Blow-up for a non-local diffusion problem with Neumann boundary conditions and a reaction term.

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Abstract

In this paper we study the blow-up problem for a non-local diffusion equation with a reaction term,

$$u_t(x,t) = \int_\Omega J(x-y)(u(y,t) - u(x,t))\, dy + u^p(x,t).$$

We prove that nonnegative and nontrivial solutions blow up in finite time if and only if $p > 1$. Moreover, we find that the blow-up rate is the same that the one that holds for the ODE $u_t = u^p$, that is, $\lim_{t \nearrow T}(T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_\infty = (\frac{1}{p-1})^{\frac{1}{p-1}}$. Next, we deal with the blow-up set. We prove single point blow-up for radially symmetric solutions with a single maximum at the origin, as well as the localization of the blow-up set near any prescribed point, for certain initial conditions in a general domain with $p > 2$. Finally, we show some numerical experiments which illustrate our results.

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1 Introduction and main results

Our purpose in this work is to analyze some features of the blow-up phenomenon arising from the following non-local diffusion problem in $\Omega \times (0,T)$,

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\Omega} J(x-y)(u(y,t)-u(x,t))\,dy + u^p(x,t), \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Here Ω is a bounded connected and smooth domain and the kernel $J : \mathbb{R}^N \to \mathbb{R}$ is assumed to be a nonnegative, bounded (let us denote by $K = \|J\|_\infty$) and symmetric function ($J(z) = J(-z)$), such that $\int_{\mathbb{R}^N} J(z)dz = 1$. We take the initial datum, $u_0(x)$, nonnegative and nontrivial.

Recently non-local diffusion processes have taken some attention in the literature. Nonlocal evolution equations of the form $u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t)$, and variations of it, have been widely used to model diffusion processes, see [AMTR], [AMRT2], [BCh], [BFRW], [CF], [ChChR], [C], [CER], [CERW], [CERW2], [F], [FW] and [IR].

As stated in [F], $u(x,t)$ can be interpreted as the density of a single population at the point x at time t, and $J(x-y)$ as the probability distribution of jumping from location y to location x. Then, the convolution $(J * u)(x,t) = \int_{\mathbb{R}^N} J(y-x)u(y,t)\,dy$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to travel to any other site. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies the nonlocal equation $u_t = J * u - u$. This equation is called a nonlocal diffusion equation, since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$, but on all the values of u in a neighborhood of x through the convolution term $J * u$. This equation shares many properties with the classical heat equation, $u_t = \Delta u$, such as the fact that bounded stationary solutions are constant, a maximum principle holds for both of them and perturbations propagate with infinite speed, [F]. However, there is no regularizing effect in general (see [ChChR]).

Note that in our problem (1.1) we are integrating in Ω. As we have explained the integral $\int J(x-y)(u(y,t)-u(x,t))\,dy$ takes into account the individuals arriving or leaving position x from other places. Therefore, we are imposing that the diffusion takes place only in Ω. No individual may enter or leave the domain. This is what is called Neumann boundary conditions, see [CERW]. Moreover, in this work we add a reaction term $+u^p(x,t)$ in the equation and look for possible blow-up singularities for the resulting problem.

As we will see through these pages, problem (1.1) shares many important properties with the corresponding local diffusion problem

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u^p(x,t), & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \eta}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

such as blowing-up conditions, blow-up rates or blow-up sets (see [BB], [CM], [FM]). For general references on blow-up problems we refer to the surveys [BB], [GV], the book [SGKM] and references therein.

We begin our study of (1.1) with a result of existence and uniqueness of continuous solutions and a comparison principle.

Theorem 1.1 *For every $u_0 \in C(\overline{\Omega})$ there exists a time $T > 0$ and a unique solution $u \in C([0,T); C(\overline{\Omega}))$ to (1.1). If the maximal existence time of the solution, T, is finite then the solution blows up in $L^\infty(\overline{\Omega})$-norm, that is*

$$\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\overline{\Omega})} = +\infty.$$

The total mass in Ω verifies the following identity

$$\int_\Omega u(y,t)\,dy = \int_\Omega u_0(y)\,dy + \int_0^t \int_\Omega u^p(y,s)\,dy\,ds.$$

In addition, the following comparison property holds: if $u_0 \le v_0$ in Ω, then $u(x,t) \le v(x,t)$ for every $(x,t) \in \Omega \times [0,T)$.

Once we have ensured existence and uniqueness of the solutions the next step is to determine the values of the parameters for which blow up occurs.

Theorem 1.2 *Let $u_0 \in C(\overline{\Omega})$ be nonnegative and nontrivial. If $p > 1$ the corresponding solution to (1.1) blows up. Conversely, if $p \le 1$ every solution to (1.1) is global. Moreover, if $p > 1$ we have the following estimate for the blow-up time,*

$$T \le \frac{1}{(p-1)} \left(\frac{|\Omega|}{\int_\Omega u_0(x)\,dx} \right)^{p-1}. \tag{1.3}$$

Concerning the blow up rate, that is the speed at which solutions are blowing up, we find the following result,

Theorem 1.3 (Blow-up rates) *Let $p > 1$ and u be a solution to* (1.1) *blowing up at time T. Then*

$$\lim_{t \nearrow T^-} (T-t)^{\frac{1}{p-1}} \max_{x \in \Omega} u(x,t) = \left(\frac{1}{p-1} \right)^{\frac{1}{p-1}}. \qquad (1.4)$$

Note that, as it happens for the local problem (1.2), the blow-up rate is the same as the one that holds for the ODE $u_t = u^p$, hence the diffusion term (local or nonlocal) plays no role when determining the blow-up rate.

Next we deal with the spacial location of the set where the solution blows up, the blow-up set, defined as follows:

$$B(u) = \{x \in \overline{\Omega}\,;\ \text{there exist } x_n \to x,\ t_n \nearrow T,\ \text{with } u(x_n, t_n) \to \infty\}.$$

There are several situations where the blow-up set is a single point (single point blow-up), for instance for the local model (1.2) with $p > 1$ and radially symmetric initial conditions with only one maximum at the origin, [CM], [FM]. Our next result reproduces this phenomenon of single point blow-up for symmetric data with $p > 2$, we use ideas from [GR]. It is not clear for us how to adapt the arguments given in [FM] to the nonlocal problem.

Theorem 1.4 (Blow-up sets: symmetric case.) *Let us consider problem* (1.1) *with $p > 2$ in $\Omega = B_R = \{|x| < R\}$. Let $u_0 \in C^1(\overline{B_R})$ be a radial nonnegative function, with a unique maximum at the origin, that is*

$$u_0 = u_0(r) \geq 0, \quad u_0'(r) < 0 \quad \textit{if} \quad 0 < r \leq R, \quad u_0''(0) < 0. \qquad (1.5)$$

Then, the blow-up set of the solution consists only of the point $x = 0$.

Note that the flat solution (a solution that does not depend on x) blows up in the whole $\overline{\Omega}$. Hence, for any domain Ω we have initial conditions (positive constants) producing global blow-up, $B(u) = \overline{\Omega}$. However, for $p > 2$ we can also localize the blow-up set near any point in $\overline{\Omega}$ just by taking an initial condition being very large near that point and not so large in the rest of the domain. This is the content of our last result.

Theorem 1.5 (Blow-up sets: general case.) *Let us consider problem* (1.1) *in a general domain Ω with $p > 2$. Given $x_0 \in \Omega$ and $\varepsilon > 0$ there exists an initial condition, u_0, such that $B(u) \subset B_\varepsilon(x_0) = \{x \in \overline{\Omega}\,;\ \|x - x_0\| < \varepsilon\}$.*

The rest of the paper is organized as follows: in Section 2 we prove existence and uniqueness of a continuous solution as well as a comparison lemma; in Section 3 we deal with blow-up versus global existence and we find the blow-up rates; Section 4 we show our results concerning the blow-up sets and finally in Section 5 we discretize the spacial variable and include some numerical experiments which illustrate our results.

2 Local existence of solutions. Main properties

We devote this section to the proof of Theorem 1.1, concerning existence, uniqueness and the validity of the comparison principle of solutions. Existence and uniqueness will be obtained via Banach's fixed point theorem. First, let us give some necessary preliminaries. Let $t_0 > 0$ be fixed and consider $X_{t_0} = C([0, t_0]; C(\overline{\Omega}))$, a Banach space with the norm

$$\|\omega\|_{X_{t_0}} = \max_{0\le t\le t_0} \|\omega(\cdot, t)\|_{L^\infty(\overline{\Omega})} = \max_{0\le t\le t_0} \max_{\overline{\Omega}} |u(x, t)|.$$

We define the following operator $\mathfrak{D} : X_{t_0} \longrightarrow X_{t_0}$,

$$\begin{aligned}\mathfrak{D}_{\omega_0}(\omega)(x, t) = \omega_0(x) + \int_0^t \int_\Omega J(x - y)(\omega(y, s) - \omega(x, s))\, dy\, ds \\ + \int_0^t |\omega|^{p-1}\omega(x, s)\, ds.\end{aligned}$$

The solution to problem (1.1) will be obtained as a fixed point of the previous operator in a convenient ball of X_{t_0}. In the next lemma we show that this operator is well defined and give conditions assuring that it is strictly contractive.

Lemma 2.1 *The operator $\mathfrak{D}_{\omega_0}$ is well defined, mapping X_{t_0} into X_{t_0}. Moreover, let $\omega_0, z_0 \in C(\overline{\Omega})$ and $w, z \in X_{t_0}$. Then there exists a positive constant $C = C(p, \|\omega\|_{X_{t_0}}, \|z\|_{X_{t_0}}, \|J\|_\infty, \Omega)$ such that*

$$\|\mathfrak{D}_{\omega_0}(\omega) - \mathfrak{D}_{z_0}(z)\|_{X_{t_0}} \le \|\omega_0 - z_0\|_{L^\infty(\overline{\Omega})} + Ct\|\omega - z\|_{X_{t_0}}. \tag{2.6}$$

Thus, $\mathfrak{D}_{u_0}$ is a strict contraction in the ball $B(u_0, 2\|u_0\|_{L^\infty(\overline{\Omega})}) \subset X_{t_0}$, if t_0 is small enough.

Proof. We begin by checking that $\mathfrak{D}_{\omega_0}$ maps X_{t_0} into X_{t_0}. For any $(x,t) \in \overline{\Omega} \times [0,t_0]$ we have,

$$\begin{aligned}\Big|\mathfrak{D}_{\omega_0}(\omega(x,t)) - \omega_0\Big| &\le \left|\int_0^t \int_\Omega J(x-y)\Big(\omega(y,s) - \omega(x,s)\Big)dy\, ds\right| \\ &\quad + \left|\int_0^t |\omega|^{p-1}\omega(x,s)\, ds\right| \\ &\le \max\{1, K|\Omega|\} t\Big(\|\omega\|_{X_{t_0}} + \|\omega\|^p_{X_{t_0}}\Big),\end{aligned}$$

which assures that $\mathfrak{D}_{\omega_0}(\omega)$ is continuous at $t = 0$. Now, for any $(x,t_1),(x,t_2) \in \overline{\Omega} \times [0,t_0]$ it holds that

$$\begin{aligned}&\Big|\mathfrak{D}_{\omega_0}(\omega(x,t_1)) - \mathfrak{D}_{\omega_0}(\omega(x,t_2))\Big| \\ &\quad = \left|\int_{t_1}^{t_2}\left(\int_\Omega J(x-y)\Big(\omega(y,s) - \omega(x,s)\Big)dy + |\omega|^{p-1}\omega(x,s)\right) ds\right| \\ &\quad \le 2\left|\max\{1, K|\Omega|\}\int_{t_1}^{t_2}\Big(\|\omega(\cdot,s)\|_{L^\infty(\overline{\Omega})} + |\omega|^{p-1}\omega(x,s)\Big)ds\right| \\ &\quad \le 2C(t_2 - t_1)\Big(\|\omega\|_{X_{t_0}} + \|\omega\|^p_{X_{t_0}}\Big),\end{aligned}$$

which completes the proof of the continuity in time for any $t \in (0, t_0]$.

Note that $\mathfrak{D}_{\omega_0}(\omega)$ is continuous as a function of x, since the convolution in space with the function J is also uniformly continuous. Then, for any $\omega_0 \in C(\overline{\Omega})$ and $\omega \in X_{t_0}$ we conclude that $\mathfrak{D}_{\omega_0}(\omega) \in C([0,t_0]; C(\overline{\Omega}))$. Thus $\mathfrak{D}_{\omega_0}$ maps X_{t_0} into X_{t_0}.

To prove the estimate (2.6) we argue as follows: for any $(x,t) \in \overline{\Omega} \times [0,t_0]$, it holds

$$
\begin{aligned}
\Big|\Big(\mathfrak{D}_{\omega_0}(\omega) - \mathfrak{D}_{z_0}(z)\Big)(x,t)\Big| &\le \|\omega_0 - z_0\|_{L^\infty(\overline{\Omega})} \\
&\quad + \left|\int_0^t (|\omega|^{p-1}\omega(x,s) - |z|^{p-1}z(x,s))ds\right| \\
&\quad + \left|\int_0^t \int_\Omega J(x-y)\Big(\omega(y,s) - z(y,s) - (\omega(x,s) - z(x,s))\Big)dy\, ds\right| \\
&\le \|\omega_0 - z_0\|_{L^\infty(\overline{\Omega})} + p\eta^{p-1}\int_0^t |\omega(x,s) - z(x,s)|\, ds \\
&\quad + \left|2\int_0^t \|\omega(\cdot,s) - z(\cdot,s)\|_{L^\infty(\overline{\Omega})} \int_\Omega J(x-y)dy\, ds\right| \\
&\le \|\omega_0 - z_0\|_{L^\infty(\overline{\Omega})} + (p\eta^{p-1} + 2K|\Omega|)t\|\omega - z\|_{X_{t_0}},
\end{aligned}
$$

where $\eta \le \max\{\|\omega\|_{X_{t_0}}, \|z\|_{X_{t_0}}\}$. The arbitrariness of $(x,t) \in \overline{\Omega} \times [0,t_0]$ gives the desired estimate (2.6).

Finally, choosing t_0 such that $Ct_0 < 1$ and taking $\omega_0 \equiv z_0$, (2.6) ensures that $\mathfrak{D}_{\omega_0}$ is a strict contraction in the ball $B(u_0, 2\|u_0\|_{L^\infty(\overline{\Omega})})$ in X_{t_0}. Indeed, for ω and z in such a ball we have that $|\eta| \le C\|u_0\|_{L^\infty(\overline{\Omega})}$ and therefore we conclude that there exists a constant C that only depends on J and u_0, such that

$$
\|\mathfrak{D}_{u_0}(\omega) - \mathfrak{D}_{u_0}(z)\|_{X_{t_0}} \le Ct_0\|\omega - z\|_{X_{t_0}}.
$$

Hence it is enough to choose t_0 such that $Ct_0 < 1/2$ to obtain a strict contraction in the ball $B(u_0, 2\|u_0\|_{L^\infty(\overline{\Omega})})$. The proof is finished. □

Remark 1 *If $p > 1$, we can define the operator $\mathfrak{D}_{\omega_0}$ in the space $Y_{t_0} = C([0,t_0]; C^k(\overline{\Omega}))$, with the norm*

$$
\|\omega\|_{Y_{t_0}} = \max_{0\le t\le t_0} \sum_{\alpha=0}^{k} \|D_x^\alpha \omega\|_{L^\infty(\overline{\Omega})}.
$$

Arguing as in (2.6) we find a similar estimate

$$
\|\mathfrak{D}_{\omega_0}(\omega) - \mathfrak{D}_{z_0}(z)\|_{Y_{t_0}} \le \|\omega_0 - z_0\|_{C^k(\overline{\Omega})} + Ct\|\omega - z\|_{Y_{t_0}},
$$

where $C = C(p, \|\omega\|_{Y_{t_0}}, \|z\|_{Y_{t_0}}, \|J\|_\infty, \Omega)$. Thus, for any $\omega_0 \in C^k(\overline{\Omega})$, $\mathfrak{D}_{\omega_0}$ is a strict contraction in a ball of Y_{t_0} for t_0 small enough.

We have all the ingredients to prove the first statements of Theorem 1.1, concerning existence and uniqueness of solutions.

Proof of Theorem 1.1(existence and uniqueness). As a consequence of the Banach's fixed point theorem and the previous lemma we get the existence and uniqueness of solutions to (1.1) in the time interval $[0, t_0]$. If $\|u\|_{X_{t_0}} < \infty$, taking as initial datum $u(\cdot, t_0) \in C(\overline{\Omega})$ and arguing as before, it is possible to extend the solution up to some interval $[0, t_1)$, for certain $t_1 > t_0$. Hence, we conclude that if the maximal existence time of the solution, T, is finite then the solution blows up in $L^\infty(\overline{\Omega})$-norm, that is

$$\limsup_{t \nearrow T} \|u(\cdot, t)\|_{L^\infty(\overline{\Omega})} = +\infty.$$

Finally, from the equation $(1.1)_1$ it easily follows that u verifies the following identity,

$$u(x,t) - u_0(x) = \int_0^t \left(\int_\Omega J(x-y)(u(y,s) - u(x,s))\, dy + u^p(x,s) \right) ds.$$

Integrating in the x variable and applying Fubini's theorem we get

$$\int_\Omega u(x,t)\, dx - \int_\Omega u_0(x)\, dx = \int_0^t \int_\Omega u^p(x,s)\, dx\, ds,$$

and the proof is completed. □

Remark 2 *If $p > 1$ and $u_0 \in C^k(\overline{\Omega})$, $0 \le k \le \infty$, then the solution u to* (1.1) *belongs to $C([0,T); C^k(\overline{\Omega}))$, see Remark 1.*

We conclude this section with the statement of the comparison principle for the solutions to (1.1). To this end we introduce the concept of sub and supersolutions for this problem.

Definition 2.1 *A function $\overline{u} \in C^1([0,T); C(\overline{\Omega}))$ is a supersolution of* (1.1) *if it satisfies*

$$\begin{cases} \overline{u}_t(x,t) \ge \displaystyle\int_\Omega J(x-y)(\overline{u}(y,t) - \overline{u}(x,t))\, dy + \overline{u}^p(x,t), \\ \overline{u}(x,0) \ge u_0(x). \end{cases}$$

Subsolutions are defined similarly by reversing the inequalities.

Now, we state two lemmas with a comparison principle and the maximum principle for sub and supersolutions. The proofs can be found in [CERW], and hence we omit them.

Lemma 2.2 *Let $\overline{u}, \underline{u}$ be super and subsolutions to* (1.1)*, respectively. Then, $\overline{u}(x,t) \geq \underline{u}(x,t)$, for every $(x,t) \in \overline{\Omega} \times [0,T)$.*

Lemma 2.3 *Let $\overline{u}$ be a supersolution to* (1.1)*. Then, if $u_0 \geq 0$ we have that $\overline{u}(x,t) \geq 0$, for every $(x,t) \in \overline{\Omega} \times [0,T)$ and moreover, a strict inequality holds if u_0 is nontrivial.*

An analogous statement holds for subsolutions by reversing the inequalities.

3 Blow-up versus global existence. Blow-up rates.

We begin by determining the conditions that ensuring blow-up occurrence, that is, Theorem 1.2.

Proof of Theorem 1.2 Assume that $p > 1$. Integrating in $x \in \Omega$ the equation $(1.1)_1$ and applying Fubini's theorem, we get

$$\frac{\partial}{\partial t} \int\limits_{\Omega} u(x,t)\, dx = \int\limits_{\Omega} u^p(x,t)\, dx \geq |\Omega|^{1-p} \left(\int\limits_{\Omega} u(x,t)\, dx \right)^p .$$

Since $p > 1$ we have that $\int_\Omega u(x,t)\, dx$ cannot be global; thus u cannot be global either. Note that, by Theorem 1.1, in this case we have blow-up in $L^\infty(\overline{\Omega})$-norm. Moreover, integrating the above inequality we obtain the following estimate for the blow-up time

$$T \leq \frac{1}{(p-1)} \left(\frac{|\Omega|}{\int_\Omega u_0(x)\, dx} \right)^{p-1} .$$

Conversely, suppose now that $p \leq 1$. Let us consider the ODE problem

$$\begin{cases} z'(t) = z(t), \\ z(0) = \max\limits_{x \in \overline{\Omega}} \{u_0(x), 1\}. \end{cases}$$

Observe that $z(t) \geq z^p(t)$, since $z(t) > 1$ for every $t > 0$ and $p \geq 1$. Therefore, z is a global supersolution of our problem (1.1). Thus, u is global by comparison. $\square$

Now we proceed with the proof of Theorem 1.3, which gives the blow up rate.

Proof of Theorem 1.3 Let $T < \infty$ the maximal time of existence of a blowing up solution. Let $x_0 \in \overline{\Omega}$ be such that $\max_{x\in\overline{\Omega}} u(\cdot,t) = u(x_0,t)$. From the equation $(1.1)_1$ for this point, the following estimate follows

$$u_t(x_0,t) = \int_\Omega J(x-y)\Big(u(y,t)-u(x_0,t)\Big)dy + u^p(x_0,t) \le u^p(x_0,t). \quad (3.7)$$

Integrating (3.7) in (t,T), and taking into account that $p>1$, we obtain

$$\max_{x\in\overline{\Omega}} u(x,t) \ge \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} (T-t)^{-\frac{1}{p-1}}. \quad (3.8)$$

To get the upper estimate we observe that for any $(x,t) \in \overline{\Omega}\times[0,T)$ it holds

$$u_t(x,t) \ge -u(x,t) + u^p(x,t) = u^p(x,t)(1-u^{-(p-1)}(x,t)).$$

In particular

$$\max_{x\in\overline{\Omega}} u_t(x,t) \ge \max_{x\in\overline{\Omega}} u^p(x,t)\left(1-\left(\max_{x\in\overline{\Omega}} u(x,t)\right)^{-(p-1)}\right).$$

Taking into account (3.8) in this expression we get

$$\max_{x\in\overline{\Omega}} u_t(x,t) \ge \max_{x\in\overline{\Omega}} u^p(x,t)\Big(1-(p-1)(T-t)\Big).$$

We integrate as before in (t,T) to obtain

$$\max_{x\in\overline{\Omega}} u(x,t) \le \left((p-1)(T-t) - \frac{1}{2}(p-1)^2(T-t)^2\right)^{-\frac{1}{p-1}}.$$

Taking limit as $t\to T$ (1.4) is proved. $\square$

4 Blow-up sets

In this section we give some results concerning the blow-up sets for the solutions to problem (1.1). In the following, we will assume that $p>1$ to ensure blow-up occurrence, and u will be a solution to (1.1) blowing up at time T. We begin with the symmetric case, that is, the proof of Theorem 1.4. To simplify we will consider only the one-dimensional case, $\Omega = (-L, L)$. The radial case is analogous, we leave the details to the reader.

First, we prove a lemma that says that if the initial condition has a unique maximum at the origin, then the solution has a unique maximum at this point for every $t \in (0, T)$.

Lemma 4.1 *For any p, under the hypothesis on the initial condition imposed in Theorem 1.4 we have that the solution is symmetric and such that $u_x < 0$ in $(0, L] \times (0, T)$.*

Proof. Symmetry follows from uniqueness since $w(x, t) = u(-x, t)$ is also a solution to (1.1).

Denote $v = u_x$. Then v verifies the following equation

$$\begin{aligned} v_t(x,t) = & \int_{-L}^{L} J'(x-y)(u(y,t) - u(x,t))\, dy - v(x,t) \int_{-L}^{L} J(x-y)\, dy \\ & + p u^{p-1}(x,t) v(x,t). \end{aligned}$$

From this equation it is easy to obtain a contradiction, if we assume that there exists a point $(x_0, t_0) \in (0, L] \times (0, T)$ at which $v(x_0, t_0) = 0$. We use here that J' is odd and the symmetry of u. $\square$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. The proof consists of several steps, following the ideas of [GR] for numerical approximations of the corresponding local diffusion problem.

Step 1. First, we prove that the only blow-up point that verifies the blow-up estimate (1.4) is $x = 0$. For a fixed $x_0 > 0$, let $w(t) = u(0, t) - u(x_0, t)$. This function w verifies

$$\begin{aligned} w'(t) = & \int_{-L}^{L} J(-y)(u(y,t) - u(0,t))\, dy - \int_{-L}^{L} J(x_0 - y)(u(y,t) - u(x_0,t))\, dy \\ & + p\xi^{p-1}(t) w(t), \end{aligned}$$

where $\xi(t)$ is a point between $u(0, t)$ and $u(x_0, t)$. Hence

$$\begin{aligned} w'(t) \geq & \int_{-L}^{L} (J(-y) - J(x_0 - y)) u(y,t)\, dy - w(t) + p\xi^{p-1}(t) w(t) \\ \geq & -w(t) + p\xi^{p-1}(t) w(t). \end{aligned}$$

Integrating we have

$$\ln(w)(t) - \ln(w)(t_0) \geq \int_{t_0}^{t} (-1 + p\xi^{p-1}(s))\, ds.$$

Now we argue by contradiction. Assume that $(T-t)^{\frac{1}{p-1}} u(x_0, t) \to C_p$. Since $u(x_0, t) \leq \xi(t) \leq u(0, t)$, we get,

$$\lim_{t \to T} \xi(t)(T-t)^{\frac{1}{p-1}} = C_p\,,$$

and then we just have to observe that

$$\int_{t_0}^{t} (-1 + p\xi^{p-1}(s))\, ds \geq p \int_{t_0}^{t} \frac{(C_p^{p-1} - \varepsilon)}{(T-s)}\, ds - C = -p(C_p^{p-1} - \varepsilon)\ln(T-t) - C.$$

Hence

$$w(t) \geq C(T-t)^{-p(C_p^{p-1} - \varepsilon)} = C(T-t)^{-\frac{p}{p-1} + p\varepsilon}.$$

Using this fact, we have

$$0 = \lim_{t \to T} (T-t)^{\frac{1}{p-1}} w(t) \geq C \lim_{t \to T} (T-t)^{\frac{1}{p-1} - \frac{p}{p-1} + p\varepsilon} = +\infty,$$

a contradiction that proves our claim.

Step 2. We conclude by showing that the only possible blowing up point is the origin. To this end, let us perform the following change of variables

$$z(x,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \qquad (T-t) = e^{-s}. \tag{4.9}$$

This function z verifies

$$z_s(x,s) = e^{-s} \int_{-L}^{L} J(x-y)(z(y,s) - z(x,s))\, dy - \frac{1}{p-1} z(x,s) + z^p(x,s).$$

Note that the blow-up rate of u implies that $z(x,s) \leq C$ for every $(x,s) \in [-L, L] \times (-\ln(T), \infty)$.

Now we observe that, if there exists s_0 such that $z^p(x, s_0) - \frac{1}{p-1} z(x, s_0) < -Ce^{-s_0}$, then $z(x,s) \to 0$ as $s \to \infty$. This fact can be proved as in Lemma 4.2 of [GR] using that $z(x,s)$ is bounded and verifies

$$z_s(x,s) \leq Ce^{-s} - \frac{1}{p-1} z(x,s) + z^p(x,s). \tag{4.10}$$

Moreover, if there exists s_0 such that $z^p(x,s_0) - \frac{1}{p-1}z(x,s_0) > Ce^{-s_0}$, then $z(x,s)$ blows up in finite time $\tilde{s}$. This follows from Lemma 4.3 in [GR] using that

$$z_s(x,s) \ge -Ce^{-s} - \frac{1}{p-1}z(x,s) + z^p(x,s).$$

Since for $x \neq 0$, $z(x,s)$ is bounded and does not converge to C_p (thanks to step 1) we conclude that $z(x,s) \to 0$, as $s \to +\infty$.

We study next the asymptotic behaviour of $z(x,s)$. To this end, given $\varepsilon > 0$, using that $z(x,s) \to 0$ in (4.10), we get

$$z_s(x,s) \le Ce^{-s} - \left(\frac{1}{p-1} - \varepsilon\right) z(x,s).$$

By a comparison argument, as in [GR], it follows that

$$z(x,s) \le C_1 e^{-s} + C_2 e^{-(\frac{1}{p-1}-\varepsilon)s}. \tag{4.11}$$

Now going back to the equation verified by $z(x,t)$ we obtain,

$$(e^{\frac{1}{p-1}s} z(x,s))_s = e^{\frac{1}{p-1}s}\left(e^{-s}\int_{-L}^{L} J(x-y)(z(y,s)-z(x,s))\,dy + z^p(x,s)\right).$$

Integrating we get

$$z(x,s) = e^{-\frac{1}{p-1}s}\left(C_1 + \int_{s_0}^{s} e^{-\frac{p-2}{p-1}\sigma}\left(\int_{-L}^{L} J(x-y)(z(y)-z(x))dy + e^{\sigma}z^p\right)d\sigma\right).$$

From (4.11) it follows that $e^s z^p(x,s) \to 0$, as $s \to \infty$ and since z is bounded, we conclude,

$$z(x,s) \le e^{-\frac{1}{p-1}s}\left(C_1 + C_2\int_{s_0}^{s} e^{-\frac{p-2}{p-1}\sigma}d\sigma\right).$$

Using that $p > 2$, we have

$$z(x,s) \le C_3 e^{-\frac{1}{p-1}s}.$$

This implies that $u(x,t)$ verifies

$$u(x,t) = e^{\frac{1}{p-1}s}z(x,s) \le C_3.$$

The proof is now complete. $\square$

Now we show that in a general domain Ω and $p > 2$ we can find an initial condition with blow-up set localized around a given point in $\overline{\Omega}$.

Proof of Theorem 1.5 Given $x_0 \in \overline{\Omega}$ and $\varepsilon > 0$ we want to construct an initial condition u_0 such that

$$B(u) \subset B_\varepsilon(x_0) = \{x \in \overline{\Omega} \,:\, \|x - x_0\| < \varepsilon\}. \tag{4.12}$$

To this end we will consider u_0 concentrated near x_0 and small away from x_0.

Let φ be a nonnegative smooth function such that

$$supp(\varphi) \subset B_{\varepsilon/2}(x_0), \qquad \text{and} \qquad \varphi(x) > 0 \text{ for } x \in B_{\varepsilon/2}(x_0).$$

Now, let

$$u_0(x) = M\varphi(x) + \delta.$$

We want to choose M large and δ small in such a way that (4.12) holds.

First, note that, thanks to the estimate (1.3),

$$T \leq \frac{1}{(p-1)} \left(\frac{|\Omega|}{\int_\Omega u_0(x)\,dx} \right)^{p-1} \leq \frac{C(\Omega, p, \varphi)}{M^{p-1}},$$

taking M large enough we can assume that T is as small as we need.

Now, using the upper bound for the blow-up rate

$$\begin{aligned} \max_{x \in \overline{\Omega}} u(x,t) &\leq \left((p-1)(T-t) - \frac{1}{2}(p-1)^2 (T-t)^2 \right)^{-\frac{1}{p-1}} \\ &\leq C\,(T-t)^{-\frac{1}{p-1}}, \end{aligned}$$

we obtain, for any $\bar{x} \in \Omega$,

$$\begin{aligned} u_t(\bar{x},t) &= \int_\Omega J(\bar{x}-y)(u(y,t) - u(\bar{x},t))\,dy + u^p(\bar{x},t) \\ &\leq \int_\Omega J(\bar{x}-y)u(y,t)\,dy + u^p(\bar{x},t) \\ &\leq C(\Omega, J, p)\,(T-t)^{-\frac{1}{p-1}} + u^p(\bar{x},t). \end{aligned}$$

Therefore $u(\bar{x},t)$ is a subsolution to

$$w_t(t) = C(\Omega, J, p)\,(T-t)^{-\frac{1}{p-1}} + w^p(t), \tag{4.13}$$

and hence, if $u(\bar{x},0) \leq w(0)$, we have

$$u(\bar{x},t) \leq w(t). \tag{4.14}$$

Now we just have to prove that a solution w to (4.13) beginning with $w(0) = \delta$ remains bounded up to $t = T$, provided that δ and T are small enough. To see this we use ideas from [GR]. Let

$$z(s) = (T - t)^{1/(p-1)} w(t), \qquad s = -\ln(T - t).$$

It is not difficult to see that $z(s)$ verifies

$$z'(s) = Ce^{-s} - \frac{1}{p-1} z(s) + z^p(s), \qquad z(-\ln T) = T^{1/(p-1)}\delta.$$

Note that for T and δ small (T is small if M is large) it holds that $z'(-\ln T) < 0$. Indeed, we need

$$CT - \frac{1}{p-1}\delta T^{\frac{1}{p-1}} + \delta^p T^{\frac{p}{p-1}} < 0.$$

Here, we are using that $p > 2$. From this fact it is easy to prove that $z'(s) < 0$ for all $s > -\ln T$ and therefore $z(s) \to 0$ as $s \to \infty$, see [GR]. Going back to the equation verified by z we obtain that, using again that $p > 2$,

$$z(s) \leq Ce^{-\frac{1}{p-1}s}.$$

In terms of $w(t)$ this bound implies that $w(t) \leq C$, for $0 \leq t < T$. From the boundedness of w and (4.14) we get $u(\bar{x}, t) \leq w(t) \leq C$ for every $\bar{x} \in \overline{\Omega} \setminus B_\varepsilon(x_0)$, as we wished. $\square$

5 Numerical experiments.

Finally, we discretize the problem in the spacial variable and obtain an ODE system. We take $\Omega = [-3, 3]$ and $-3 = x_{-N} < .... < x_N = 3$, $N = 100$, a partition of Ω. We consider the following ODE system

$$\begin{cases} u_i'(t) = \displaystyle\sum_{j=-N}^{N} J(x_i - x_j)(u_j(t) - u_i(t)) + (u_i)^p(t) \\ u_i(0) = u_0(x_i). \end{cases}$$

We choose $p = 3$ and

$$J(s) = \begin{cases} 1 & \text{for } |s| \leq 1/10, \\ 0 & \text{for } 1/10 \leq |s| \leq 3. \end{cases}$$

In Figure 1 we show the evolution in time of a solution beginning with a symmetric initial condition with a unique maximum, as required in the hypothesis of Theorem 1.4, $u_0(x) = 9 - x^2$. The computed blow-up time is $T \approx 6.092 \times 10^{-3}$. We observe that the solution is blowing up only at the origin.

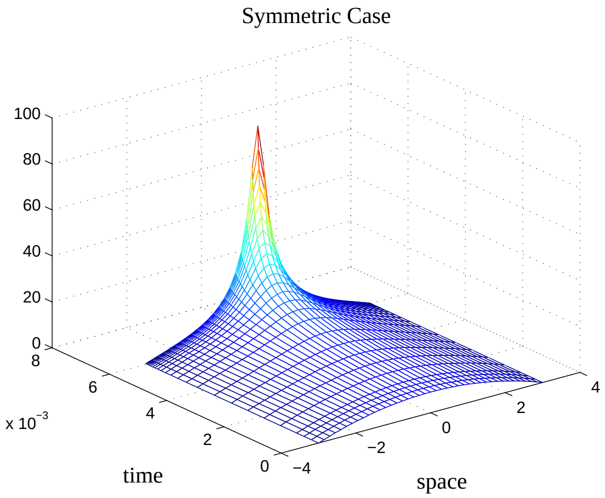

Figure 1. Evolution in time, symmetric datum.

In Figure 2 we show the evolution of the logarithm of the maximum of the solution, $u(0,t)$, vs. the logarithm of $(T-t)^{-1}$, in dashed line. We can appreciate that the slope of the graph is approximately $1/2 = 1/(p-1)$, the exponent that appears in the blow-up rate. Also, with a continuous line and in a dashed-pointed line we show the evolution of the two adjacent nodes, u_1 and u_2 (also in logarithmic scale). We can appreciate that they are bounded (the slopes of the curves become horizontal near $t = T$).

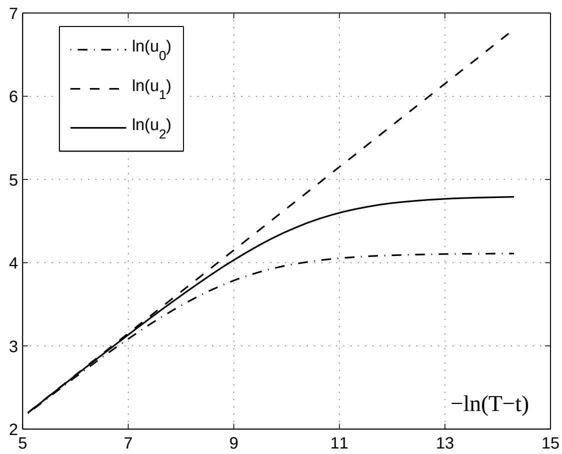

Figure 2. Blow-up rate.

Next, we choose a non-symmetric initial condition very large near the point $x_0 = 1$, $u_0(x) = 1/2 + 100(1 - |x - 1|)_+$, and show that the blow-up set is localized in a small neighborhood of that point $x_0 = 1$, Figure 3.

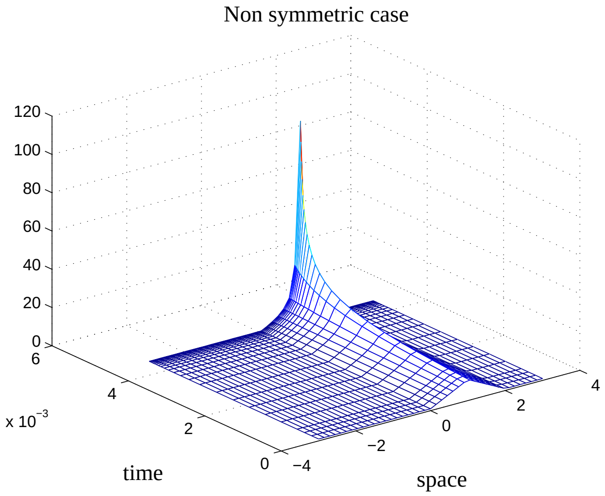

Figure 3. Evolution in time, non-symmetric datum.

Finally, we take $p = 3/2$ and the same non-symmetric initial condition. We denote by k the index of the node where the solution attains its maximum. As can be observed in Figure 4 the solution blows up (with the precise blow-up rate $1/(p-1) = 2$) at a single point, x_k. Therefore, our numerical experiments support the conjecture that Theorem 1.5 also holds for $1 < p \leq 2$.

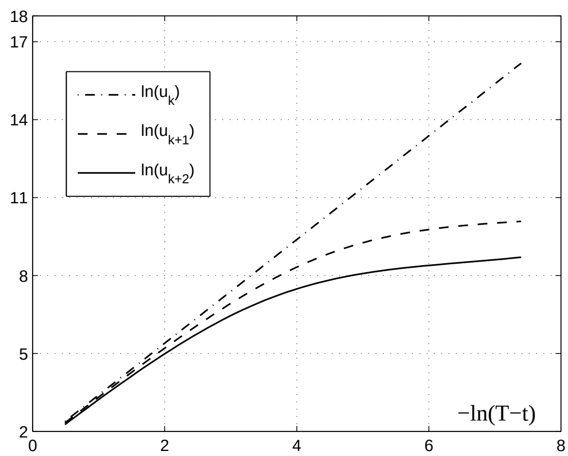

Figure 4. Blow-up rates of the maximum and the two adjacent nodes for $p = 3/2$ and non-symmetric datum.

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