set of conditions:

- (i) $F(0, \dots, 0) = 0$ and $F(1, \dots, 1) = 1$;
- (ii) $F(tx_1, \dots, tx_m) = tF(x_1, \dots, x_m)$ for all $t \in \mathbb{R}$;
- (iii) $F(t + x_1, \dots, t + x_m) = t + F(x_1, \dots, x_m)$ for all $t \in \mathbb{R}$;
- (iv) $F(x_1, \dots, x_m) < \max\{x_1, \dots, x_m\}$ if not all x_j 's are equal;
- (v) F is nondecreasing with respect to each variable.

Remark 2.2. It holds that, if $(x_1, \dots, x_m), (y_1, \dots, y_m) \in \mathbb{R}^m$, then

$$x_j \leq y_j + \max_{1 \leq j \leq m} \{x_j - y_j\}$$

for all $j \in \{1, \dots, m\}$. Let F be an averaging operator. Then, by (iii) and (v),

$$F(x_1, \dots, x_m) \leq F(y_1, \dots, y_m) + \max_{1 \leq j \leq m} \{x_j - y_j\}.$$

Therefore

$$F(x_1, \dots, x_m) - F(y_1, \dots, y_m) \leq \max_{1 \leq j \leq m} \{x_j - y_j\}.$$

Remark 2.3. If F is an averaging operator then, using (ii) and (iii),

$$\begin{aligned} F(a, \dots, a, b) &= F(a, \dots, a, a + (b - a)) \\ &= a + F(0, \dots, 0, b - a) \\ &= a + (b - a)F(0, \dots, 0, 1) \\ &= a(1 - F(0, \dots, 0, 1)) + bF(0, \dots, 0, 1) \end{aligned}$$

for all $a, b \in \mathbb{R}$.

For the proof of the following proposition see [9].

Proposition 2.4. *If F is an averaging operator then*

- (1) $F(1 - x_1, \dots, 1 - x_m) = 1 - F(x_1, \dots, x_m)$ for all $(x_1, \dots, x_m) \in \mathbb{R}^m$;
- (2) *There exists $b > 0$ such that whenever $F(x_1, \dots, x_m) \geq 0$ and $\max\{x_1, \dots, x_m\} \leq 1$, then $\min\{x_1, \dots, x_m\} \geq -b$.*

In sections 4 and 5 we require, in addition, for F to be permutation invariant, that is,

$$F(x_1, \dots, x_m) = F(x_{\tau(1)}, \dots, x_{\tau(m)})$$

for any permutation τ of $\{1, \dots, m\}$.

Remark 2.5. If F is a permutation invariant averaging operator then we have that

$$F(1, 0, \dots, 0, -1) = F(-1, 0, \dots, 0, 1) = -F(1, 0, \dots, 0, -1).$$

Therefore $F(1, 0, \dots, 0, -1) = 0$.

In Section 5, we will also need the following assumption: F is a permutation invariant averaging operator with the property that there exists $0 < \kappa < 1$ such that

$$(2.2) \quad F(x_1 + c, \dots, x_m) \leq F(x_1, \dots, x_m) + c\kappa,$$

for all $(x_1, \dots, x_m) \in \mathbb{R}^m$ and for all $c > 0$.

Remark 2.6. If F is a permutation invariant averaging operator with the property (2.2), then

$$\begin{aligned} F(x_1 + c, x_2 + c, x_3, \dots, x_m) &\leq F(x_1, x_2 + c, x_3, \dots, x_m) + c\kappa \\ &= F(x_2 + c, x_1, x_3, \dots, x_m) + c\kappa \leq F(x_2, x_1, x_3, \dots, x_m) + 2c\kappa \\ &= F(x_1, \dots, x_m) + 2c\kappa \end{aligned}$$

for all $(x_1, \dots, x_m) \in \mathbb{R}^m$ and for all $c > 0$.

Now we give some examples.

Example 2.7. This example is taken from [9]. For $1 < p < +\infty$, the operator

$$F^p(x_1, \dots, x_m) = t$$

from \mathbb{R}^m to \mathbb{R} defined implicitly by

$$\sum_{j=1}^m (x_j - t)|x_j - t|^{p-2} = 0$$

is a permutation invariant averaging operator.

Example 2.8. For $0 \leq \alpha, \beta \leq 1$ with $\alpha + \beta = 1$, let us consider

$$\begin{aligned} F_0(x_1, \dots, x_m) &= \frac{\alpha}{2} \left(\max_{1 \leq j \leq m} \{x_j\} + \min_{1 \leq j \leq m} \{x_j\} \right) + \frac{\beta}{m} \sum_{j=1}^m x_j, \\ F_1(x_1, \dots, x_m) &= \alpha \operatorname{median}_{1 \leq j \leq m} \{x_j\} + \frac{\beta}{m} \sum_{j=1}^m x_j, \\ F_2(x_1, \dots, x_m) &= \alpha \operatorname{median}_{1 \leq j \leq m} \{x_j\} + \frac{\beta}{2} \left(\max_{1 \leq j \leq m} \{x_j\} + \min_{1 \leq j \leq m} \{x_j\} \right), \end{aligned}$$

where

$$\operatorname{median}_{1 \leq j \leq m} \{x_j\} := \begin{cases} y_{\frac{m+1}{2}} & \text{if } m \text{ is even,} \\ \frac{y_{\frac{m}{2}} + y_{(\frac{m}{2}+1)}}{2} & \text{if } m \text{ is odd,} \end{cases}$$

with $\{y_1, \dots, y_m\}$ a nondecreasing rearrangement of $\{x_1, \dots, x_m\}$.

It holds that F_0, F_1 and F_2 are permutation invariant averaging operators. Moreover, F_0, F_1 and F_2 satisfy (2.2) with $\kappa_0 = \alpha/2 + \beta/m$, $\kappa_1 = \alpha + \beta/m$ and $\kappa_2 = \alpha + \beta/2$, due to the fact that for any $c > 0$

$$\begin{aligned} \max Y + \min Y &\leq \max X + \min X + c, \\ \operatorname{median} Y &\leq \operatorname{median} X + c, \end{aligned}$$

where $Y = \{x_1 + c, x_2, \dots, x_m\}$ and $X = \{x_1, x_2, \dots, x_m\}$.

2.3. F -harmonic Functions. In this subsection we will present the definition and some properties of F -harmonic functions.

Let F be an averaging operator. A function $u: \mathbb{T}_m \rightarrow \mathbb{R}$ is called F -subharmonic function if the inequality

$$u(x) \leq F(u(x, 0), \dots, u(x, m-1))$$

holds for all $x \in \mathbb{T}_m$ and F -superharmonic if the opposite inequality holds for all $x \in \mathbb{T}_m$.

We say that u is F -harmonic if u is both F -subharmonic and F -superharmonic.

Example 2.9. For $1 < p < +\infty$, a function $u: \mathbb{T}_m \rightarrow \mathbb{R}$ is p -harmonic if

$$\sum_{j=0}^{m-1} (u(x, j) - u(x)) |u(x, j) - u(x)|^{p-2} = 0, \quad \forall x \in \mathbb{T}_m$$

that is

$$F^p(u(x, 0), \dots, u(x, m-1)) = u(x), \quad \forall x \in \mathbb{T}_m.$$

Thus the p -harmonic functions and F^p -harmonic functions are the same.

Example 2.10. For $1 < p < +\infty$, a function $u: \mathbb{T}_m \rightarrow \mathbb{R}$ is called p -harmonious if

$$u(x) = \frac{\alpha}{2} \left(\max_{1 \leq j \leq m} \{u(x, j)\} + \min_{1 \leq j \leq m} \{u(x, j)\} \right) + \frac{\beta}{m} \sum_{j=1}^m u(x, j), \quad \forall x \in \mathbb{T}_m$$

that is

$$F_0(u(x, 0), \dots, u(x, m-1)) = u(x), \quad \forall x \in \mathbb{T}_m.$$

These functions are related to game theory, see [13] for the continuous case and [22, 23] for trees.

Remark 2.11. Let F be an averaging operator and u be a F -harmonic function. Then

- (1) $au + b$ is a F -harmonic function for all $a, b \in \mathbb{R}$;
- (2) $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ are F -subharmonic functions.

Next, we collect some properties.

Lemma 2.12. *Let F be an averaging operator. If u is a bounded above F -subharmonic function and there exists $x \in \mathbb{T}_m$ such that $u(x) = \max_{y \in \mathbb{T}_m} u(y)$ then $u(y) = u(x)$ for any $y \in \mathbb{T}_m^x$.*

Proof. Let $M = u(x) = \max_{y \in \mathbb{T}_m} u(y)$. We first observe that it is sufficient to show that $u(y) = M$ for all $y \in \mathcal{S}(x)$. Since u is a F -subharmonic function and F is an averaging operator, we have that

$$M = u(x) \leq F(u(x, 0), \dots, u(x, m-1)) \leq \max_{y \in \mathcal{S}(x)} u(y) \leq M.$$

Then

$$F(u(x, 0), \dots, u(x, m-1)) = \max_{y \in \mathcal{S}(x)} u(y) = M.$$

Therefore, by property (iv), we have that $u(x, i) = M$ for all $0 \leq i \leq m-1$, i.e. $u(y) = M$ for all $y \in \mathcal{S}(x)$. \square

In the same manner, we can prove the following lemma.

Lemma 2.13. *Let F be an averaging operator. If u is a bounded below F -superharmonic function and there exists $x \in \mathbb{T}_m$ such that $u(x) = \min_{y \in \mathbb{T}_m} u(y)$ then $u(y) = u(x)$ for any $y \in \mathbb{T}_m^x$.*

If F is an averaging operator then F is a continuous function and therefore the following result holds.

Lemma 2.14. *Let F be an averaging operator and $\{u_n\}_{n \in \mathbb{N}}$ be a sequence of F -harmonic functions. If*

$$u(x) = \lim_{n \rightarrow +\infty} u_n(x)$$

for all $x \in \mathbb{T}_m$, then u is a F -harmonic function.

The Fatou set $\mathcal{F}(u)$ of a function u is the set of the branches $\pi = (x_1, \dots, x_k, \dots)$ on which

$$\lim_{k \rightarrow +\infty} u(x_k)$$

exists and is finite, and $BV(u)$ is the set of the branches $\pi = (x_1, \dots, x_k, \dots)$ on which u has finite variation

$$\sum_{k=1}^{\infty} |u(x_{k+1}) - u(x_k)|.$$

Clearly $BV(u) \subseteq \mathcal{F}(u)$.

In [9, Theorem A], the authors show that: If F is an averaging operator and \mathcal{H}_F^m is the set of bounded F -harmonic functions on \mathbb{T}_m , then

$$\min_{\mathcal{H}_F^m} \dim \mathcal{F}(u) = \min_{\mathcal{H}_F^m} \dim BV(u) = \log_m(\tau(m, F)),$$

where

$$\tau(m, F) = \min \left\{ \sum_{j=1}^m e^{x_j} : F(x_1, \dots, x_m) = 0 \right\}$$

and \dim denotes the usual Hausdorff dimension.

In [10], for the classical p -harmonic functions on trees (Example 2.9), the authors prove that

$$\lim_{m \rightarrow +\infty} \min_{\mathcal{H}_{Fp}^m} \dim \mathcal{F}(u) = \lim_{m \rightarrow +\infty} \min_{\mathcal{H}_{Fp}^m} \dim BV(u) = 1.$$

While from [4], for the p -harmonious functions on trees (Example 2.10), we have that

$$\lim_{m \rightarrow +\infty} \min_{\mathcal{H}_{F_0}^m} \dim \mathcal{F}(u) = \lim_{m \rightarrow +\infty} \min_{\mathcal{H}_{F_0}^m} \dim BV(u) = \frac{1}{2} + \frac{\beta}{2}.$$

In the case $F = F_1$, we observe that the minimum $\tau(m, F_1)$ is attained at

$$x_i = \begin{cases} -\frac{m+(1-s)(1-\alpha)}{m} \log \gamma & \text{if } 1 \leq i \leq s-1, \\ \frac{(s-1)(1-\alpha)}{m} \log \gamma & \text{if } s \leq i \leq m, \end{cases}$$

where

$$\gamma = \frac{m + (1-s)(1-\alpha)}{(m-s+1)(1-\alpha)} \quad \text{and} \quad s = \begin{cases} \left[\frac{m}{2}\right] + 1 & \text{if } m \text{ is odd,} \\ \frac{m}{2} & \text{if } m \text{ is even.} \end{cases}$$

Therefore

$$\begin{aligned} \min_{\mathcal{H}_{F_1}} \dim \mathcal{F}(u) &= \min_{\mathcal{H}_{F_1}} \dim BV(u) \\ &= \log_m \left(\frac{m}{1-\alpha} \left(\frac{(m-s+1)(1-\alpha)}{m+(1-s)(1-\alpha)} \right)^{1-\frac{s-1}{m}(1-\alpha)} \right). \end{aligned}$$

Finally, in the case $F = F_2$, the minimum $\tau(m, F_2)$ is attained at

$$\begin{aligned} x_j &= \frac{1-\alpha}{2} \log \eta, \quad 1 \leq j \leq m-1, \quad x_m = -\frac{1+\alpha}{2} \log \eta, & \text{if } 0 \leq \alpha \leq \frac{m-2}{m}, \\ x_1 &= -\frac{1+\alpha}{2} \log \eta, \quad x_j = \frac{1-\alpha}{2} \log \eta, \quad 2 \leq j \leq m, & \text{if } \frac{m-2}{m} < \alpha < 1, \end{aligned}$$

where

$$\eta = \frac{1+\alpha}{(m-1)(1-\alpha)}.$$

Then

$$\min_{\mathcal{H}_{F_2}} \dim \mathcal{F}(u) = \min_{\mathcal{H}_{F_2}} \dim BV(u) = \log_m \left(\frac{2(m-1)(1+\alpha)^{-\frac{1+\alpha}{2}}}{((m-1)(1-\alpha))^{\frac{1-\alpha}{2}}} \right).$$

Thus, we can compute the following limits as the number of branches go to infinity,

$$\begin{aligned} \lim_{m \rightarrow +\infty} \min_{\mathcal{H}_{F_1}} \dim \mathcal{F}(u) &= \lim_{m \rightarrow +\infty} \min_{\mathcal{H}_{F_1}} \dim BV(u) = 1, \\ \lim_{m \rightarrow +\infty} \min_{\mathcal{H}_{F_2}} \dim \mathcal{F}(u) &= \lim_{m \rightarrow +\infty} \min_{\mathcal{H}_{F_2}} \dim BV(u) = \frac{1}{2} + \frac{\alpha}{2}. \end{aligned}$$

2.4. F -harmonic Measure. Let F be an averaging operator, $f : [0, 1] \rightarrow \mathbb{R}$ be a bounded function, $c > 0$ and $E \subset \partial \mathbb{T}_m$.

We define $U_F(f, E, c)$ as the set of all F -superharmonic functions u such that

$$\liminf_{k \rightarrow +\infty} u(x_k) \geq f(\pi) + c\chi_E(\pi),$$

for all $\pi = (x_1, \dots, x_k, \dots) \in \partial \mathbb{T}_m$.

When $f \equiv 0$ and $c = 1$, we say that $U_F(0, E, 1)$ is the upper class of E , and

$$\omega_F(x, E) := \inf \{u(x) : u \in U_F(0, E, 1)\}$$

is the F -harmonic measure function for E . We call $\omega_F(E) := \omega_F(\emptyset, E)$ the F -harmonic measure of E .

Let E be a subset of $\partial \mathbb{T}_m$, following the arguments in [6], we have that

- (a) $0 \leq \omega_F(\cdot, E) \leq 1$ on \mathbb{T}_m ;
- (b) $\omega_F(E) \leq \omega_F(G)$ when $E \subset G$;
- (c) $\omega_F(\cdot, E)$ is F -harmonic on \mathbb{T}_m ;
- (d) If E is compact, then $\lim_{k \rightarrow +\infty} \omega_F(x_k, E) = 0$ for $\pi = (x_1, \dots, x_k, \dots) \in \partial \mathbb{T}_m \setminus E$;
- (e) If E is compact, then $\omega_F(E) + \omega_F(\partial \mathbb{T}_m \setminus E) = 1$.

Moreover, if E and G are disjoint compact sets on $\partial \mathbb{T}_m$ and $\omega_F(E) = \omega_F(G) = 0$, then $\omega_F(E \cup G) = 0$. Lastly if $E_1 \supseteq E_2 \supseteq \dots \supseteq E_j \supseteq \dots$ are compact sets then

$$\lim_{j \rightarrow +\infty} \omega_F(E_j) = \omega_F \left(\bigcap_{j=1}^{\infty} E_j \right).$$

In [9], the authors show that F -harmonious measures on trees lack many desirable properties of set valued functions find in classical analysis. More precisely, if F is a permutation invariant averaging operator, not equal to the usual average, then ω_F is not a Choquet capacity, union of sets of ω_F measure zero can have positive ω_F measure and there exist sets of full ω_F measure having small dimension. See also [1].

3. THE DIRICHLET PROBLEM

We now introduce what we understand by the Dirichlet problem in this work.

Dirichlet Problem (DP). Given an averaging operator F and a bounded function $f: [0, 1] \rightarrow \mathbb{R}$, find a F -harmonic function u such that

$$\lim_{k \rightarrow +\infty} u(x_k) = f(\pi) \quad \forall \pi = (x_1, \dots, x_k, \dots) \in \partial \mathbb{T}_m.$$

We say that u is a supersolution of (DP) if u is a F -superharmonic function and

$$\liminf_{k \rightarrow +\infty} u(x_k) \geq f(\pi) \quad \forall \pi = (x_1, \dots, x_k, \dots) \in \partial \mathbb{T}_m.$$

We say that u is a subsolution of (DP) if $-u$ is a supersolution of (DP) with boundary data $-f$.

3.1. Existence. In this subsection, following [4, Section 4], we give a proof of existence of solutions of (DP) when the boundary data is a continuous function.

Let $f: [0, 1] \rightarrow \mathbb{R}$ be a bounded function and $n \in \mathbb{N}$, we define $f_n: [0, 1] \rightarrow \mathbb{R}$ as

$$f_n(t) := \sum_{j=0}^{m^n-1} f(j/m^n) \chi_{I_{nj}}(t)$$

where $I_{nj} := [j/m^n, (j+1)/m^n)$ for all $j \in \{0, \dots, m^n - 1\}$ and $I_{n(m^n-1)} := [(m^n-1)/m^n, 1]$. Note that this function is piecewise constant.

Our next goal is to construct a F -harmonic function u_n such that $u_n(x) = f_n(x)$ for all $x \in \mathbb{T}_m^k$ for any $k \geq n$.

We first observe that, for all $j \in \{0, \dots, m^n - 1\}$ there exists $x_{nj} \in \mathbb{T}_m^n$ such that $I_{x_{nj}} = \overline{I_{nj}}$. Then, for all $k \in \{1, \dots, n\}$, we take $\{x_{(n-k)j}\}_{j=0}^{m^{n-k}-1} \subset \mathbb{T}_m$ such that

$$\mathcal{S}(x_{(n-k)j}) = \{x_{(n-k+1)\tau} : 1 + (j-1)m \leq \tau \leq jm\} \quad \forall j \in \{0, \dots, m^{n-k} - 1\}.$$

Let $u_n: \mathbb{T}_m \rightarrow \mathbb{R}$ be such that

$$(3.3) \quad u_n(y) := f(j/m^n),$$

for all $y \in \mathbb{T}_m^{x_{nj}}$ for some $j \in \{1, \dots, m^n - 1\}$, and

$$u_n(x_{(n-k)j}) := F(u_n(x_{(n-k)j}, 0), \dots, u_n(x_{(n-k)j}, m-1)),$$

for any $k \in \{1, \dots, n\}$ and for all $j \in \{0, \dots, m^{n-k} - 1\}$.

It is easy to check that u_n is a F -harmonic function. Moreover, $\{u_n\}_{n \in \mathbb{N}}$ is uniformly bounded on \mathbb{T}_m due to the fact that f is bounded.

Remark 3.1. Let f be a continuous function on $[0, 1]$. Then, given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x) - f(y)| \leq \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{\delta} |x - y|,$$

for all $x, y \in [0, 1]$. Therefore, for any $n \in \mathbb{N}$ and $j \in \{0, \dots, m^n - 1\}$ we have that

$$|f_n(x) - f(y)| \leq \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{\delta m^n},$$

for all $x, y \in I_{nj}$. Then $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly to f .

We are now ready to state our existence result for the Dirichlet problem.

Theorem 3.2. *Let F be an averaging operator and $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Then the sequence $\{u_n\}_{n \in \mathbb{N}}$ converges uniformly to a solution u of (DP) with boundary data f . Moreover, if f is a Lipschitz function we have a bound for the error, it holds that*

$$|u_n(x) - u(x)| \leq \frac{L}{m^n},$$

for all $x \in \mathbb{T}_m$, where L is the Lipschitz constant of f .

Proof. The proof is divided into 3 steps.

Step 1. First, we prove that $\{u_n\}_{n \in \mathbb{N}}$ is a uniformly Cauchy sequence. Let $h, k, n \in \mathbb{N}$ and $x \in \mathbb{T}_m^h$. If $n \leq k \leq h$, there exist $i \in \{0, \dots, m^n - 1\}$ and $j \in \{0, \dots, m^k - 1\}$ such that $u_n(x) = f_n(x) = f(i/m^n)$ and $u_k(x) = f_k(x) = f(j/m^k)$. Moreover, $I_x \subset I_{x_{kj}} \subset I_{x_{ni}}$. Then, given $\varepsilon > 0$, by Remark 3.1, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|u_n(x) - u_k(x)| \leq \frac{\varepsilon}{2} + \frac{2\|f\|_\infty}{\delta m^n}, \quad \forall x \in \mathbb{T}_m^h.$$

Thus, there exists n_0 such that if $k \geq n \geq n_0$,

$$|u_n(x) - u_k(x)| \leq \varepsilon, \quad \forall x \in \mathbb{T}_m^h.$$

Then for any $x \in \mathbb{T}_m^{h-1}$, by the above inequality,

$$u_k(y) - \varepsilon \leq u_n(y) \leq u_k(y) + \varepsilon, \quad \forall y \in \mathcal{S}(x).$$

Therefore, since u_n is a F -harmonic function and using (iii) and (v), we have

$$u_k(x) - \varepsilon \leq u_n(x) \leq u_k(x) + \varepsilon \quad \forall x \in \mathbb{T}_m^{h-1},$$

that is,

$$|u_n(x) - u_k(x)| \leq \varepsilon \quad \forall x \in \mathbb{T}_m^{h-1}.$$

In the same manner, in $(h-1)$ -steps, we can see that

$$|u_n(x) - u_k(x)| \leq \varepsilon \quad \forall x \in \mathbb{T}_m.$$

Therefore $\{u_n\}_{n \in \mathbb{N}}$ is a uniformly Cauchy sequence.

Step 2. Now we show that

$$u(x) := \lim_{n \rightarrow +\infty} u_n(x) \quad \forall x \in \mathbb{T}_m$$

is a solution of (DP) with boundary data f . By step 1, $\{u_n\}_{n \in \mathbb{N}}$ converges uniformly to u . Therefore, by Lemma 2.14, u is a F -harmonic function. Then, we only need to show that

$$\lim_{k \rightarrow +\infty} u(x_k) = f(\pi) \quad \forall \pi = (x_1, \dots, x_k, \dots) \in \partial \mathbb{T}_m.$$

Let $\pi = (x_1, \dots, x_k, \dots) \in \partial \mathbb{T}_m$ and $\varepsilon > 0$. Since $\{u_n\}_{n \in \mathbb{N}}$ converges uniformly to u , there exists $n_0 = n_0(\varepsilon)$ such that

$$(3.4) \quad |u_n(x_j) - u(x_j)| \leq \frac{\varepsilon}{2}, \quad \forall j \in \mathbb{N}$$

if $n \geq n_0$.

On the other hand, we can observe that there exists $n_1 = n_1(\varepsilon)$ such that

$$|f_n(\pi) - f(\pi)| \leq \frac{\varepsilon}{2}$$

if $n \geq n_1$. Then, since $u_n(x) = f_n(x)$ for all $x \in \mathbb{T}_m^h$ for any $h \geq n$, if $n \geq n_1$ we have that

$$(3.5) \quad |u_n(x_j) - f(\pi)| \leq \frac{\varepsilon}{2} \quad \forall j \geq n.$$

Finally, taking $n \geq \max\{n_0, n_1\}$ and $j \geq n$, by (3.4) and (3.5), we get

$$|u(x_j) - f(\pi)| \leq |u(x_j) - u_n(x_j)| + |u_n(x_j) - f(\pi)| \leq \varepsilon.$$

Step 3. We observe that if f is a Lipschitz function, in the same manner as in step 1, we obtain that, if $k, n \in \mathbb{N}$,

$$|u_n(x) - u_k(x)| \leq \frac{L}{m^n} \quad \forall x \in \mathbb{T}_m.$$

Therefore,

$$|u_n(x) - u(x)| \leq \frac{L}{m^n} \quad \forall x \in \mathbb{T}_m,$$

where L is the Lipschitz constant of f . This completes the proof. \square

3.2. Uniqueness. The comparison principle and the uniqueness of our Dirichlet problem follow immediately from the following lemma.

Lemma 3.3. *Let F be an averaging operator and $f, g: [0, 1] \rightarrow \mathbb{R}$ be bounded functions. If u is a subsolution of (DP) with boundary data f and v is a supersolution of (DP) with boundary data g then*

$$\sup_{x \in \mathbb{T}_m} \{u(x) - v(x)\} \leq \sup_{x \in [0, 1]} \{f(x) - g(x)\}.$$

Proof. Let $M = \sup_{x \in \mathbb{T}_m} \{u(x) - v(x)\}$. Then, given $\varepsilon > 0$ there exists $x_{k_0} \in \mathbb{T}_m$ such that

$$\begin{aligned} M - \varepsilon &\leq u(x_{k_0}) - v(x_{k_0}) \\ &\leq F(u(x_{k_0}, 0), \dots, u(x_{k_0}, m-1)) - F(v(x_{k_0}, 0), \dots, v(x_{k_0}, m-1)). \end{aligned}$$

Using Remark 2.2, we get

$$M - \varepsilon \leq \max_{y \in \mathcal{S}(x_{k_0})} \{u(y) - v(y)\}.$$

Thus, taking $x_{k_1} \in \mathcal{S}(x_{k_0})$ such that $\max_{y \in \mathcal{S}(x_{k_0})} \{u(y) - v(y)\} = u(x_{k_1}) - v(x_{k_1})$, we have that

$$M - \varepsilon \leq u(x_{k_1}) - v(x_{k_1}).$$

Continuing this reasoning, we obtain by induction that for all $j \geq 1$ there exists $x_{k_j} \in \mathcal{S}(x_{k_{j-1}})$ such that

$$(3.6) \quad M - \varepsilon \leq u(x_{k_j}) - v(x_{k_j}).$$

Now, since

$$\limsup_{k \rightarrow +\infty} u(x_k) \leq f(\pi) \quad \text{and} \quad \liminf_{k \rightarrow +\infty} v(x_k) \geq g(\pi)$$

for all $\pi = (x_1, \dots, x_k, \dots) \in \partial \mathbb{T}_m$, by (3.6), we have that

$$M - \varepsilon \leq f(\pi_0) - g(\pi_0)$$

where $\pi_0 = (x_{k_0}, \dots, x_{k_j}, \dots)$. Therefore

$$M - \varepsilon \leq \sup_{x \in [0, 1]} \{f(x) - g(x)\}.$$

Since ε is arbitrary, the proof is complete. \square

The above lemma implies the comparison principle for solutions of (DP).

Theorem 3.4 (Comparison Principle). *Let F be an averaging operator and $f, g: [0, 1] \rightarrow \mathbb{R}$ be bounded functions. If v is a supersolution (resp. subsolution) of (DP) with boundary data g , u is a solution of (DP) with boundary data f and $f \leq g$ (resp. $f \geq g$), we have that $u \leq v$ (resp. $u \geq v$).*

Now, we arrive to the main result of this section.

Theorem 3.5. *Let F be an averaging operator and $f: [0, 1] \rightarrow \mathbb{R}$ be a bounded function. There exists a unique bounded solution of (DP) with boundary data f .*

Proof. Theorem 3.2 gives a solution of (DP) and the comparison principle implies the uniqueness. \square

Remark 3.6. Observe that the sequence $\{u_n\}_{n \in \mathbb{N}}$ given by (3.3) converges uniformly to the unique solution of (DP).

4. UNIQUE CONTINUATION PROPERTY

Now, following [4], we give a characterization for the subsets of \mathbb{T}_m that have the unique continuation property.

Definition 4.1 (UCP). Let F be an averaging operator. We say that a subset U of \mathbb{T}_m satisfies the unique continuation property if for any bounded F -harmonic function u such that $u = 0$ in U , we have that $u \equiv 0$ in \mathbb{T}_m .

Let us first prove that the density of the set $\psi(U)$ in $[0, 1]$ is a necessary condition for UCP.

Theorem 4.2. *Let F be an averaging operator such that $F(1, 0, \dots, 0, -1) = 0$. If $U \subset \mathbb{T}_m$ satisfies UCP then $\psi(U)$ is dense in $[0, 1]$.*

Proof. The proof is similar to that of Theorem 5.2 in [4] but we include the details here for completeness. We will show that if $\psi(U)$ is not dense in $[0, 1]$, then there exists a F -harmonic function u such that $u \neq 0$ in \mathbb{T}_m and $u = 0$ in U . Since $\psi(U)$ is not dense in $[0, 1]$ there exist $\tau > 0$ and $r \in [0, 1]$ such that

$$(4.7) \quad (r - \tau, r + \tau) \cap \psi(U) = \emptyset.$$

Then there exist $k \in \mathbb{N}$ and $x = (a_1, \dots, a_k) \in \mathbb{T}_m$ such that $1/m^k < \tau$ and $I_x \subset (r - \tau, r + \tau)$. Therefore, using (4.7) and the fact that I_x is the subset of $\partial\mathbb{T}_m$ consisting of all branches that start at x , we have that $(x, b_1, \dots, b_s) \notin U$ for all $s \in \mathbb{N}$. Now, we construct u as follows

$$u(y) = \begin{cases} 1 & \forall y \in \mathbb{T}_m^{(x,0)}, \\ -1 & \forall y \in \mathbb{T}_m^{(x,m-1)}, \\ 0 & \text{otherwise.} \end{cases}$$

Using that $F(1, 0, \dots, 0, -1) = 0$, we obtain that u is a bounded F -harmonic function such that $u = 0$ in U and $u \neq 0$. This finishes the proof. \square

To state the characterization for the subsets of \mathbb{T}_m that have the unique continuation property, we need the following definition.

Definition 4.3. Let U be a subset of \mathbb{T}_m such that $\mathbb{T}_m^n \setminus U \neq \emptyset$ for all $n \in \mathbb{N}$. We define the sequence $\{\rho_k(U)\}_{k \in \mathbb{N}} \subset \mathbb{N}$ as follows:

$$\rho_1(U) := \min\{n \in \mathbb{N} : \exists x \in \mathbb{T}_m^n \cap U\},$$

and for all $k \in \mathbb{N}_{\geq 2}$,

$$\rho_k(U) := \min\{n \in \mathbb{N} : \exists y \in \mathbb{T}_m^{\eta_{k-1}(U)} \setminus U \text{ and } x \in \mathbb{T}_m^{\eta_{k-1}(U)+n} \cap U \cap \mathbb{T}_m^y\},$$

where

$$\eta_{k-1}(U) := \sum_{j=1}^{k-1} \rho_j(U).$$

In addition, for all $k \in \mathbb{N}_{\geq 2}$, we define the sets

$$\mathcal{A}_k(U) := \{y \in \mathbb{T}_m^{\eta_{k-1}(U)} \setminus U : I_y \cap I_{x_j} = \emptyset \forall j \in \{1, \dots, k-1\}\}.$$

We will simply write ρ_k , η_{k-1} and \mathcal{A}_k when no confusion arises.

We can now formulate the main result of this section.

Theorem 4.4. *Let F be a permutation invariant averaging operator and U be a subset of \mathbb{T}_m such that*

- (P0) $\psi(U)$ is dense in $[0, 1]$ and $\mathbb{T}_m^n \setminus U \neq \emptyset$ for all $n \in \mathbb{N}$;
- (P1) There exists a unique $x_1 \in U \cap \mathbb{T}_m^{\rho_1}$;
- (P2) For all $k \in \mathbb{N}_{\geq 2}$ and for all $y \in \mathcal{A}_{k-1}$ there exists a unique $x \in \mathbb{T}_m^{\eta_{k-1} + \rho_k} \cap U \cap \mathbb{T}_m^y$.

Then U satisfies UCP if and only if

$$\sum_{k=1}^{\infty} \delta^{\rho_k} = +\infty$$

where $\delta = F(0, \dots, 0, 1)$.

Proof. We use ideas from [4]. We will proceed in two steps.

Step 1. First we will prove that if U satisfies UCP, then

$$\sum_{k=1}^{\infty} \delta^{\rho_k} = +\infty.$$

Arguing by contradiction, we suppose that $\sum_{k=1}^{\infty} \delta^{\rho_k} < +\infty$. By (P1), there exists a unique $x_1 = (a_1, \dots, a_{\rho_1}) \in U$ such that $\tau_{1i} = (a_1, \dots, a_i) \notin U$ for any $1 \leq i < \rho_1$.

Since F is a permutation invariant averaging operator, there is no loss of generality in assuming that $a_i = m-1$ for all $1 \leq i \leq \rho_1$.

We now construct a F -harmonic function u such that $u = 0$ in U as follows:

$$\begin{aligned} u(\emptyset) &= 1, \\ u(m-1) &= \mathfrak{m}_{11} = \min_{y \in S(\emptyset)} u(y) \\ u(b_1, \dots, b_j) &= \mathfrak{M}_{11} = \max_{y \in S(\emptyset)} u(y) \text{ if } b_1 \neq m-1 \quad \forall 1 \leq j \leq \rho_1, \end{aligned}$$

and for any $2 \leq i < \rho_1$

$$\begin{aligned} u(\tau_{1i}) &= \mathfrak{m}_{1i} = \min_{y \in S(\tau_{1i})} u(y) \\ u(\tau_{1(i-1)}, b_i, \dots, b_j) &= \mathfrak{M}_{1i} = \max_{y \in S(\tau_{1(i-1)})} u(y) \text{ if } b_i \neq m-1 \quad \forall i \leq j \leq \rho_1. \end{aligned}$$

Since $x_1 \in U$ and we need that $u = 0$ in U , we define

$$u(x_1) = 0 = \mathfrak{m}_{1, \rho_1} = \min_{y \in S(\tau_{1(\rho_1-1)})} u(y).$$

We also take $u(y) = 0$ for all $y \in \mathbb{T}_m^{x_1}$. Thus, in order for u to be a F -harmonic function, we need to take $\mathfrak{M}_{11}, \dots, \mathfrak{M}_{1, \rho_1}$ and $\mathbf{m}_{11}, \dots, \mathbf{m}_{1, \rho_1-1}$ such that

$$\begin{aligned} 1 &= F(\mathfrak{M}_{11}, \dots, \mathfrak{M}_{11}, \mathbf{m}_{11}), \\ \mathbf{m}_{1i} &= F(\mathfrak{M}_{1(i+1)}, \dots, \mathfrak{M}_{1(i+1)}, \mathbf{m}_{1(i+1)}), \quad \forall 1 \leq i < \rho_1. \end{aligned}$$

Then, by Remark 2.3, we have that

$$(4.8) \quad 1 = (1 - \delta)\mathfrak{M}_{11} + \delta \mathbf{m}_{11},$$

$$(4.9) \quad \mathbf{m}_{1i} = (1 - \delta)\mathfrak{M}_{1(i+1)} + \delta \mathbf{m}_{1(i+1)}, \quad \forall 1 \leq i < \rho_1$$

where $\delta = F(0, \dots, 0, 1)$.

Now, using that $\mathbf{m}_{1\rho_1} = 0$, we have that

$$\mathfrak{M}_{1\rho_1} = \frac{\mathbf{m}_{1(\rho_1-1)}}{1 - \delta}.$$

If we take

$$\mathfrak{M}_{1i} = \mathfrak{M}_{1\rho_1} = \frac{\mathbf{m}_{1(\rho_1-1)}}{1 - \delta} = \mathfrak{M}_1, \quad \forall 1 \leq i \leq \rho_1,$$

by (4.9), we obtain

$$\mathbf{m}_{1i} = \mathbf{m}_{1(\rho_1-1)} + \delta \mathbf{m}_{1(i+1)}, \quad \forall 1 \leq i < \rho_1.$$

Using the above equality, we have

$$\mathbf{m}_{1(\rho_1-2)} = \mathbf{m}_{1(\rho_1-1)} + \delta \mathbf{m}_{1(\rho_1-1)} = (1 + \delta)\mathbf{m}_{1(\rho_1-1)},$$

and, for any $2 < j \leq \rho_1 - 1$,

$$(4.10) \quad \mathbf{m}_{1(\rho_1-j)} = \sum_{k=0}^{j-1} \delta^k \mathbf{m}_{1(\rho_1-1)} = \frac{1 - \delta^j}{1 - \delta} \mathbf{m}_{1(\rho_1-1)}.$$

Thus, by (4.8) and (4.10), we have that

$$\mathbf{m}_{1(\rho_1-1)} = \frac{1 - \delta}{1 - \delta^{\rho_1}}.$$

In addition, since $\mathfrak{M}_1 = \frac{\mathbf{m}_{1(\rho_1-1)}}{1 - \delta}$, we obtain

$$\mathfrak{M}_1 = \frac{1}{1 - \delta^{\rho_1}}.$$

On the other hand,

$$\mathcal{A}_2 = \{y_j\}_{j=1}^{m^{\rho_1}-1} \text{ and } u(y_j) = \mathfrak{M}_{11} \quad \forall j \in \{1, \dots, m^{\rho_1} - 1\}.$$

Furthermore, by (P2), for all $j \in \{1, \dots, m^{\rho_1} - 1\}$ there exists a unique

$$x_2^j = (y_j, a_{\rho_1+1}^j, \dots, a_{\rho_1+\rho_2}^j) \in \mathbb{T}_m^{\rho_1+\rho_2} \cap U$$

with $\tau_{2i}^j = (y_j, a_{\rho_1+1}^j, \dots, a_{\rho_1+i}^j) \notin U$ for any $i \in \{1, \dots, \rho_2\}$.

Again, since F is a permutation invariant averaging operator, there is no loss of generality in assuming that $a_{\rho_1+i}^j = m - 1$ for all $1 \leq i \leq \rho_2$.

Let $j \in \{1, \dots, m^{\rho_1} - 1\}$. We define u as follows

$$u(y_j, m - 1) = \mathbf{m}_{21} = \min_{y \in S(y_j)} u(y)$$

$$u(y_j, b_{\rho_1+1}, \dots, b_{\rho_1+l}) = \mathfrak{M}_{21} = \max_{y \in S(y_j)} u(y) \text{ if } b_{\rho_1+1} \neq m - 1 \quad \forall l \in \{1, \dots, \rho_2\},$$

and for any $2 \leq i < \rho_2$,

$$\begin{aligned} u(\tau_{2i}^j) &= \mathbf{m}_{2i} = \min_{y \in S(\tau_{2(i-1)}^j)} u(y), \\ u(\tau_{2(i-1)}^j, b_{\rho_1+i}, \dots, b_{\rho_1+j}) &= \mathfrak{M}_{2i} \\ &= \max_{y \in S(\tau_{2(i-1)}^j)} u(y) \text{ if } b_{\rho_1+l} \neq m-1 \quad \forall l \in \{i, \dots, \rho_2\}. \end{aligned}$$

Since $x_2^j \in U$ and we need that $u = 0$ in U , we define

$$u(x_2^j) = 0 = \mathbf{m}_{2, \rho_2} = \min_{y \in S(\tau_{2(\rho_2-1)}^j)} u(y).$$

We also take $u(y) = 0$ for all $y \in \mathbb{T}_m^{x_2^j}$. Arguing as before, taking

$$\mathfrak{M}_{2i} = \mathfrak{M}_{2\rho_2} = \frac{\mathbf{m}_{2(\rho_2-1)}}{1-\delta} = \mathfrak{M}_2 \quad \forall 1 \leq i \leq \rho_2,$$

we get

$$\begin{aligned} \mathbf{m}_{2(\rho_2-1)} &= \frac{\mathfrak{M}_1}{\rho_2-1} = \frac{1-\delta^{\rho_2}}{1-\delta}, \\ &\quad \sum_{k=0}^{\rho_2-1} \delta^k \\ \mathbf{m}_{2(\rho_2-l)} &= \sum_{k=0}^{l-1} \delta^k \mathbf{m}_{2(\rho_2-1)} = \frac{1-\delta^l}{1-\delta} \mathbf{m}_{2(\rho_2-l)}, \quad \forall l \in \{2, \dots, \rho_2-1\}, \end{aligned}$$

and

$$\mathfrak{M}_2 = \frac{\mathfrak{M}_1}{1-\delta^{\rho_2}} = \frac{1}{(1-\delta^{\rho_1})(1-\delta^{\rho_2})}.$$

By induction in k , we construct u so that u is F -harmonic in \mathbb{T}_m , $u = 0$ in U , $u \neq 0$ in \mathbb{T}_m and

$$\mathfrak{M}_k = \prod_{i=1}^k \frac{1}{1-\delta^{\rho_i}}, \quad \forall k \in \mathbb{N}.$$

Since

$$\sum_{i=1}^{\infty} \delta^{\rho_i} < +\infty,$$

we have that

$$\sum_{i=1}^{\infty} \frac{\delta^{\rho_i}}{1-\delta^{\rho_i}} < +\infty \Leftrightarrow \sum_{i=1}^{\infty} \log \left(1 + \frac{\delta^{\rho_i}}{1-\delta^{\rho_i}} \right) = \sum_{i=1}^{\infty} \log \left(\frac{1}{1-\delta^{\rho_i}} \right) < +\infty.$$

Thus,

$$\prod_{i=1}^{\infty} \frac{1}{1-\delta^{\rho_i}} < +\infty.$$

Therefore u is a bounded F -harmonic function such that $u = 0$ in U and $u \neq 0$ in \mathbb{T}_m . This is a contradiction.

Step 2. We assume that

$$\sum_{i=1}^{\infty} \delta^{\rho_i} = +\infty$$

and we prove that U satisfies the UCP .

Suppose that there exists a F -harmonic function $v \neq 0$ such that $v = 0$ in U . We will prove that v is unbounded. Multiplying v by a suitable constant, we can assume that $v(\emptyset) = 1$. Let u be defined as in the above step. First, we need to show that

$$(4.11) \quad \mathfrak{M}_k \leq \max\{v(y) : y \in \mathbb{T}_m^{\rho_k}\} \quad \forall k \in \mathbb{N}.$$

To this end, we observe that

$$(1 - \delta)\mathfrak{M}_1 + \delta\mathfrak{m}_{11} = u(\emptyset) = 1 = v(\emptyset) \leq (1 - \delta) \max_{y \in \mathcal{S}(\emptyset)} v(y) + \delta \min_{y \in \mathcal{S}(\emptyset)} v(y),$$

due to (v), Remark 2.3 and the fact that F is a permutation invariant averaging operator. Then

$$\mathfrak{M}_1 \leq \max_{y \in \mathcal{S}(\emptyset)} v(y) \quad \text{or} \quad \mathfrak{m}_{11} \leq \min_{y \in \mathcal{S}(\emptyset)} v(y).$$

If $\mathfrak{M}_1 \leq \max_{y \in \mathcal{S}(\emptyset)} v(y)$ then $\mathfrak{M}_1 \leq \max\{v(y) : y \in \mathbb{T}_m^k \text{ with } k \in \{1, \dots, \rho_1\}\}$, and therefore $\mathfrak{M}_1 \leq \max\{v(y) : y \in \mathbb{T}_m^{\rho_1}\}$.

Now we consider the case $\mathfrak{M}_1 > \max_{y \in \mathcal{S}(\emptyset)} v(y)$ and $\mathfrak{m}_{11} \leq \min_{y \in \mathcal{S}(\emptyset)} v(y)$.

By (P1), there exists a unique $x_1 = (a_1, \dots, a_{\rho_1}) \in U$ such that $\tau_{1i} = (a_1, \dots, a_i) \notin U$ for any $1 \leq i < \rho_1$. Then, since $\mathfrak{m}_{11} \leq \min_{y \in \mathcal{S}(\emptyset)} v(y) \leq v(a_1)$, we have that

$$(1 - \delta)\mathfrak{M}_1 + \delta\mathfrak{m}_{12} = \mathfrak{m}_{11} \leq v(a_1) \leq (1 - \delta) \max_{y \in \mathcal{S}(a_1)} v(y) + \delta \min_{y \in \mathcal{S}(a_1)} v(y),$$

and then

$$\mathfrak{M}_1 \leq \max_{y \in \mathcal{S}(a_1)} v(y) \quad \text{or} \quad \mathfrak{m}_{12} \leq \min_{y \in \mathcal{S}(a_1)} v(y).$$

Again, if $\mathfrak{M}_1 \leq \max_{y \in \mathcal{S}(a_1)} v(y)$, then we have that $\mathfrak{M}_1 \leq \max\{v(y) : y \in \mathbb{T}_m^{\rho_1}\}$. If $\mathfrak{m}_{12} \leq \min_{y \in \mathcal{S}(a_1)} v(y) \leq v(\tau_{12})$, then we can prove as before that

$$\mathfrak{M}_1 \leq \max_{y \in \mathcal{S}(\tau_{12})} v(y) \quad \text{or} \quad \mathfrak{m}_{13} \leq \min_{y \in \mathcal{S}(\tau_{12})} v(y).$$

In the same manner, using $(\rho_1 - 1)$ steps, we show that

$$\mathfrak{M}_1 \leq \max\{v(y) : y \in \mathbb{T}_m^{\rho_1}\} \quad \text{or} \quad \mathfrak{m}_{1(\rho-1)} \leq \min_{y \in \mathcal{S}(\tau_{1(\rho-2)})} v(y).$$

If $\mathfrak{m}_{1(\rho-1)} \leq \min_{y \in \mathcal{S}(\tau_{1(\rho-2)})} v(y) \leq v(\tau_{1(\rho-1)})$, then

$$(1 - \delta)\mathfrak{M}_1 = \mathfrak{m}_{1\rho_1} \leq v(\tau_{1(\rho-1)}) = (1 - \delta) \max_{y \in \mathcal{S}(\tau_{1(\rho-1)})} v(y) + \delta \min_{y \in \mathcal{S}(\tau_{1(\rho-1)})} v(y).$$

Since $x_1 = (\tau_{1(\rho-1)}, a_{\rho_1}) \in U$ and $v = 0$ in U , $\min_{y \in \mathcal{S}(\tau_{1(\rho-1)})} v(y) \leq 0$ and then

$$\mathfrak{M}_1 \leq \max_{y \in \mathcal{S}(\tau_{1(\rho-1)})} v(y).$$

Therefore

$$\mathfrak{M}_1 \leq \max\{v(y) : y \in \mathbb{T}_m^{\rho_1}\}.$$

Then, by induction on k , using (P2), we have that (4.11) holds. Now, we observe that, since

$$\sum_{j=1}^{\infty} \delta^{b_j} = +\infty,$$

we have that

$$\lim_{k \rightarrow +\infty} \mathfrak{M}_k = \lim_{k \rightarrow +\infty} \prod_{i=1}^k \frac{1}{1 - \delta^{\rho_i}} = +\infty.$$

Therefore, by (4.11), v is unbounded. The proof is complete. \square

5. F -HARMONIC MEASURE ESTIMATES

In this section we give some estimates for F -harmonic measures. First we introduce some definitions.

Definition 5.1. Let F be an averaging operator, $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function, $u : \mathbb{T}_m \rightarrow \mathbb{R}$ the solution of (DP) with boundary data f , $c > 0$ and $n \in \mathbb{N}$. Given $I = \bigcup_{j=k_0}^{k_1} \left[\frac{j}{m^n}, \frac{j+1}{m^n} \right]$ with $0 \leq k_0 \leq k_1 \leq m^n - 1$, we define $\mathbf{v}_{(f,I,c)} = \mathbf{v} : \mathbb{T}_m \rightarrow \mathbb{R}$ by

$$\mathbf{v}(x) = \begin{cases} u(x) + c & \text{if } x \in K, \\ u(x) & \text{if } x \in \mathbb{T}_m^y \text{ with } y \in \mathbb{T}_m^n \setminus \{x_j\}_{j=k_0}^{k_1}, \\ F(\mathbf{v}(x, 0), \dots, \mathbf{v}(x, m-1)) & \text{if } x \in \mathbb{T}_m^j, 0 \leq j \leq n-1, \end{cases}$$

where $\{x_j\}_{j=k_0}^{k_1}$ is the unique set of vertices of \mathbb{T}_m^n such that $I = \bigcup_{j=k_0}^{k_1} I_{x_j}$ and $K = \bigcup_{j=k_0}^{k_1} \mathbb{T}_m^{x_j}$.

Remark 5.2. The function $\mathbf{v}_{(f,I,c)}$ is F -harmonic.

We now prove some technical results.

Lemma 5.3. *Let F be a permutation invariant averaging operator with the property (2.2), $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function and $c > 0$. If $I = \bigcup_{j=k_0}^{k_1} \left[\frac{j}{m^n}, \frac{j+1}{m^n} \right]$ with $0 \leq k_0 \leq k_1 \leq m^n - 1$ and $k_0 + k_1 \neq m^n - 1$, then*

$$(5.12) \quad \mathbf{v}_{(f,I,c)}(\emptyset) = \inf \{w(\emptyset) : w \in U_F(f, I, c)\}.$$

Proof. Let $\{x_j\}_{j=k_0}^{k_1} \subset \mathbb{T}_m^n$ such that

$$I = \bigcup_{j=k_0}^{k_1} I_{x_j}.$$

Given $w \in U_F(f, I, c)$, we have that:

- For each $j \in \{k_0, k_0+1, \dots, k_1\}$, $\mathbf{v}_{(f,I,c)}$ and w are the solution and a supersolution of (DP) on $\mathbb{T}_m^{x_j}$ with boundary data $f + c\chi_I$, respectively;
- For any $z \in \mathbb{T}_m^n \setminus \{x_j\}_{j=k_0}^{k_1}$, $\mathbf{v}_{(f,I,c)}$ and w are the solution and a supersolution of (DP) on \mathbb{T}_m^z with boundary data f , respectively.

Thus, by the comparison principle,

$$\mathbf{v}_{(f,I,c)}(x) \leq w(x), \quad \forall x \in \mathbb{T}_m^y$$

for any $y \in \mathbb{T}_m^n$. Therefore, using that $\mathbf{v}_{(f,I,c)}$ is F -harmonic, w is F -superharmonic and (v), we have that

$$\mathbf{v}_{(f,I,c)}(x) \leq w(x),$$

for all $x \in \mathbb{T}_m$. In particular

$$\mathbf{v}_{(f,I,c)}(\emptyset) \leq w(\emptyset).$$

Since $w \in U_F(f, I, c)$ is arbitrary, we obtain that

$$\mathbf{v}_{(f,I,c)}(\emptyset) \leq \inf \{w(\emptyset) : w \in U_F(f, I, c)\}.$$

To prove the opposite inequality, we will construct a sequence $\{w_l\}_{l \in \mathbb{N}} \subset U_F(f, I, c)$ such that

$$\mathbf{v}_{(f, I, c)}(\emptyset) = \lim_{l \rightarrow +\infty} w_l(\emptyset).$$

To this end, we need to study three cases.

Case 1. First we study the case $k_0 = 0$.

Let $l \in \mathbb{N}$. We define $I_l := \left[\frac{k_1+1}{m^n}, \frac{k_1+1}{m^n} + \frac{1}{m^{n+l}} \right]$ and $w_l(x) := \mathbf{v}_{(f, I \cup I_l, c)}(x)$. Observe that w_l is a F -harmonic function such that

$$\liminf_{k \rightarrow +\infty} w_l(x_k) \geq f(\pi) + c\chi_I(\pi) \quad \forall \pi = (x_1, \dots, x_k, \dots) \in \partial \mathbb{T}_m.$$

Then $w_l \in U_F(f, I, c)$ for all $l \in \mathbb{N}$. Moreover, by (v), $\{w_l\}_{l \in \mathbb{N}}$ is a nonincreasing sequence and

$$w_l(x) \geq \mathbf{v}_{(f, I, c)}(x)$$

for all $x \in \mathbb{T}_m$ and $l \in \mathbb{N}$.

Finally, we will prove the following inequality

$$(5.13) \quad \mathbf{v}_{(f, I, c)}(\emptyset) \leq w_l(\emptyset) \leq \mathbf{v}_{(f, I, c)}(\emptyset) + c\kappa^{n+l}$$

for all $l \in \mathbb{N}$.

Let $l \in \mathbb{N}$ and $z_1 \in \mathbb{T}_m^{n+l-1}$ such that $I_l = I_{z_0}$ where $z_0 = (z_1, 0)$. Then, for any $x \in \mathbb{T}_m^{n+r}$ with $r \in \mathbb{N}_l$, we have that

$$w_l(x) = \begin{cases} \mathbf{v}_{(f, I, c)}(x) & \text{if } x \in \mathbb{T}_m^y \text{ with } y \in \mathbb{T}_m^{n+l} \setminus \{z_0\}, \\ \mathbf{v}_{(f, I, c)}(x) + c & \text{if } x \in \mathbb{T}_m^{z_0}. \end{cases}$$

Then

$$(5.14) \quad w_l(x) = \mathbf{v}_{(f, I, c)}(x) \quad \forall x \in \mathbb{T}_m^{n+l-1} \setminus \{z_1\},$$

and, by (v), (2.2) and the fact that $\mathbf{v}_{(f, I, c)}$ is F -harmonic, we get

$$(5.15) \quad \begin{aligned} w_l(z_1) &= F(w_l(z_1, 0), w_l(z_1, 1), \dots, w_l(z_1, m-1)) \\ &= F(\mathbf{v}_{(f, I, c)}(z_0) + c, \mathbf{v}_{(f, I, c)}(z_1, 1), \dots, \mathbf{v}_{(f, I, c)}(z_1, m-1)) \\ &\leq F(\mathbf{v}_{(f, I, c)}(z_1, 0), \mathbf{v}_{(f, I, c)}(z_1, 1), \dots, \mathbf{v}_{(f, I, c)}(z_1, m-1)) + c\kappa \\ &= \mathbf{v}_{(f, I, c)}(z_1) + c\kappa. \end{aligned}$$

Let $z_2 \in \mathbb{T}_m^{n+l-2}$ such that $z_1 = (z_2, 0)$. Then, by (5.14),

$$w_l(x) = \mathbf{v}_{(f, I, c)}(x) \quad \forall x \in \mathbb{T}_m^{n+l-1} \setminus \{z_2\},$$

and, using that w_l is F -harmonic, (5.15), (v), (2.2) and the fact that $\mathbf{v}_{(f, I, c)}$ is F -harmonic, we get

$$\begin{aligned} w_l(z_2) &= F(w_l(z_2, 0), w_l(z_2, 1), \dots, w_l(z_2, m-1)) \\ &= F(w_l(z_1), \mathbf{v}_{(f, I, c)}(z_2, 1), \dots, \mathbf{v}_{(f, I, c)}(z_2, m-1)) \\ &\leq F(\mathbf{v}_{(f, I, c)}(z_1) + c\kappa, \mathbf{v}_{(f, I, c)}(z_2, 1), \dots, \mathbf{v}_{(f, I, c)}(z_2, m-1)) \\ &\leq F(\mathbf{v}_{(f, I, c)}(z_2, 0), \mathbf{v}_{(f, I, c)}(z_2, 1), \dots, \mathbf{v}_{(f, I, c)}(z_2, m-1)) + c\kappa^2 \\ &= \mathbf{v}_{(f, I, c)}(z_2) + c\kappa^2. \end{aligned}$$

By repeating this procedure $n + l - 2$ times we can obtain (5.13). Therefore taking limit as $l \rightarrow +\infty$ in (5.13), we have that

$$\lim_{l \rightarrow +\infty} w_l(\emptyset) = \mathbf{v}_{(f, I, c)}(\emptyset).$$

Case 2. $k_1 = m^n - 1$. The proof of this case is similar to the previous one.

Case 3. Finally we will study the case $0 < k_0 \leq k_1 < m^n - 1$.

Let $l \in \mathbb{N}$. We define $I_l^1 = [\frac{k_0}{m^n} - \frac{1}{m^{n+l}}, \frac{k_0}{m^n}]$, $I_l^2 = [\frac{k_1+1}{m^n}, \frac{k_1+1}{m^n} + \frac{1}{m^{n+l}}]$ and $w_l(x) := \mathbf{v}_{(f, I_l^1 \cup I_l^2, c)}$. As in case 1, $w_l \in U_F(f, I, c)$ for all $l \in \mathbb{N}$ and

$$w_l(x) \geq \mathbf{v}_{(f, I, c)}(x)$$

for all $x \in \mathbb{T}_m$ and $l \in \mathbb{N}$.

We will prove the following inequality

$$\mathbf{v}_{(f, I, c)}(\emptyset) \leq w_l(\emptyset) \leq \mathbf{v}_{(f, I, c)}(\emptyset) + 2c\kappa^{n+l},$$

for all $l \in \mathbb{N}$.

Let $l \in \mathbb{N}$ and $z_1^1, z_1^2 \in \mathbb{T}_m^{n+l-1}$ such that $I_l^1 = I_{z_1^1}$ and $I_l^2 = I_{z_1^2}$ where $z_0^1 = (z_1^1, m^n - 1)$ and $z_0^2 = (z_1^2, 0)$. Observe that $z_1^1 \neq z_1^2$.

Then, for any $x \in \mathbb{T}_m^{n+r}$ with $r \in \mathbb{N}_l$, we have that

$$w_l(x) = \begin{cases} \mathbf{v}_{(f, I, c)}(x) & \text{if } x \in \mathbb{T}_m^y \text{ with } y \in \mathbb{T}_m^{n+l} \setminus \{z_0^1, z_0^2\}, \\ \mathbf{v}_{(f, I, c)}(x) + c & \text{if } x \in \mathbb{T}_m^{z_0^1} \cup \mathbb{T}_m^{z_0^2}. \end{cases}$$

Then

$$(5.16) \quad w_l(x) = \mathbf{v}_{(f, I, c)}(x) \quad \forall x \in \mathbb{T}_m^{n+l-1} \setminus \{z_1^1, z_1^2\}$$

and, by (v), (2.2), using that F is a permutation invariant averaging operator and the fact that $\mathbf{v}_{(f, I, c)}$ is F -harmonic, we get

$$(5.17) \quad \begin{aligned} w_l(z_1^j) &= F(w_l(z_1^1, 0), w_l(z_1^1, 1), \dots, w_l(z_1^1, m-1)) \\ &= F(\mathbf{v}_{(f, I, c)}(z_1^1, 0), \mathbf{v}_{(f, I, c)}(z_1^1, 1), \dots, \mathbf{v}_{(f, I, c)}(z_1^1, m-1) + c) \\ &\leq F(\mathbf{v}_{(f, I, c)}(z_1^1, 0), \mathbf{v}_{(f, I, c)}(z_1^1, 1), \dots, \mathbf{v}_{(f, I, c)}(z_1^1, m-1)) + c\kappa \\ &= \mathbf{v}_{(f, I, c)}(z_1^1) + c\kappa \end{aligned}$$

and

$$\begin{aligned} w_l(z_1^2) &= F(w_l(z_1^2, 0), w_l(z_1^2, 1), \dots, w_l(z_1^2, m-1)) \\ &= F(\mathbf{v}_{(f, I, c)}(z_0^2) + c, \mathbf{v}_{(f, I, c)}(z_1^2, 1), \dots, \mathbf{v}_{(f, I, c)}(z_1^2, m-1)) \\ &\leq F(\mathbf{v}_{(f, I, c)}(z_1^2, 0), \mathbf{v}_{(f, I, c)}(z_1^2, 1), \dots, \mathbf{v}_{(f, I, c)}(z_1^2, m-1)) + c\kappa \\ &= \mathbf{v}_{(f, I, c)}(z_1^2) + c\kappa. \end{aligned}$$

Let $z_2^1, z_2^2 \in \mathbb{T}_m^{n+l-2}$ such that $z_1^1 = (z_2^1, m-1)$ and $z_1^2 = (z_2^2, 0)$. Then, by (5.16),

$$w_l(x) = \mathbf{v}_{(f, I, c)}(x) \quad \forall x \in \mathbb{T}_m^{n+l-1} \setminus \{z_2^1, z_2^2\}.$$

Using that w_l is F -harmonic, (5.17), (v), (2.2), F is a permutation invariant averaging operator and the fact that $\mathbf{v}_{(f, I, c)}$ is F -harmonic, we get

$$\begin{aligned} w_l(z_2^1) &= F(w_l(z_2^1, 0), w_l(z_2^1, 1), \dots, w_l(z_2^1, m-1)) \\ &= F(\mathbf{v}_{(f, I, c)}(z_2^1, 0), \mathbf{v}_{(f, I, c)}(z_2^1, 1), \dots, w_l(z_1^1)) \\ &\leq F(\mathbf{v}_{(f, I, c)}(z_2^1, 0), \mathbf{v}_{(f, I, c)}(z_2^1, 1), \dots, \mathbf{v}_{(f, I, c)}(z_1^1) + c) \\ &\leq F(\mathbf{v}_{(f, I, c)}(z_2^1, 0), \mathbf{v}_{(f, I, c)}(z_2^1, 1), \dots, \mathbf{v}_{(f, I, c)}(z_2^1, m-1)) + c\kappa^2 \\ &= \mathbf{v}_{(f, I, c)}(z_2^1) + c\kappa^2 \end{aligned}$$

and, by (5.17),

$$\begin{aligned}
w_l(z_2^2) &= F(w_l(z_2^2, 0), w_l(z_2^2, 1), \dots, w_l(z_2^2, m-1)) \\
&= F(\mathbf{v}_{(f,I,c)}(z_1^2), \mathbf{v}_{(f,I,c)}(z_2^2, 1), \dots, w_l(z_2^2, m-1)) \\
&\leq F(\mathbf{v}_{(f,I,c)}(z_1^2) + c, \mathbf{v}_{(f,I,c)}(z_2^2, 1), \dots, \mathbf{v}_{(f,I,c)}(z_2^2, m-1)) \\
&\leq F(\mathbf{v}_{(f,I,c)}(z_2^2, 0), \mathbf{v}_{(f,I,c)}(z_2^2, 1), \dots, \mathbf{v}_{(f,I,c)}(z_2^2, m-1)) + c\kappa^2 \\
&= \mathbf{v}_{(f,I,c)}(z_2^2) + c\kappa^2.
\end{aligned}$$

There exists $z \in \mathbb{T}_m^k$, $0 \leq k \leq n+l-1$, such that $z_1^1, z_2^2 \in \mathbb{T}_m^z$ and there exists $\{z_j^i\}_{j=3}^{n+l-1}$, $i = 1, 2$, such that

$$\begin{aligned}
z_j^i &\in \mathcal{S}(z_{j+1}^i) \quad \forall j \in \{2, \dots, n+l-k-1\} \quad \forall i \in \{1, 2\}, \\
z_{n+l-k}^1 &= z_{n+l-k}^2 = z \text{ and } z_{n+l-k-1}^1 \neq z_{n+l-k-1}^2.
\end{aligned}$$

Arguing as before, for any $j \in \{0, \dots, n+l-k-1\}$ we have that

$$\begin{aligned}
w_l(x) &= \mathbf{v}_{(f,I,c)}(x), \quad \forall x \in \mathbb{T}_m^{n+l-j} \setminus \{z_j^1, z_j^2\}, \\
w_l(z_j^i) &\leq \mathbf{v}_{(f,I,c)}(z_j^i) + c\kappa^j, \quad i \in \{1, 2\}.
\end{aligned}$$

Therefore

$$w_l(x) = \mathbf{v}_{(f,I,c)}(x), \quad \forall x \in \mathbb{T}_m^k \setminus \{z\},$$

and using that w_l is F -harmonic, (v), Remark 2.6, that F is a permutation invariant averaging operator and the fact that $\mathbf{v}_{(f,I,c)}$ is F -harmonic, we get

$$w_l(z) \leq \mathbf{v}_{(f,I,c)}(z) + 2c\kappa^{n+l-k}.$$

Then, the following inequality

$$\mathbf{v}_{(f,I,c)}(\emptyset) \leq w_l(\emptyset) \leq \mathbf{v}_{(f,I,c)}(\emptyset) + 2c\kappa^{n+l}, \forall l \in \mathbb{N}$$

can be proved in the same way as in the case 1. Therefore

$$\lim_{l \rightarrow +\infty} w_l(\emptyset) = \mathbf{v}_{(f,I,c)}(\emptyset).$$

The proof is now complete. \square

Lemma 5.4. *Let F be a permutation invariant averaging operator with the property (2.2), $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function, $I = [\frac{k}{m^n}, \frac{k+1}{m^n}]$ with $k \in \{0, \dots, m^n - 1\}$ ($n \in \mathbb{N}$) and $c > 0$. If u is the solution of (DP) with boundary data f , then*

$$(5.18) \quad 0 \leq \mathbf{v}_{(f,I,c)}(\emptyset) - u(\emptyset) \leq c|I|^\gamma$$

for all $\gamma \leq -\log_m(\kappa)$.

Proof. In a similar way to the proof of Lemma 5.3 (case 1), we can prove that

$$(5.19) \quad u(\emptyset) \leq \mathbf{v}_{(f,I,c)}(\emptyset) \leq u(\emptyset) + c\kappa^n.$$

On the other hand, it is easy to check that

$$(5.20) \quad \kappa^n \leq \left(\frac{1}{m^n}\right)^\gamma$$

for all $\gamma \leq -\log_m(\kappa)$. Therefore, by (5.19), (5.20) and using that $c > 0$, the inequality (5.18) holds. \square

Now we consider the case where $I = [\frac{k}{m^n}, \frac{k+1}{m^n}] \cup [\frac{k+1}{m^n}, \frac{k+2}{m^n}]$ with $k \in \{0, 1, \dots, m^n - 2\}$ ($n \in \mathbb{N}$).

Lemma 5.5. *Let F be a permutation invariant averaging operator with the property (2.2), $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function, $I = \left[\frac{k}{m^n}, \frac{k+1}{m^n}\right] \cup \left[\frac{k+1}{m^n}, \frac{k+2}{m^n}\right]$ with $k \in \{0, 1, \dots, m^n - 2\}$ ($n \in \mathbb{N}$) and $c > 0$. If u is the solution of (DP) with boundary data f , then*

$$0 \leq \mathbf{v}_{(f, I, c)}(\emptyset) - u(\emptyset) \leq 2^{1-\gamma} c |I|^\gamma$$

for all $\gamma \leq -\log_m(\kappa)$.

Proof. In a similar way to the proof of Lemma 5.3 (case 3), we can show that

$$(5.21) \quad 0 \leq \mathbf{v}_{(f, I, c)}(\emptyset) \leq u(\emptyset) + 2c\kappa^n.$$

Then, by (5.20) and (5.21), we have that

$$0 \leq \mathbf{v}_{(f, I, c)}(\emptyset) - u(\emptyset) \leq 2c \left(\frac{1}{m^n}\right)^\gamma = 2^{1-\gamma} c \left(\frac{2}{m^n}\right)^\gamma$$

for all $\gamma \leq -\log_m(\kappa)$. Thus

$$0 \leq \mathbf{v}_{(f, I, c)}(\emptyset) - u(\emptyset) \leq 2^{1-\gamma} c |I|^\gamma$$

for all $\gamma \leq -\log_m(\kappa)$. □

Now, we are able to prove Theorem 1.1.

Proof of Theorem 1.1. We begin by taking

$$n = \min \left\{ l \in \mathbb{N} : \exists k \in \{0, \dots, m^l - 1\} \text{ such that } \left[\frac{k}{m^l}, \frac{k+1}{m^l} \right] \subset I \right\}.$$

Observe that

$$(5.22) \quad \frac{1}{m^n} \leq |I| < \frac{2}{m^{n-1}}$$

and there exists $k_{n-1} \in \{0, \dots, m^{n-1} - 2\}$ such that

$$I \not\subseteq J_{n-1} := \left[\frac{k_{n-1}}{m^{n-1}}, \frac{k_{n-1}+1}{m^{n-1}} \right] \cup \left[\frac{k_{n-1}+1}{m^{n-1}}, \frac{k_{n-1}+2}{m^{n-1}} \right].$$

Then $U_F(f, J_{n-1}, c) \subset U_F(f, I, c)$ and therefore

$$\inf \{w(\emptyset) - u(\emptyset) : w \in U_F(f, I, c)\} \leq \inf \{w(\emptyset) - u(\emptyset) : w \in U_F(f, J_{n-1}, c)\}.$$

Then, by Lemma 5.3 and Theorem 3.4, we have that

$$\begin{aligned} \inf \{w(\emptyset) - u(\emptyset) : w \in U_F(f, I, c)\} &\leq \inf \{w(\emptyset) - u(\emptyset) : w \in U_F(f, J_{n-1}, c)\} \\ &= \mathbf{v}_{(f, J_{n-1}, c)}(\emptyset) - u(\emptyset) \\ &\leq 2^{1-\gamma} c |J_{n-1}|^\gamma, \end{aligned}$$

for all $\gamma \leq -\log_m(\kappa)$. Therefore, using the fact that

$$|J_{n-1}| = \frac{2m}{m^n}$$

and (5.22), we have that

$$0 \leq \inf \{w(\emptyset) - u(\emptyset) : w \in U_F(f, I, c)\} \leq 2c(m|I|)^\gamma,$$

for all $\gamma \leq -\log_m(\kappa)$. □

If F is a permutation invariant averaging operator with the property that there exists $0 < \eta < 1$ such that

$$(5.23) \quad F(x_1 + c, x_2, \dots, x_m) \geq F(x_1, x_2, \dots, x_m) + c\eta$$

for all $(x_1, \dots, x_m) \in \mathbb{R}^m$ and for all $c > 0$, arguing as in Theorem 1.1, we can show the following result.

Theorem 5.6. *Let F be a permutation invariant averaging operator with the property (5.23), $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function, I be a subinterval of $[0, 1]$ and $c > 0$. If u is the solution of (DP) with boundary data f , then*

$$\inf \{w(\emptyset) - u(\emptyset) : w \in U_F(f, I, c)\} \geq c \left(\frac{|I|}{2m} \right)^\theta$$

for all $\theta \geq -\log_m(\eta)$.

Example 5.7. The permutation invariant averaging operator F_0 and F_1 satisfy (5.23) with $\eta = \frac{\beta}{m}$.

Finally, we prove Corollary 1.3.

Proof of Corollary 1.3. We begin by observing that $g \leq f + M\chi_I$ due to $\|f\|_\infty + \|g\|_\infty \leq M$. Then, by Theorem 3.4, $v(\emptyset) - u(\emptyset) \leq w(\emptyset) - u(\emptyset)$ for all $w \in U_F(f, I, M)$. Then $v(\emptyset) - u(\emptyset) \leq \inf\{w(\emptyset) - u(\emptyset) : w \in U_F(f, I, M)\}$. Therefore, using Theorem 1.1,

$$v(\emptyset) - u(\emptyset) \leq 2M(m|I|)^\gamma$$

for all $\gamma \leq -\log_m(\kappa)$. By a similar argument, we have that

$$u(\emptyset) - v(\emptyset) \leq 2M(m|I|)^\gamma,$$

for all $\gamma \leq -\log_m(\kappa)$. Thus,

$$|v(\emptyset) - u(\emptyset)| \leq 2M(m|I|)^\gamma = 2M(m\delta)^\gamma, \quad \forall \gamma \leq -\log_m(\kappa).$$

Finally, taking $\delta < \frac{1}{m} \left(\frac{\varepsilon}{2M} \right)^\frac{1}{\gamma}$, we get $|v(\emptyset) - u(\emptyset)| < \varepsilon$, which completes the proof. \square

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