ON THE FIRST NONTRIVIAL EIGENVALUE OF THE ∞-LAPLACIAN WITH NEUMANN BOUNDARY CONDITIONS.

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Abstract. We study the limit as $p \to \infty$ of the first non-zero eigenvalue λ_p of the p-Laplacian with Neumann boundary conditions in a smooth bounded domain $U \subset \mathbb{R}^n$. We prove that $\lambda_\infty := \lim_{p\to+\infty} \lambda_p^{1/p} = 2/\mathrm{diam}(U)$, where $\mathrm{diam}(U)$ denotes the diameter of U with respect to the geodesic distance in U. We can think of λ_∞ as the first eigenvalue of the ∞-Laplacian with Neumann boundary conditions. We also study the regularity of λ_∞ as a function of the domain U proving that under a smooth perturbation U_t of U by diffeomorphisms close to the identity there holds that $\lambda_\infty(U_t) = \lambda_\infty(U)+O(t)$. Although $\lambda_\infty(U_t)$ is in general not differentiable at $t = 0$, we prove that in some cases it is so with an explicit formula for the derivative.

1. INTRODUCTION

Denote by λ_p the first non-zero eigenvalue of the p-Laplacian with Neumann boundary conditions in a smooth bounded domain $U \subset \mathbb{R}^n$. The aim of this paper is two-fold. We first study the asymptotic behaviour of λ_p as $p \to \infty$, obtaining that

$$\lambda_\infty := \lim_{p\to+\infty} \lambda_p^{1/p} = \frac{2}{\mathrm{diam}(U)},$$

where $\mathrm{diam}(U)$ denotes the diameter of U with respect to the geodesic distance in U (see (12) below), and also identify the variational limit problem defining λ_∞. Analogous results have been obtained previously for the first eigenvalue of the p-Laplacian with Dirichlet or Steklov boundary conditions. Next, we study the regularity of $\lambda_\infty = \lambda_\infty(U)$ with respect to U. Considering smooth perturbations U_t of U by diffeomorphisms close to the identity, we prove that $\lambda_\infty(U_t) = \lambda_\infty(U) + O(t)$. Notice that $\lambda(U_t)$ is in general not differentiable at $t = 0$. However, we prove that it is when $\mathrm{diam}(U)$ is reached at a unique pair of points.

The limit as $p \to \infty$ of the first eigenvalue $\lambda_{p,D}$ of the p-Laplacian with Dirichlet boundary condition was studied in [15], [14] (see also [3] for an anisotropic version). In those papers the authors prove that

$$\lambda_{\infty,D} := \lim_{p\to\infty} \lambda_{p,D}^{1/p} = \inf_{v\in W_0^{1,\infty}(\Omega)} \frac{\|\nabla v\|_{L^\infty(\Omega)}}{\|v\|_{L^\infty(\Omega)}} = \frac{1}{R}, \tag{1}$$

where R is the largest possible radius of a ball contained in U. In addition, we have existence of extremals, i.e., functions where the above infimum is

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attained. These extremals can be constructed taking the limit as $p \to \infty$ in the eigenfunctions of the $p-$laplacian eigenvalue problem (see [14]) and are viscosity solutions of the following eigenvalue problem (called the infinity eigenvalue problem in the literature):

$$\begin{cases} \min\{|Du| - \lambda_{\infty,D} u,\, \Delta_\infty u\} = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

The *limit operator* $\lim_{p\to\infty} \Delta_p = \Delta_\infty$ is the ∞-Laplacian given by

$$\Delta_\infty u = -\langle D^2 u Du, Du\rangle = -\sum_{i,j=1}^{N} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial x_i} \frac{\partial u}{\partial x_i}.$$

This fact can be understood in the sense that solutions to $\Delta_p v_p = 0$ with a Dirichlet data $v_p = f$ on $\partial\Omega$ converge as $p \to \infty$ to the solution to $\Delta_\infty v = 0$ with $v = f$ on $\partial\Omega$ in the viscosity sense (see [2], [5] and [7]). This operator appears naturally when one considers absolutely minimizing Lipschitz extensions in Ω of a boundary data f (see [1], [2], and [13]).

Recently the authors in [6] relate $\lambda_{\infty,D}$ with the Monge-Kantorovich distance W_1. Recall that the Monge-Kantorovich distance $W_1(\mu,\nu)$ between two probability measures μ and ν over $\bar U$ is defined by

$$W_1(\mu,\nu) = \max_{v\in W^{1,\infty}(U),\, \|\nabla v\|_\infty \le 1} \int_U v\,(d\mu - d\nu). \tag{2}$$

It was proved in [6] that

$$\lambda_{\infty,D}^{-1} = \sup_{\mu\in P(U)} W_1(\mu, P(\partial U)), \tag{3}$$

where $P(U)$ and $P(\partial U)$ denotes the set of probability measures over $\bar U$ and ∂U. Notice that the maximum is easily seen to be reached at δ_x where $x \in U$ is a most inner point so that we can recover (1) from (3).

The case of Steklov boundary condition has also been investigated recently. Indeed the authors in [9] (see also [17] for a slightly different problem) studied the behaviour as $p \to +\infty$ of the so-called variational eigenvalues $\lambda_{k,p,S}$, $k \ge 1$, of the p-Laplacian with a Steklov boundary condition. In particular they proved that

$$\lim_{p\to+\infty} \lambda_{1,p,S}^{1/p} = 1 \quad \text{and} \quad \lambda_{2,\infty,S} := \lim_{p\to+\infty} \lambda_{2,p,S}^{1/p} = \frac{2}{\operatorname{diam}(U,\mathbb{R}^n)},$$

where here $\operatorname{diam}(U,\mathbb{R}^n)$ denotes the diameter of U for the usual Euclidean distance in $\mathbb{R}^n$, and also identify the limit problem defining $\lambda_{2,\infty,S}$.

The purpose of this paper is to complete these studies considering the case of the Neumann boundary condition. It is known (see [16]) that the first eigenvalue of the p-Laplacian with Neumann boundary condition in a smooth bounded domain $U \subset \mathbb{R}^n$ is 0 with eigenspace $\sim \mathbb{R}$, and that it is isolated. The first non-zero eigenvalue λ_p of the p-Laplacian is then defined by the minimization problem

$$\lambda_p = \inf_{u\in W^{1,p}(U)} \left\{ \int_U |\nabla u|^p\,dx \,:\, \int_U |u|^p\,dx = 1,\, \int_U |u|^{p-2}u\,dx = 0 \right\}. \tag{4}$$

According to [16], λ_p can also be characterised using Ljusternik-Schnirelman's genus by the following min-max formula

$$\lambda_p = \inf_{A \in \mathcal{A}_{p,2}} \max_{u \in A} \frac{\displaystyle\int_U |\nabla u|^p \, dx}{\displaystyle\int_U |u|^p \, dx} \tag{5}$$

where

$$\mathcal{A}_{p,2} = \{A \subset W^{1,p}(U),\ A \text{ is compact},\ A = -A,\ \gamma(A) \geq 2\},$$

and $\gamma(A) = \inf\{n \in \mathbb{N}, \exists\, \phi \in C(A, \mathbb{R}^n \backslash \{0\}) \text{ odd }\}$ is the genus of A. By standard arguments the infimum in (4) is attained by some $u_p \in W^{1,p}(U)$ satisfying the problem

$$\begin{aligned} \Delta_p u_p &= \lambda_p |u_p|^{p-2} u_p \quad \text{ in } U, \\ |\nabla u_p|^{p-2} \partial_\nu u_p &= 0 \quad \text{ on } \partial U, \end{aligned} \tag{6}$$

where $\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2} \nabla u)$. According to [16][thm 4.1] and [18], $u_p \in C^{1,\alpha}(\bar{U})$ for some $\alpha > 0$.

We first identify the limit problem obtained by taking the limit $p \to +\infty$ in (4) and provide some information on the asymptotic behaviour of the u_p.

Theorem 1. *There holds*

$$\lim_{p \to +\infty} \lambda_p^{1/p} = \lambda_\infty, \tag{7}$$

where λ_p is defined by (4)*, and*

$$\lambda_\infty := \inf \left\{ \|\nabla u\|_{L^\infty(U)};\ u \in W^{1,\infty}(U) \ s.t.\ \max_U u = -\min_U u = 1 \right\}. \tag{8}$$

Moreover if u_p is a normalized minimizer for λ_p, then, up to a subsequence, u_p converge in $C(\bar{U})$ to some minimizer $u_\infty \in W^{1,\infty}(U)$ of λ_∞ which is a solution of

$$\begin{aligned} F(u, \nabla u, D^2 u) &= 0 \quad \text{in } U, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial U, \end{aligned} \tag{9}$$

in the viscosity sense, where

$$F(u, \eta, A) = \begin{cases} \min\{-(A\eta, \eta), |\eta| - \lambda_\infty u\} & \text{in } \{u > 0\}, \\ \max\{-(A\eta, \eta), -|\eta| - \lambda_\infty u\} & \text{in } \{u < 0\}, \\ -(A\eta, \eta) \quad \text{in } \{u = 0\}. \end{cases} \tag{10}$$

Our second result gives the value of λ_∞. First notice that if U is not connected then considering a constant function equal to 1 in one connected component and -1 in another one, we obtain that $\lambda_\infty = 0$. Thus, from now on we will assume that U is connected. The value of λ_∞ turns out to be related to the intrinsic or geodesic diameter of U that we now define. Given two points $x, y \in \bar{U}$ we denote by $d(x, y)$ their intrinsic or geodesic distance defined by

$$d(x, y) = \inf_{\gamma \in \Gamma(x,y)} \mathrm{Long}(\gamma), \tag{11}$$

the infimum being taken over the class $\Gamma(x,y)$ of Lipschitz curves in $\bar{U}$ joining x and y. The intrinsic diameter $\mathrm{diam}(U)$ of U is then defined as

$$\mathrm{diam}(U) := \max_{(x,y)\in\bar{U}} d(x,y) = \max_{(x,y)\in\partial U} d(x,y). \tag{12}$$

We have the following result:

Theorem 2. *There holds*

$$\frac{2}{\lambda_\infty} = diam(U), \tag{13}$$

where λ_∞ is defined in (8)*, and $diam(U)$ in* (12)*.*

Consider for example the bounded lipschitz open subset $U \subset \mathbb{R}^2$ defined in $\mathbb{R}^2_+$ as the intersection of the sets $\mathbb{R}^2_+ \setminus D((0,0),1)$ and $D((0,1/2),\sqrt{5}/2)$. Then $\mathrm{diam}(U)$ is attained by the arc of circle $C((0,0),1)$ so that $\mathrm{diam}(U) = \pi$. Moreover the function defined in polar coordinate by $u(x,y) = \frac{2}{\pi}\theta - 1$ is admissible for λ_∞ so that $\lambda_\infty \le \|\nabla u\|_\infty = \frac{2}{\pi} = 2/\mathrm{diam}(U)$. The reverse inequality is easy to obtain (see Step 3.1 below).

We also expresses λ_∞ as the value of a maximization problem involving the Monge-Kantorovich distance in the spirit of (3). We denote by $M(\bar{U})$ the space of bounded measures over $\bar{U}$. Given $\sigma \in M(\bar{U})$, we denote its positive and negative part by σ^+ and σ^- so that $\sigma = \sigma^+ - \sigma^-$, and $|\sigma| = \sigma^+ + \sigma^-$. We then have

Theorem 3. *There holds*

$$\frac{2}{\lambda_\infty} = \max_{\sigma\in M(U),\, \int_U \sigma^+ = \int_U \sigma^- = 1} W_1(\sigma^+,\sigma^-) \tag{14}$$

where λ_∞ is defined in (8)*, and W_1 in* (2)*.*

As a corollary we can recover the value of λ_∞ given in theorem 2.

We now turn our attention to the study of the regularity of $\lambda_\infty(U)$ as a function of U. Maximization or minimization of eigenvalues with respect to the domain is an active area of research; see the survey [11]. Notice that the equation (9) for the eigenfunctions is not linear, not in divergence form, and, in addition, no regularity result is known for the eigenfunctions (further that they belong to $W^{1,\infty}(U)$). Also remark that the variational quotient (8) does not involve L^p-integrals but the L^∞-norm that is not differentiable, and that the diameter of U is defined by a sup inf problem. All these facts make the study of the dependence of λ_∞ with respect to the domain a nontrivial task.

From now on we assume that U is connected. Given a smooth vector field V on $\bar{U}$, we consider the perturbed subset U_t defined for small t by

$$U_t = \phi_t(U) \quad \text{with} \quad \phi_t(x) = x + tV(x). \tag{15}$$

Our purpose is to study the regularity of the map $t \to \lambda_\infty(U_t)$ at $t=0$, and in particular to study the existence of its derivative at $t=0$, the so-called shape-derivative. In the case of Dirichlet boundary condition this study has been done recently in [20].

We first prove, following ideas from [20], that

Theorem 4. *There exists a contant $C > 0$ such that for $|t|$ small*

$$|\lambda_\infty(U_t) - \lambda_\infty(U)| \leq Ct.$$

Notice that in general the function $t \to \operatorname{diam}(U_t)$ is not differentiable at $t = 0$ when $\operatorname{diam}(U)$ is attained at at least two pairs of points. For example take $U = B(0,1) \subset \mathbb{R}^2$ and $V(x) = 2x\eta(|x - e_2|)$ where $e_2 = (0,1)$ and $\eta : [0, +\infty) \to [0,1]$ is a smooth cut-off function equal to 1 near 0. Then, $\operatorname{diam}(U) = 2$ and

$$\operatorname{diam}(U_t) = \begin{cases} |(1+2t)e_2 - (-e_2)| = 2(1+t) & \text{if } t \geq 0, \\ 2 \quad \text{otherwise}, \end{cases}$$

so that $t \to \operatorname{diam}(U_t)$ is not differentiable at $t = 0$. When $\operatorname{diam}(U)$ is attained at an unique pair of points but with at least two extremal curves, the function $t \to \operatorname{diam}(U_t)$ is still not differentiable at $t = 0$. Consider for example the domain $U \subset \mathbb{R}^2$ bounded by the circle $x_1^2 + x_2^2 = 1$ and the ellipse $\frac{x_1^2}{4} + \frac{4x_2^2}{9} = 1$. Then $\operatorname{diam}(U)$ is attained at the pair of points $\{(-2,0),(2,0)\}$ two extremal curves: the first one is composed of the union of the segment $[(-2,0),(-\frac{1}{2},\frac{\sqrt{3}}{2})]$, the arc of the circle $C((0,0),1)$ from $(-\frac{1}{2},\frac{\sqrt{3}}{2})$ to $(\frac{1}{2},\frac{\sqrt{3}}{2})$ and the segment $[(\frac{1}{2},\frac{\sqrt{3}}{2}),(0,2)]$. The second one is its reflection through $\{y = 0\}$. Then $\operatorname{diam}(U) = 2(\sqrt{3} + \frac{\pi}{6})$. We now consider the diffeoemorphism ϕ_t defined to be the identity except in a small neighborhood of $C((0,0),1)$ where it is

$$\phi_t(x) = \begin{cases} (1 - \lambda_t(x_2))x, & \text{if } x_2 \geq 0, \\ x, & \text{if } x_2 < 0 \end{cases}$$

where λ_t is chosen so that $\phi_t(C((0,0),1) \cap \{x_2 \geq 0\}) = \mathcal{E}_t \cap \{x_2 \geq 0\})$ with $\mathcal{E}_t : x^2 + \frac{y^2}{(1-t)^2} = 1$. A short computation show that $\lambda_t(x_2) = tx_2^2 + O(t^2)$. The shortest-path in $\phi_t(U) \cap \mathbb{R}^2_+$ from $(-2,0)$ to $(2,0)$ is composed of the segment $[(-2,0),(-\frac{1}{2},\frac{\sqrt{3}}{2}(1-t))]$, the arc of the ellipse $\mathcal{E}_t$ from $(-\frac{1}{2},\frac{\sqrt{3}}{2}(1-t))$ to $(\frac{1}{2},\frac{\sqrt{3}}{2}(1-t))$ and the segment $[(\frac{1}{2},\frac{\sqrt{3}}{2}(1-t)),(0,2)]$. Its length is $\operatorname{diam}(U) - t(\frac{\pi}{6} + \frac{\sqrt{3}}{4}) + O(t^2)$ which is less that $\operatorname{diam}(U)$ when $t > 0$. Hence we can see that

$$\operatorname{diam}(\phi_t(U)) = \begin{cases} \operatorname{diam}(U) - t(\frac{\pi}{6} + \frac{\sqrt{3}}{4}) + O(t^2) & \text{if } t > 0, \\ \operatorname{diam}(U) \quad \text{if } t \leq 0, \end{cases}$$

It follows that $t \to \operatorname{diam}(U_t)$ is not differentiable at $t = 0$. As a conclusion for the function $t \to \operatorname{diam}(U_t)$ to be differentiable at $t = 0$ we must assume at least that $\operatorname{diam}(U)$ is attained at an unique pair of points with an unique shortest-curve. Indeed we can prove that under a slightly stronger assumption the function $t \to \lambda_\infty(U_t)$ is differentiable at $t = 0$ with an explicit formula for the derivative.

Theorem 5. *Assume that*

1. *$diam(\bar{U})$ is attained at an unique pair of points (x^*, y^*),*
2. *for any $(x,y) \in \partial U \times \partial U$ close to (x^*, y^*), there exists an unique curve γ joining x to y such that $d(x,y) = Long(\gamma)$.*

Then $t \to \lambda_\infty(U_t)$ is differentiable at $t = 0$ with derivative

$$\frac{d}{dt}\lambda_\infty(U_t)_{|t=0} = -\frac{2}{diam(U)^3}\int_0^1 (DV(\gamma^*(s))\gamma^{*'}(s), \gamma^{*'}(s))\, ds, \tag{16}$$

where $\gamma^* : [0,1] \to \bar{U}$ is the unique constant-speed curve joining x^* to y^* such that $diam(\bar{U}) = d(x^*, y^*) = Long(\gamma^*)$.

In the particular case where γ^* is the segment $[x^*, y^*]$, e.g. if U is convex, then $\gamma^*(s) = x^* + t(y^* - x^*)$, $s \in [0,1]$, and

$$\begin{aligned}\int_0^1 (DV(\gamma^*(s))\gamma^{*'}(s), \gamma^{*'}(s))\, ds &= \int_0^1 DV(\gamma^*(s))\gamma^{*'}(s)\, ds.(y^* - x^*) \\ &= \int_0^1 \frac{d}{ds}V(\gamma^*(s))\, ds.(y^* - x^*),\end{aligned}$$

so that, in that case, formula (16) becomes

$$\frac{d}{dt}\lambda_\infty(U_t)_{|t=0} = -2\frac{(V(y^*) - V(x^*))(y^* - x^*)}{\mathrm{diam}(U)^3}. \tag{17}$$

Notice that if the segment (x^*, y^*) is strictly included in U then the extremal curve for $\mathrm{diam}(U_t)$ is also a segment $[x_t^*, y_t^*]$ with $x_t^* \to x^*$, $y_t^* \to y^*$ and then writing

$$\mathrm{diam}(U_t) = \max_{(x,y)\in\partial U\times\partial U \text{ close to } (x^*,y^*)} |\phi_t(x) - \phi_t(y)|,$$

formula (17) is an easy consequence of (37) and lemma 4 below.

We eventually provide the shape-derivative formula for λ_p since we were not able to find it in the literature.

Proposition 1.1. *If λ_p is simple, then the function $t \to \lambda_p(U_t)$ is differentiable at $t = 0$ with*

$$\frac{d}{dt}\lambda_p(U_t)_{|t=0} = \int_{\partial U} (|\nabla u_p|^p - \lambda_p |u_p|^p)(V, \nu)\, d\sigma \tag{18}$$

where u_p is a normalized extremal for λ_p, and ν is the exterior unit normal vector to ∂U.

In the case $p = 2$ we recover the usual formula (see [12] thm 5.7.2 p210).

2. Proof of theorem 1.

We split the proof in several steps. We first prove that

Step 2.1. *There holds*

$$\limsup_{p\to+\infty} \lambda_p^{1/p} \le \lambda_\infty. \tag{19}$$

Proof. Let $w \in W^{1,\infty}(U)$ be admissible for λ_∞ i.e. $\max w = -\min_U w = 1$. Since w^+ and w^- are linearly independent, the set

$$A_p := \operatorname{span}\{w^-, w^+\} \cap \{u \in W^{1,p}(U),\ \|u\|_{W^{1,p}} = 1\}$$

belongs to $\mathcal{A}_{p,2}$. It then follows from (5) that

$$(\lambda_p + 1)^{-1} \ge \min_{u\in A_p} \int_U |u|^p\, dx = \min_{\{G=1\}} F(a,b)$$

where $F, G : \mathbb{R}^2 \to \mathbb{R}$ are defined by

$$F(a,b) = |a|^p \|w^+\|_p^p + |b|^p \|w^-\|_p^p, \quad G(a,b) = |a|^p \|w^+\|_{W^{1,p}}^p + |b|^p \|w^-\|_{W^{1,p}}^p.$$

Assume that $\|\nabla w^+\|_\infty < \|\nabla w^-\|_\infty$. Writing $|b|^p$ in function of $|a|^p$ in $G = 1$ we obtain

$$(\lambda_p + 1)^{-1} \ge \min_{|a|\le \|w^+\|_{W^{1,p}}^{-1}} C_p \|w^-\|_p^p |a| + \frac{\|w^-\|_p^p}{\|w^-\|_{W^{1,p}}^p},$$

$$C_p = \frac{\|w^+\|_p^p}{\|w^-\|_p^p} - \frac{\|w^+\|_{W^{1,p}}^p}{\|w^-\|_{W^{1,p}}^p}.$$

Recalling that $\max w = -\min_U w = 1$, we see that for $p \to +\infty$ we have

$$C_p > 0 \Leftrightarrow \frac{\|w^+\|_{W^{1,\infty}}}{\|w^-\|_{W^{1,\infty}}} < 1 + o(1) \Leftrightarrow \|\nabla w^+\|_\infty < \|\nabla w^-\|_\infty + o(1).$$

which is true. Hence $C_p > 0$ for large p so that the minimum is reached at $a = 0$. It follows that for p large,

$$\lambda_p^{\frac{1}{p}} \le \frac{\|\nabla w^-\|_p}{\|w^-\|_p}$$

Since $\|\nabla w^+\|_\infty < \|\nabla w^-\|_\infty$ and $\min w = -1$, we get

$$\limsup_{p\to+\infty} \lambda_p^{\frac{1}{p}} \le \frac{\|\nabla w^-\|_\infty}{\|w^-\|_\infty} \le \|\nabla w\|_\infty.$$

If $\|\nabla w^+\|_\infty > \|\nabla w^-\|_\infty$, then writing $|a|^p$ in function of $|b|^p$ in $G = 1$ we obtain the same as before interchanging w^+ and w^-. We thus obtain that $\limsup_{p\to+\infty} \lambda_p^{\frac{1}{p}} \le \lambda'_\infty$ where λ'_∞ is defined as λ_∞ by (8) with the additional constraint that either $\|\nabla u^+\|_\infty > \|\nabla u^-\|_\infty$ or $\|\nabla u^+\|_\infty < \|\nabla u^-\|_\infty$. Notice that if u is admissible for λ_∞ then for an appropriate function η, $u_\varepsilon = u + \varepsilon\eta$, $\varepsilon > 0$, is admissible for λ'_∞ and $\lim_{\varepsilon\to 0} u_\varepsilon = u$ in $W^{1,\infty}(U)$. Hence $\lambda_\infty = \lambda'_\infty$, which ends the proof of (19). Concerning η, if for example, $\|\nabla u^+\|_\infty = \|\nabla u^-\|_\infty$, given $x_0 \in \operatorname{argmax} |\nabla u^+|$, take $\eta \in C^\infty(U, [0,1])$ with compact support in a sufficiently small neighborhood of x_0 and such that $\eta(x_0) = 0$, $\nabla\eta(x_0) = \nabla u(x_0)$. Then $u_\varepsilon^- = u^-$ and $|\nabla u_\varepsilon(x_0)|^2 = (1+2\varepsilon+\varepsilon^2)|\nabla u(x_0)|^2 > |\nabla u(x_0)|^2$ so that $\|\nabla u_\varepsilon^+\|_\infty > \|\nabla u^+\|_\infty = \|\nabla u^-\|_\infty = \|\nabla u_\varepsilon^-\|_\infty$. $\square$

As a second step, we prove that, up to a subsequence, u_p converges uniformly to a minimizer of λ_∞.

Step 2.2. *Up to a subsequence, u_p converge uniformly in $\bar{U}$ to some $u_\infty \in W^{1,\infty}(U)$ which is a minimizer of λ_∞ defined by* (8)*. Moreover* (7) *holds.*

Proof. Let u_p be a normalized minimizer for λ_p. We first notice that $(u_q)_{q\geq p}$ is bounded in $W^{1,p}(U)$ for any p. Indeed by Hölder's inequality,

$$\int_U |\nabla u_q|^p \leq \|\nabla u_q\|_q^p |U|^{1-p/q}$$

so that by (19),

$$\|\nabla u_q\|_p \leq \lambda_q^{1/q} |U|^{1/p-1/q} \leq C_p. \tag{20}$$

In the same way

$$\|u_q\|_p \leq \|u_q\|_q |U|^{1/p-1/q} = |U|^{1/p-1/q} \leq C_p. \tag{21}$$

Taking $p > n$ it follows by Morrey´s inequality that $(u_q)_{q>p}$ is bounded in some Hölder space $C^{0,\alpha}(\bar{U})$, and then, up to a subsequence, that $u_q \to u$ in $C(\bar{U})$ according to Arzela-Ascoli theorem. We can also assume that this convergence holds weakly in $W^{1,p}(U)$ for any p.

Let us prove that $\|u\|_\infty = 1$. Letting $q \to +\infty$ and then $p \to +\infty$ in (21), we see that $\|u\|_\infty \leq 1$. Suppose that $\|u\|_\infty \leq 1 - 2\varepsilon < 1$ for some $\varepsilon > 0$. Since $\lim_{p\to\infty} \|u_p\|_\infty = \|u\|_\infty$, we have $\|u_p\|_\infty \leq 1 - \varepsilon$ for p large. Then

$$1 = \int_U |u_p|^p \, dx \leq (1-\varepsilon)^p |U| \to 0$$

as $p \to +\infty$, which is absurd.

We now verify that $\max u + \min u = 0$. From $\int_U |u_p|^{p-2} u_p \, dx = 0$ we obtain that

$$\int_{\{u_p \geq 0\}} |u_p|^{p-1} \, dx = \int_{\{u_p \leq 0\}} |u_p|^{p-1} \, dx.$$

We already know that $\|u\|_\infty = 1$. Assume e.g. that $\max_{\bar{U}} u = 1$ but that $\min_{\bar{U}} u \geq -1 + 2\varepsilon$ for some $\varepsilon > 0$. Since $u_p \to u$ in $C(\bar{U})$, we also have $\min_{\bar{U}} u_p \geq -1 + \varepsilon$ for p big. Then

$$\int_{\{u_p \geq 0\}} |u_p|^{p-1} \, dx = \int_{\{u_p \leq 0\}} |u_p|^{p-1} \, dx \leq (1-\varepsilon)^{p-1} |U| \to 0$$

as $p \to \infty$. Since (u_p) is bounded in $C(\bar{U})$ (because it converges), we obtain

$$1 = \int_U |u_p|^p \, dx \leq C \int_U |u_p|^{p-1} \, dx \to 0$$

which is a contradiction.

As $\|u\|_\infty = 1$ and $\max u + \min u = 0$, u is an admissible test-function for λ_∞ as defined in (8). It follows that $\lambda_\infty \leq \|\nabla u\|_\infty$. Independently since $u_q \to u$ weakly in $W^{1,p}(U)$ for any $p \geq 1$, we also have from (20) that

$$\|\nabla u\|_p \leq \liminf_{q\to+\infty} \|\nabla u_q\|_p \leq |U|^{1/p} \liminf_{q\to+\infty} \lambda_q^{1/q},$$

Letting $p \to +\infty$, we obtain, using (19), that

$$\lambda_\infty \leq \|\nabla u\|_\infty \leq \liminf_{q\to+\infty} \lambda_q^{1/q} \leq \limsup_{p\to+\infty} \lambda_p^{1/p} \leq \lambda_\infty$$

from where we deduce the claim. $\square$

The proof that u_∞ is a viscosity solution of (9) is by now standard. We briefly sketch it for the reader´s convenience and refer to [14], [9], [10] for more details. As a preliminary step we verify that

Step 2.3. *For $p > 2$, any continuous weak solution of* (6) *is a viscosity solution of* (6).

Before doing the proof we introduce some notations. Denote by S the space of symmetric matrices $n \times n$, and consider the functions $F_p : \mathbb{R} \times \mathbb{R}^n \times S \to \mathbb{R}$ and $B_p : \partial U \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined for $p > 2$ by

$$F_p(u, \eta, A) = \begin{cases} -|\eta|^{p-2}Tr(A) - (p-2)|\eta|^{p-4}(A\eta, \eta) - \lambda_p|u|^{p-2}u, & \text{if } \eta \neq 0, \\ -\lambda_p|u|^{p-2}u & \text{otherwise,} \end{cases}$$

and $B_p(x, u, \eta) = |\eta|^{p-2}\eta.\nu(x)$. Observe that $F_p(u, \eta, B) \leq F_p(u, \eta, A)$ if $B \geq A$.

Proof. Let u be a weak continuous solution of (6). We only verify that u is a viscosity super-solution. The proof that u is also a sub-solution is similar. Fix some point $x_0 \in \bar{U}$ and a smooth function ϕ such that $u - \phi$ has a strict minimum at x_0 with $u(x_0) = \phi(x_0)$. We have to prove that

$$\begin{aligned} &F_p(u(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \geq 0 \quad \text{if } x_0 \in U \\ &\max\{F_p(u(x_0), \nabla\phi(x_0), D^2\phi(x_0)), B_p(x_0, u(x_0), \nabla\phi(x_0))\} \geq 0 \text{ if } x_0 \in \partial U \end{aligned} \tag{22}$$

Assume first that $x_0 \in U$ but that (22) does not hold. Then, since u, ϕ and F_p are continuous we have that

$$\Delta_p\phi(x) - \lambda|u(x)|^{p-2}u(x) = F_p(u(x), \nabla\phi(x), D^2\phi(x)) < 0 \quad \text{in } B_{x_0}(r) \tag{23}$$

for some $r > 0$. Let $\psi = \phi + m/2$ with $m = \min_{|x-x_0|=r}\{u(x) - \phi(x)\} > 0$. Then $\psi(x_0) - u(x_0) > 0$ and $\psi - u < 0$ on $\partial B_{x_0}(r)$, so that $(\psi - u)^+$, when extended by 0 outside $B_{x_0}(r)$, has support in $B_{x_0}(r)$. Using it as a test-function in (23) and (6) gives

$$\int_{\{\psi > u\}} (|\nabla\psi|^{p-2}\nabla\psi - |\nabla u|^{p-2}\nabla u)(\nabla\psi - \nabla u)\, dx < 0.$$

We obtain a contradiction using the inequality $(|X|^{p-2}X - |Y|^{p-2}Y)(X - Y) \geq C|X - Y|^p$ which holds for some $C > 0$ and for any $X, Y \in \mathbb{R}^n \backslash \{0\}$. The case $x_0 \in \partial U$ is handled in the same way. $\square$

We can now pass to the limit $p \to +\infty$ in (22) (and also in the corresponding inequality for the subsolution case) to obtain the equation satisfied by u_∞.

Step 2.4. *The limit u_∞ of the u_p obtained in the first step is a viscosity solution of* (9).

Proof. We prove that u_∞ is a supersolution of (9). The proof of the subsolution property is similar. Fix some point $x_0 \in \bar{U}$ and a smooth function ϕ such that $u_\infty - \phi$ has a strict minimum at x_0 with $u_\infty(x_0) = \phi(x_0)$. Since $u_p \to u_\infty$ uniformly there exist $x_p \in argmax\,\{u_p - \phi\}$ such that $x_p \to x_0$.

Assume first that $x_0 \in U$, so that $x_p \in U$ for p large. If $\nabla\phi(x_0) = 0$ then by definition of Δ_∞ we have $\Delta_\infty\phi(x_0) = 0$. We assume now that $\nabla\phi(x_0) \neq 0$. As u_p is a viscosity solution of (6) according to the previous step, we have

$$F_p(x_p, u_p(x_p), \nabla\phi(x_p), D^2\phi(x_p)) \geq 0. \tag{24}$$

Dividing this inequality by $(p-2)|\nabla\phi(x_p)|^{p-4}$ we obtain

$$\Delta_\infty\phi(x_0) + o(1) \geq u_p(x_p)|\nabla\phi(x_p)|^2 \left(\frac{\lambda_p^{\frac{1}{p-2}} u_p(x_p)}{|\nabla\phi(x_p)|(p-2)^{\frac{1}{p-2}}} \right)^{p-2}. \tag{25}$$

If $u_\infty(x_0) > 0$, then, recalling that $\lambda_p^{\frac{1}{p-2}} \to \lambda_\infty$ (see first step), it follows that we must have $\frac{\lambda_\infty u_\infty(x_0)}{|\nabla\phi(x_0)|} \leq 1$ i.e. $|\nabla\phi(x_0)| - \lambda_\infty u_\infty(x_0) \geq 0$. Going back to (25) we also get $\Delta_\infty\phi(x_0) \geq 0$. If $u_\infty(x_0) < 0$ then we rewrite (24) as

$$-|\nabla\phi(x_p)|^{-3} \left(\frac{(p-2)^{\frac{1}{p-1}}|\nabla\phi(x_p)|}{\lambda_p^{\frac{1}{p-1}}|u_p(x_p)|} \right)^{p-1} (\Delta_\infty\phi(x_0) + o(1)) \leq 1.$$

If $\frac{|\nabla\phi(x_0)|}{\lambda_\infty|u_\infty(x_0)|} > 1$ then we must have $\Delta_\infty\phi(x_0) \geq 0$. Otherwise we have $-|\nabla\phi(x_0)| - \lambda_\infty u_\infty(x_0) \geq 0$. Eventually if $u_\infty(x_0) = 0$, then $u_p(x_p) \to 0$ so that $|u_p(x_p)|^{p-2}u_p(x_p) \leq u_p(x_p) \to 0$. It then follows from (24) that

$$|\nabla\phi(x_p)|^{p-2}\Delta\phi(x_p) + (p-2)|\nabla\phi(x_p)|^{p-4}\Delta_\infty\phi(x_p) \geq o(1).$$

Dividing this inequality by $(p-2)|\nabla\phi(x_p)|^{p-4}$ and letting $p \to +\infty$ yield $\Delta_\infty\phi(x_0) \geq 0$.

Assume now that $x_0 \in \partial U$. We have to prove that

$$\max\{F(x_0, \nabla\phi(x_0), D^2\phi(x_0)), \partial_\nu\phi(x_0)\} \geq 0.$$

If $x_p \in U$ for some subsequence then we can proceed as before to get $F(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \geq 0$. Assume that $x_p \in \partial U$ for p big. If $\nabla\phi(x_0) = 0$ then $\partial_\nu\phi(x_0) = 0$. Otherwise, (22) holds with x_p in place of x_0. If (24) holds for a subsequence we are done as before. Otherwise

$$B_p(x_p, u(x_p), \nabla\phi(x_p)) = |\nabla\phi(x_p)|^{p-2}\partial_\nu\phi(x_p) \geq 0 \quad \text{for } p \text{ large}$$

so that $\partial_\nu\phi(x_0) = \lim_{p\to\infty} \partial_\nu\phi(x_p) \geq 0$. □

3. Proof of theorem 2

Again we divide the proof into several steps. As a first step, we prove that

Step 3.1. *There holds $\lambda_\infty \geq 2/diam(U)$.*

Proof. Given some admissible test-function u for λ_∞, let $x \in \bar{U}$ be a point of maximum of u, and $y \in \bar{U}$ a point of minimum so that $u(x) = 1$ and $u(y) = -1$. Consider also some curve $\gamma : [0,T] \to U$ joining y to x. Then

$$\begin{aligned} 2 = u(x) - u(y) = u(\gamma(T)) - u(\gamma(0)) &= \int_0^T \nabla u(\gamma(s))\gamma'(s)\, ds \\ &\leq \|\nabla u\|_\infty \int_0^T |\gamma'(s)|\, ds = \|\nabla u\|_\infty \text{Long}(\gamma). \end{aligned}$$

Taking the infimum over all such curves γ and all admissible u, we obtain $2 \leq \lambda_\infty d(x,y)$, from which we deduce the claim. $\square$

We now prove the reverse inequality.

Step 3.2. *There holds* $\lambda_\infty \leq 2/diam(U)$.

Proof. We are able to prove this inequality in an elementary way only when U is convex. Indeed in that case pick two points $x_0, y_0 \in \partial U$ such that $diam(U) = |x_0 - y_0|$. By extremality the vector $y_0 - x_0$ must be orthogonal to the tangent spaces $T_{x_0}\partial U$ and $T_{y_0}\partial U$ of ∂U at x and y. Moreover $T_{x_0}\partial U \cap \partial U = \{x_0\}$ and $T_{y_0}\partial U \cap \partial U = \{y_0\}$ so that U lies strictly between $T_{x_0}\partial U$ and $T_{y_0}\partial U$. Indeed if there exists $z \in T_{x_0}\partial U \cap \partial U$, $z \neq x$, then $|z-y|^2 = |z-x|^2 + |x-y|^2$ so that $|z-y| > |x-y|$ - a contradiction. It follows that the planes orthogonal to $n = \frac{y_0 - x_0}{|y_0 - x_0|}$ which intersects U have an equation of the form $(z - x_0)n = s$ with $s \in (0,d)$, $d = diam(U)$. Hence the function

$$u(z) = \frac{2}{d}\left((z - x_0)n - \frac{d}{2}\right), \quad z \in U,$$

is admissible for λ_∞. This yields the upper bounds. Another possible choice of test-function is $u(z) = C_y(z)_+ - C_x(z)_+$ where

$$C_y(z) = 1 - \frac{2}{d}|z-y|, \quad C_x(z) = 1 - \frac{2}{d}|z-y|$$

are the cones centered at x and y of height 1 and slope $\frac{d}{2}$.

To obtain the result in the general case we consider the tug-of-war game described in [21]. We use the notation from [21]. Let Y be a curve joining $x_0, y_0 \in \partial U$ extremal for $\mathrm{diam}(U)$. We consider the function $F : Y \to [-1,1]$ given by $F(x) = -1 + Ld(x_0, x)$, $L = 2/\mathrm{diam}(U)$. Then $F(x_0) = -1 \leq F(x) \leq F(y_0) = 1$ for any $x \in Y$, and F is Lipschitz with Lipschitz constant L (w.r.t. the geodesic distance in Y). We consider the tug-of-war game with terminal set Y, pay-off F, and running cost $f \equiv 0$. It is proved in [21] that this game has a value u which turns out to be an extension of F to U satisfying $|u(x) - u(x)| \leq Ld(x,y)$ for any $x \in U \backslash Y$ and $y \in Y$ (see the proof of theorem 1.4 p190 in [21]).

We now check that u is Lipschitz in U with $Lip_U(u) = L$ using the idea of the proof of theorem 1.4 in [21]. Assume that $|u(\tilde{x}) - u(\bar{x})| > Ld(\tilde{x}, \bar{x})$ for some points $\tilde{x}, \bar{x} \in U \backslash Y$. We consider the tug-of-war game in U with terminal set $Y' = Y \cup \{\bar{x}\}$ and pay-off $F' = u$. Then u is the value of this new game so that, noting that $Lip_{Y'}F' = L$, we have $|u(x) - u(x)| \leq Ld(x,y)$ for any $x \in U \backslash Y', y \in Y'$. We obtain a contradiction taking $y = \bar{x}$, $x = \tilde{x}$.

Observe that since the terminal pay-off F takes value in $[-1,1]$, we have that $\|u\|_\infty \leq 1$, and also that $u(x_0) = F(x_0) = -1$, $u(y_0) = F(y_0) = 1$ since u extends F. We can then use u as a test-funtion in (8) to obtain that $\lambda_\infty \leq L = 2/\mathrm{diam}(U)$. $\square$

4. Proof of theorem 3

The proof of theorem 3 follows closely the lines of [6]. Let u_p be an extremal for λ_p normalized by $\|u_p\|_p = 1$. Then $f_p := |u_p|^{p-2}u_p \in L^{p'}(U)$

satisfies

$$\|f_p\|_{p'} = 1, \quad \text{and} \quad \int_U f_p = 0. \tag{26}$$

The first step consists in extracting from (f_p) a subsequence converging weakly to some measure $f_\infty \in M(\bar{U})$, the weak convergence meaning that $\lim_{p\to+\infty} \int_{\bar{U}} \phi f_p \, dx = \int_{\bar{U}} \phi \, df_\infty$ for any $\phi \in C(\bar{U})$.

Step 4.1. *Up to a subsequence, the measures $f_p \, dx$ converge weakly as measure in $\bar{U}$ to some measure f_∞ supported in $\bar{U}$ satisfying*

$$\int_U f_\infty = 0, \qquad \text{and} \qquad \int_U |f_\infty| = 1. \tag{27}$$

Proof. We claim that

$$\lim_{p\to+\infty} \int_U |f_p| \, dx = 1. \tag{28}$$

First, in view of (26), we have that

$$\int_U |f_p| \, dx \le \|f_p\|_{p'} |U|^{1-1/p'} = |U|^{1-1/p'} \to 1$$

and then, recalling that $u_p \to u$ in $C(\bar{U})$ with $\|u\|_\infty = 1$,

$$1 = \int_U u_p f_p \, dx \le \|u_p\|_\infty \|f_p\|_1 = (1 + o(1)) \|f_p\|_1.$$

It follows in particular that the measures $|f_p| \, dx$ are bounded in $M(\bar{U})$. Since $\bar{U}$ is compact, we can then extract from this sequence a subsequence converging weakly to some measure $f_\infty \in M(\bar{U})$. Passing to the limit in (26) and (28) gives (27). $\square$

Consider the functionals $G_p, G_\infty : (v, \sigma) \in C(\bar{U}) \times M(\bar{U}) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$G_p(v,\sigma) = \begin{cases} -\int_U v\sigma, & \text{if } \sigma \in L^{p'}(U),\ \|\sigma\|_{p'} \le 1,\ \int_U \sigma = 0, \\ & \qquad \text{and } v \in W^{1,p}(U),\ \|\nabla v\|_p \le \lambda_p^{1/p}, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$G_\infty(v,\sigma) = \begin{cases} -\int_U v \, d\sigma, & \text{if } \sigma \in M(\bar{U}),\ \int_U |\sigma| \le 1,\ \int_U \sigma = 0, \\ & \qquad \text{and } v \in W^{1,\infty}(U),\ \|\nabla v\|_\infty \le \lambda_\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Endowing the space $M(\bar{U})$ with the weak convergence of measure, and $C(\bar{U})$ with the uniform convergence, we can prove as in [6] that G_∞ is the limit of the G_p in the sense of Γ-convergence:

Step 4.2. *The functionals G_p converge in the sense of Γ-convergence to G_∞.*

The proof is similar as that of Prop. 3.7 in ([6]). We sketch it for the reader's convenience.

Proof. Assume that $(v_p,\sigma_p)\in C(\bar U)\times M(\bar U)$ converge to (v,σ). We have to prove that

$$\liminf_{p\to+\infty} G_p(v_p,\sigma_p)\geq G(v,\sigma). \tag{29}$$

We can assume that $G_p(v_p,\sigma_p)<\infty$. We then have

$$\int_U v_p\sigma_p\,dx-\int_U v\,d\sigma=\int_U (v_p-v)\sigma_p\,dx+\int_U v\,(\sigma_p\,dx-d\sigma)\to 0$$

as $p\to+\infty$. Indeed the first integral on the right hand side can be bounded by $\|v_p-v\|_\infty\|\sigma_p\|_{p'}|U|^{\frac{1}{p}}=o(1)$. Independently $\int_U\sigma=\lim_{p\to+\infty}\int_U\sigma_p=0$, and

$$\int_U|\sigma|=\int_U|\sigma_p|\,dx+o(1)\leq\|\sigma_p\|_{p'}|U|^{\frac{1}{p}}+o(1)\leq 1+o(1)$$

so that $\int_U|\sigma|\leq 1$. For any $\phi\in L^{p'}(U,\mathbb{R}^n)$ such that $\|\phi\|_{p'}\leq 1$ we have

$$\int_U\phi\nabla v\,dx=-\int_U v\operatorname{div}\phi\,dx=-\int_U v_p\operatorname{div}\phi\,dx+o(1)=\int_U\phi\nabla v_p\,dx+o(1)$$
$$\leq\|\nabla v_p\|_p+o(1)\leq\lambda_p^{\frac{1}{p}}+o(1)=\lambda_\infty+o(1),$$

where the $o(1)$ does not depend on ϕ. Taking the supremum over all such ϕ we obtain $\|\nabla v\|_p\leq\lambda_\infty+o(1)$, so that $\|\nabla v\|_\infty\leq\lambda_\infty$. It follows that (v,σ) is admissible for G_∞.

We now fix a pair (v,σ) admissible for G_∞. We have to find some pair (v_p,σ_p) admissible for G_p which converges to (v,σ) and such that

$$\limsup_{p\to+\infty}G_p(v_p,\sigma_p)\leq G_\infty(v,\sigma).$$

We define $v_p=\dfrac{\lambda^{\frac{1}{p}}}{\lambda_\infty|U|^{\frac{1}{p}}}v$. Then $v_p\in W^{1,p}(U)$ with $\|\nabla v_p\|_p\leq\lambda^{\frac{1}{p}}$, and $v_p\to v$ uniformly since $\lambda^{\frac{1}{p}}\to\lambda_\infty$. Denoting by ρ_ε the standard mollifying sequence (i.e. $\rho_\varepsilon(x)=\varepsilon^{-n}\rho(x/\varepsilon)$ where ρ is a smooth function supported in the unit ball with $\int\rho=1$), we let $\sigma_p=\sigma*\bar\rho_{1/p}$ where $\bar\rho_{1/p}=\frac{\rho_{1/p}}{\|\rho_{1/p}\|_{p'}}$. Then σ_p is a smooth function such that

$$\int_U\sigma_p(x)\,dx=\int\bar\rho_{1/p}(x-y)\,d\sigma(y)dx=\int d\sigma\int\rho_{1/p}(x)\,dx=0,$$

and, since $|\sigma|(U)|\leq 1$ and $\|\bar\rho_{1/p}\|_{p'}=1$,

$$\|\sigma_p\|_{p'}^{p'}\leq\int\int\bar\rho_{1/p}(x-y)^{p'}\,d|\sigma||\sigma(U)|^{\frac{p'}{p}}\,dx\leq 1.$$

Eventually, noticing that $\|\rho_{1/p}\|_{p'}=\|\rho\|_{p'}\to\|\rho\|_1=1$ as $p\to+\infty$, we have that $\sigma_p\to\sigma$ as measure. It follows that (σ_p,v_p) is admissible for G_p and converge to (v,σ). As before we have $G_p(v_p,\sigma_p)\to G_\infty(v,\sigma)$. ☐

As an easy corollary we obtain that

Step 4.3. *(u_p,f_p) is a minimizer for G_p, (u_∞,f_∞) is a minimizer for G_∞, and*

$$G_\infty(u_\infty,f_\infty)=\lim_{p\to+\infty}G_p(u_p,f_p)=-1 \tag{30}$$

Proof. Notice that the pair (u_p, f_p) is a minimizer of G_p. Indeed given a pair (v,σ) admissible for G_p take $\bar v \in \mathbb{R}$ such that $\int_U |v-\bar v|^{p-2}(v-\bar v)\,dx = 0$. Then, recalling that $\int \sigma = 0$ and the definition (4) of λ_p, we have

$$\begin{aligned} G_p(v,\sigma) &= -\int (v-\bar v)\sigma \geq -\|v-\bar v\|_p \|\sigma\|_{p'} \\ &\geq -\lambda_p^{-1/p}\|\nabla(v-\bar v)\|_p = -\lambda_p^{-1/p}\|\nabla(v-\bar v)\|_p \\ &\geq -1 = G_p(u_p, f_p). \end{aligned}$$

Moreover $(u_p, f_p) \to (u_\infty, f_\infty)$. It then follows from (29) that

$$\liminf_{p\to+\infty} \inf G_p = \liminf_{p\to+\infty} G_p(u_p, f_p) \geq G_\infty(u_\infty, f_\infty) \geq \inf G_\infty.$$

Moreover the lim sup property (19) implies that $\limsup \inf G_p \leq \inf G_\infty$. It follows that

$$\lim_{p\to+\infty} \inf G_p = \lim_{p\to+\infty} G_p(u_p, f_p) = G_\infty(u_\infty, f_\infty) = \inf G_\infty.$$

□

We can now relate λ_∞ to the Monge-Kantorovich distnce W_1. Recall that if $\sigma \in M(\bar U)$, then $\sigma^\pm \in M(\bar U)$ denote the positive and negative part of σ. In particular $\sigma = \sigma^+ - \sigma^-$, and $|\sigma| = \sigma^+ + \sigma^-$.

Step 4.4. *There holds*

$$\frac{2}{\lambda_\infty} = \max_{\sigma\in M(\bar U),\, \int_U \sigma^+ = \int_U \sigma^+ = 1} W_1(\sigma^+, \sigma^-). \tag{31}$$

Proof. The conditions $\int_U \sigma = 0$ and $\int_U |\sigma| = 1$ are equivalent to $\int_U \sigma^+ = \int_U \sigma^+ = 1/2$. We can therefore rewrite the fact that the pair (u_∞, f_∞) is a minimizer of G_∞ as

$$\begin{aligned} 1 &= -G_\infty(u_\infty, f_\infty) \\ &= \max_{\sigma\in M(\bar U),\, \int_U \sigma^+ = \int_U \sigma^+ = 1/2} \quad \max_{v\in W^{1,\infty}(U),\, \|\nabla v\|_\infty \leq \lambda_\infty} \int_U v(\sigma^+ - \sigma^-), \end{aligned}$$

that is,

$$\frac{2}{\lambda_\infty} = \max_{\sigma\in M(\bar U),\, \int_U \sigma^+ = \int_U \sigma^+ = 1} \quad \max_{v\in W^{1,\infty}(U),\, \|\nabla v\|_\infty \leq 1} \int_U v(\sigma^+ - \sigma^-).$$

We obtain (31) recalling the definition (2) of W_1. □

As a corollary we can easily recover the value of λ_∞ conbining (31) with the following result:

Step 4.5. *There holds*

$$\max_{\sigma\in M(\bar U),\, \int_U \sigma^+ = \int_U \sigma^+ = 1} W_1(\sigma^+, \sigma^-) = diam(U). \tag{32}$$

The proof relies on the lemma.

Lemma 1. *Given $u \in L^\infty(U)$ we have that $u \in W^{1,\infty}(U)$ with $\|\nabla u\|_\infty \leq 1$ if and only if u is 1-Lipschit witt respect to the distance d i.e. $|u(x) - u(y)| \leq d(x,y)$ for any $x, y \in U$.*

Proof. If $u \in W^{1,\infty}(U)$ with $\|\nabla u\|_\infty \leq 1$ then given $x, y \in U$ and a curve $\gamma : [0,1] \to U$ joining x to y we have

$$|u(x) - u(y)| \leq \int_0^1 |(\nabla u(\gamma(t)), \gamma'(t))|\, dt \leq \int_0^1 |\gamma'(t)|\, dt = \mathrm{Long}(\gamma). \tag{33}$$

Taking the infimum over all such γ gives the result. Conversely if u is 1-Lipschitz for d then given $x \in U$ we have $d(x,y) = |x-y|$ for $|x-y|$ small and the result follows. □

Notice that in this lemma no regularity on U is needed - in particular U need not to be an extension domain as required for the result stating that $u \in W^{1,\infty}(U)$ iff u is Lipschitz for the usual Euclidean distance (see [8][thm 4 - section 5.8]). Consider for instance the open subset $U \subset \mathbb{R}^2$ defined in polar coordinates by $U = \{a < r < b, -\pi < \theta < \pi\}$. Then U is not an extension domain. It can be verified that the function $u(x) = \theta$ belongs to $W^{1,\infty}(U)$ with $\|\nabla u\|_\infty = \frac{1}{a}$, u is not Lipschitz for the Euclidean distance but is so for the geodesic distance d with $\sup_{x,y \in U} \frac{u(x)-u(y)}{d(x,y)} = \frac{1}{a}$ (the $\leq$ follows from (33) and the $=$ is obtained considering the points (a,π) and $(a,-\pi)$).

Proof of the Step 4.5. It then follows from the Kantorovich duality theorem (see [22]) that the Monge-Kantorovich distance can also be expressed as

$$W_1(\mu,\nu) = \inf_{\pi \in \Gamma(\mu,\nu)} \int_{\bar{U}\times\bar{U}} d(x,y)\, d\pi(x,y),$$

where $\Gamma(\mu,\nu)$ is the set of probability measures on $\bar{U} \times \bar{U}$ having μ and ν for marginal distributions, and $d(x,y)$ is the geodesic distance between $x, y \in U$ defined by (11).

First since

$$d(x,y) \leq diam(U) \quad \text{ for any } x, y \in \bar{U},$$

we obtain easily that $W_1(\mu,\nu) \leq \mathrm{diam}(U)$ for any two probability measures μ and ν. In particular

$$\max_{\sigma \in M(\bar{U}),\, \int_U \sigma^+ = \int_U \sigma^+ = 1} W_1(\sigma^+, \sigma^-) \leq \mathrm{diam}(U).$$

To prove the converse inequality we pick a pair of points (x_0, y_0) such that $d(x_0,y_0) = \mathrm{diam}(U)$, and consider the measure $\sigma = \delta_{x_0} - \delta_{y_0}$. Obviously $\int \sigma = 0$, $\int |\sigma| = 2$, and $\sigma^+ = \delta_{x_0}$, $\sigma^- = \delta_{y_0}$. Noticing that

$$W_1(\delta_{x_0}, \delta_{y_0}) = d(x_0, y_0)$$

(see [22][Example 6.3] - indeed it is easily seen that $\Gamma(\delta_{x_0}, \delta_{y_0}) = \{\delta_{x_0} \otimes \delta_{y_0}\}$), it follows that

$$\max_{\sigma \in M(U),\, \int_U \sigma^+ = \int_U \sigma^+ = 1} W_1(\sigma^+, \sigma^-) \geq W_1(\delta_{x_0}, \delta_{y_0}) = \mathrm{diam}(U).$$

□

5. Proof of theorems 4 and 5

We begin this section by some general comments on the shortest-paths taken from [4]. We define the length of a Lipschitz curves $\gamma : [0,T] \to \bar{U}$ by

$$L(\gamma) = \inf \sum_{i=1}^{p-1} |\gamma(t_{i+1}) - \gamma(t_i)|,$$

where the infimum is taken over all the finite partition $0 = t_1 < .. < t_p = T$ of $[0,T]$. It follows in particular that L is lower semi-continuous with respect to the pointwise convergence of path (see [4] proposition 2.3.4). We denote by $\Gamma(x,y)$ the set of finite length Lipschitz curves connecting x to y. This set is not empty Since we assumed U connected. We then define the geodesic distance $d(x,y)$ between two points $x, y \in \bar{U}$ as $d(x,y) = \inf_{\gamma \in \Gamma(x,y)} L(\gamma)$. Following [4], $(\bar{U}, d)$ is a length space. Notice that a finite length curve γ can always be reparametrized (maybe using a nondecreasing change of parameter) in order to have constant speed v in the sense that $L(\gamma_{|[t,t']}) = v|t - t'|$ for any t, t' (see [4][prop. 2.5.9]). We can then assume that all the considered curves are defined on $[0,1]$ and have constant speed. It then follows from Arzela-Ascoli theorem and the semi-continuity of L as in [4][prop. 2.5.19] that two points $x, y \in \bar{U}$ can always be connected by a shortest path.

Let $\gamma : [0,1] \to \bar{U}$ be a shortest path. Then $\gamma_{|U}$ is a straight line (i.e. a geodesic of U with the Euclidean metric) and $\gamma_{|\partial U}$ is a (smooth) geodesic of ∂U for the induced metric otherwise. Since a shortest path enters and leaves ∂U tangentially, we have that $\gamma \in C^{1,1}(0,1)$. We will therefore restrict $\Gamma(x,y)$ to $C^{1,1}$ - curves. Notice that in general a shortest-path is not C^2. Consider for instance the shortest-path from $(-2,0)$ to $(2,0)$ in $\mathbb{R}^2_+ \backslash B_0(1)$ which is given by $y = f(x)$ with

$$f(x) = \begin{cases} -\frac{\sqrt{3}}{3}(|x| - 2) & \text{if } \frac{1}{2} \le |x| \le 2, \\ \sqrt{1 - x^2} & \text{if } |x| \le \frac{1}{2}. \end{cases}$$

Denote by n the exterior normal to ∂U. Differentiating $(\gamma', n) = 0$, we obtain the well-known relation $(\gamma'', n) = -(\nabla_{\gamma'} n, \gamma')$ where ∇ is the (covariant) derivative of $\mathbb{R}^n$. Recalling that γ on ∂U is a geodesic of ∂U if and only if it has normal acceleration, it follows that

$$\gamma'' = -(\nabla_{\gamma'} n, \gamma') n \quad \text{on } \partial U. \tag{34}$$

We first verify that

Lemma 2. *For any $x \in \bar{U}$ and any $y \in \mathbb{R}^n$, we have for $|t|$ small that*

$$|D\phi_t(x)y| = |y| + |y|(DV(x)\frac{y}{|y|}, \frac{y}{|y|})t + |y|O(t^2),$$

where ϕ_T is defined in (15) and the remainder $O(t^2)$ is uniform in $x \in \bar{U}$ and $y \in \mathbb{R}^n$.

Proof. This is a consequence of

$$\begin{aligned}|D\phi_t(x)y|^2 &= |y+tDV(x)y|^2\\ &= |y|^2+2t(DV(x)y,y)+t^2(DV(x)y,DV(x)y)\\ &= |y|^2\left(1+2t(DV(x)\frac{y}{|y|},\frac{y}{|y|})+t^2(DV(x)\frac{y}{|y|},DV(x)\frac{y}{|y|})\right)\\ &= |y|^2\left(1+2t(DV(x)\frac{y}{|y|},\frac{y}{|y|})+O(t^2)\right),\end{aligned}$$

where the coefficient of t and the $O(t^2)$ are boundad uniformly in $x \in \bar{U}$ and $y \in \mathbb{R}^n$. □

Proof of theorem 4. It suffices to prove that

$$|\mathrm{diam}(U_t) - \mathrm{diam}(U)| \le Ct. \tag{35}$$

Writing that

$$\mathrm{diam}(U_t) = \mathrm{diam}(\phi_t(U)) = \max_{x,y\in\bar{U}} \inf_{\gamma\in\Gamma(x,y)} \mathrm{Long}(\phi_t\circ\gamma),$$

it is easily seen that (35) will follow if we can prove that

$$\mathrm{Long}(\phi_t\circ\gamma) = (1+O(t))\mathrm{Long}(\gamma) \tag{36}$$

with $O(t)$ uniform in $\gamma \in \Gamma(x,y)$, $x, y \in \bar{U}$. This follows from the following lemma:

Lemma 3. *Given a C^1 curve $\gamma : [a,b] \to \bar{U}$, we have that*

$$\begin{aligned}&\mathit{Long}(\phi_t\circ\gamma) =\\ &\mathit{Long}(\gamma) + t\int_a^b (DV(\gamma(s))\gamma'(s),\gamma'(s))\,\frac{ds}{|\gamma'(s)|} + O(t^2)\int_a^b |\gamma'(s)|\,ds,\end{aligned} \tag{37}$$

where the $O(t^2)$ does not depend on γ.

Proof. Since

$$\mathrm{Long}(\phi_t\circ\gamma) = \int_a^b \left|\frac{d}{ds}\phi_t(\gamma(s))\right|\,dx = \int_a^b \left|D\phi_t(\gamma(s))\gamma'(s)\right|\,dx,$$

the result follows from lemma 2. □

Proof of theorem 5. We assume from now on that $\mathrm{diam}(U)$ has an unique extremal curve γ^*, i.e. $\mathrm{diam}(U) = \mathrm{Long}(\gamma^*)$. Up to reparametrizing, we can assume that $\gamma^* : [0,1] \to \bar{U}$ has constant-speed equal to $\mathrm{diam}(U)$. We let $x^* = \gamma^*(0)$, $y^* = \gamma^*(1)$.

Let γ_t^* be an extremal for U_t, i.e. $\mathrm{diam}(U_t) = \mathrm{Long}(\gamma_t^*)$. We can assume that $\gamma_t^* : [0,1] \to \bar{U}_t$ has constant-speed. Denote by n_t the unit exterior normal to U_t. Then $|\nabla n_t| \le Cste$ for $|t|$ small. Moreover $|{\gamma_t^*}'| = \mathrm{diam}(U_t) \le Cste$ in view of (35). It thus follows from (34) that

$$\|\gamma_t^*\|_{C^{1,1}} \le C \tag{38}$$

uniformly for $|t|$ small. We first prove that

Step 5.1. *γ_t^* C^1-converge as $t \to 0$ to $\pm\gamma^*$.*

Proof. It follows from (38) and Arzela-Ascoli theorem that there exists a curve $\tilde{\gamma} : [0,1] \to \mathbb{R}^n$ such that, up to a subsequence, $\gamma_t^* \to \tilde{\gamma}$ in C^1 as $t \to 0$. In particular $\tilde{\gamma}$ takes values in $\bar{U}$, has constant-speed, and $\lim_{t\to 0} \mathrm{Long}(\gamma_t^*) = \mathrm{Long}(\tilde{\gamma})$. According to (35), we thus have

$$\mathrm{Long}(\tilde{\gamma}) = \mathrm{Long}(\gamma_t^*) + o(1) = \mathrm{diam}(U_t) + o(1) \to \mathrm{diam}(U)$$

as $t \to 0$. Therefore $\tilde{\gamma}$ is an constant-speed extremal for $\mathrm{diam}(U)$ so that $\tilde{\gamma} = \pm\gamma^*$. □

Let us suppose that $\gamma_t^* \to \gamma^*$ in the C^1-norm. In particular $x_t^* := \gamma_t^*(0) \to x^*$ and $y_t^* := \gamma_t^*(1) \to y^*$.

Consider $K = (\bar{B}_{x^*}(\varepsilon_0) \cap \partial U) \times (\bar{B}_{y^*}(\varepsilon_0) \cap \partial U)$ where ε_0 is given in hypothesis (2) of theorem 5. In view of (38) we can write that

$$\mathrm{diam}(U_t) = \max_{x,y\in K} d(\phi_t(x), \phi_y(y)) = \max_{x,y\in K} \inf_{\gamma\in\Gamma(x,y)} \mathrm{Long}(\phi_t \circ \gamma),$$

where $\Gamma(x,y)$ is the set of constant-speed $C^{1,1}$-curve $\gamma : [0,1] \to \bar{U}$, $\gamma(0) = x$, $\gamma(1) = y$, satisfying

$$\|\gamma\|_{C^{1,1}} \leq C \tag{39}$$

for some positive constant C uniform in $\gamma \in \Gamma(x,y)$, $(x,y) \in K$. We also let $\Gamma = \cup_{(x,y)\in K}\Gamma(x,y)$. Notice that each $\Gamma(x,y)$, $(x,y) \in K$, is compact for the C^1-norm thanks to (39).

The differentiability of $t \to \mathrm{diam}(U_t)$ at $t = 0$ with formula (16) will follow from the two following lemma whose proof is similar to [19][thm.2].

Lemma 4. *Let Γ be a compact metric set. Consider a map $A : (\gamma, t) \in \Gamma \times [-\varepsilon, \varepsilon] \to A(\gamma, t) \in \mathbb{R}$ such that*

(H1) A is continuous at any point $(\gamma, 0)$, $\gamma \in \Gamma$,

(H2) for any $\gamma \in \Gamma$, there holds that

$$A(\gamma, t) = A(\gamma, 0) + tA_1(\gamma) + o(t), \tag{40}$$

where the $o(t)$ is uniform in $\gamma \in \Gamma$,

(H3) $A(\cdot, 0)$ attains its minimum at an unique point γ^*,

(H4) A_1 is continuous at γ^* and bounded over Γ.

Then the function $t \to \mu(t) := \inf_{\gamma\in\Gamma} A(\gamma, t)$ is differentiable at $t = 0$ with derivative

$$\mu'(0) = A_1(\gamma^*).$$

Proof. First since $A(\gamma, t) = A(\gamma, 0) + O(t)$ with the $O(t)$ uniform in $\gamma \in \Gamma$ in view of (H2) and (H4), it is easily seen that μ is well-defined and continuous at $t = 0$.

Consider then positive real numbers $\varepsilon_t = o(t)$ and $\gamma_t^* \in \Gamma$ such that $A(\gamma_t^*, t) \leq \mu(t) + \varepsilon_t$. Let us check that $\gamma_t^* \to \gamma^*$. Let $\bar{\gamma}^*$ be a cluster point of (γ_t^*). If $\bar{\gamma}^* \neq \gamma^*$ then $A(\bar{\gamma}^*, 0) - 2\eta \geq \mu(0)$ for some $\eta > 0$, since $A(., 0)$ has a strict minimum at γ^*. Then $A(\bar{\gamma}, 0) - \eta \geq \mu(t)$ for t small. But, using (H1), $A(\bar{\gamma}^*, 0) = A(\gamma_t^*, t) + o(1) \leq \mu(t) + \varepsilon_t + o(1)$ which yields a contradiction for t small enough.

We now prove the existence of $\mu'(0)$. First using (40)

$$\mu(t) - \mu(0) \leq A(\gamma^*, t) - A(\gamma^*, 0) = tA_1(\gamma^*) + o(t)$$

so that, with $Q(t) = \frac{\mu(t)-\mu(0)}{t}$,

$$\limsup_{t\to 0^+} Q(t) \le A_1(\gamma^*), \quad \text{and} \quad \liminf_{t\to 0^-} Q(t) \ge A_1(\gamma^*).$$

Moreover, since A_1 is continuous at γ^* and $\gamma_t^* \to \gamma^*$,

$$\begin{aligned}\mu(t) - \mu(0) &\ge A(\gamma_t^*, t) - \varepsilon_t - A(\gamma_t^*, 0) = tA_1(\gamma_t^*) + o(t)\\ &= t(A_1(\gamma^*) + o(1)) + o(t)\\ &= tA_1(\gamma^*) + o(t)\end{aligned}$$

so that

$$\limsup_{t\to 0^-} Q(t) \le A_1(\gamma^*), \quad \text{and} \quad \liminf_{t\to 0^+} Q(t) \ge A_1(\gamma^*).$$

This ends the proof of the lemma. □

Notice that under the same hypothesis an analogous result holds for a maximization problem. We keep on using the notations of the previous lemma. We now consider a family of compact subsets Γ_λ, $\lambda \in K$, of Γ, and the map A defined in (40) assuming first that

(H1') A is continuous at any point $(\gamma, 0)$, $\gamma \in \Gamma$, and (40) holds with a remainder $o(t)$ uniform in $\gamma \in \Gamma_\lambda$, $\lambda \in K$.

We also assume that the map $\lambda \to \Gamma_\lambda$ is continuous in the sense that

(H2') if $\gamma_\lambda \in \Gamma_\lambda$ converge as $\lambda \to \lambda_0$ (for some λ_0) to some γ then $\gamma \in \Gamma_{\lambda_0}$,

(H3') for any $\gamma \in \Gamma_\lambda$ and any sequence $\lambda_n \to \lambda$, there exist $\gamma_n \in \Gamma_{\lambda_n}$ s.t. $\gamma_n \to \gamma$.

We eventually make the following assumptions:

(H4') $A(\cdot, 0)$ attains its minimum over Γ_λ at an unique point denoted γ_λ^*,

(H5') the function $\mu(\lambda, 0) := \min_{\gamma\in\Gamma_\lambda} A(\gamma, 0)$ attains its maximum at an unique point λ^*,

(H6') A_1 is continuous over Γ.

Lemma 5. *Assume that assumptions (H1') - (H6') hold. Then the function $t \to m(t) := \sup_{\lambda\in K} \inf_{\gamma\in\Gamma_\lambda} A(\gamma, t)$ is differentiable at $t = 0$ with derivative*

$$m'(0) = A_1(\gamma_{\lambda^*}^*),$$

where λ^* is defined in (H5'), and $\gamma_{\lambda^*}^*$ is defined in (H4').

Proof. Let $\mu(\lambda, t) = \inf_{\gamma\in\Gamma_\lambda} A(\gamma, t)$, $\lambda \in K$, $|t| < \varepsilon$. For a fixed $\lambda \in K$, we can apply lemma 4 with $\Gamma = \Gamma_\lambda$ to obtain

$$\mu(\lambda, t) = \mu(\lambda, 0) + A_1(\gamma_\lambda^*)t + o_\lambda(t)$$

where $o_\lambda(t) \to 0$ as $t \to 0$ for a fixed λ, and γ_λ^* is defined in (H4'). We only need to apply again lemma 4 to $m(t) := \sup \lambda \in K\mu(\lambda, t)$ (more precisely the analogous version of lemma 4 for a maximisation problem). We now check that hypothesis (H1)-(H4) of lemma 4 hold in that case.

We first verify that μ is continuous at $(\lambda, 0)$, $\lambda \in K$. Fix $\lambda_n \to \lambda$ and $t_n \to 0$. First take $\gamma_n \in \Gamma_{\lambda_n}$ such that

$$A(\gamma_n, t_n) \le \mu(\lambda_n, t_n) + \frac{1}{n}. \tag{41}$$

Up to a subsequence the γ_n converge to some γ belonging to γ_λ according to (H2'). Since A is continuous at $(\gamma, 0)$ we can pass to the limit in (41) to obtain $\liminf \mu(\lambda_n, t_n) \geq A(\gamma, 0) \geq \mu(\lambda, 0)$. To prove the opposite inequality we consider, using (H3'), $\gamma_n \in \Gamma_{\lambda_n}$ such that $\gamma_n \to \gamma_\lambda^*$. Then

$$\mu(\lambda, 0) = A(\gamma_\lambda^*, 0) = A(\gamma_n, t_n) + o(1) \geq \mu(\lambda_n, t_n) + o(1).$$

Passing to the limit gives $\limsup \mu(\lambda_n, t_n) \leq \mu(\lambda, 0)$.

It remains to prove that (i) the $o_\lambda(t)$ is uniform in $\lambda \in K$, and that (ii) $A_1(\gamma_\lambda^*)$ is continuous in λ.

Concerning (i), we first write that

$$\begin{aligned} o_\lambda(t) &= \mu(\lambda, t) - \mu(\lambda, 0) - A_1(\gamma_\lambda^*)t \\ &\leq A(\gamma_\lambda^*, t) - A(\gamma_\lambda^*, 0) - A_1(\gamma_\lambda^*)t, \end{aligned}$$

where γ_λ^* is defined in (H4'). According to hypothesis (H1') the right hand side goes to 0 as $t \to 0$ uniformly in $\lambda \in K$. Independently, given $\eta > 0$ we pick some $\gamma_{\lambda,t}^* \in \Gamma_\lambda$ such that $\mu(\lambda, t) \geq A(\gamma_{\lambda,t}^*, t) - \eta$, and write

$$\begin{aligned} o_\lambda(t) &= \mu(\lambda, t) - \mu(\lambda, 0) - A_1(\gamma_\lambda^*)t \\ &\geq A(\gamma_{\lambda,t}^*, t) - \eta - A(\gamma_{\lambda,t}^*, 0) - A_1(\gamma_\lambda^*)t \\ &= (A_1(\gamma_{\lambda,t}^*) - A_1(\gamma_\lambda^*))t + o(t) - \eta, \end{aligned}$$

where the $o(t)$ in the right hand side is uniform in λ according to (H1'). Since A_1 is bounded over Γ (according to (H6') and the compactness of Γ), we get

$$o_\lambda(t) \geq -C|t| + o(t) - \eta$$

for any $\eta > 0$ with $o(t)$ uniform in λ.

Concerning (ii), it suffices to prove, in view of (H6'), that $\lambda \to \gamma_\lambda^*$ is continuous. Fix some $\lambda \in K$ and a sequence $\lambda_n \to \lambda$. Since Γ is compact, the $\gamma_{\lambda_n}^*$ converge, up to a subsequence, to some γ_λ which belongs to Γ_λ according to (H2'). Given $\tilde\gamma \in \Gamma_\lambda$ and $\tilde\gamma_{\lambda_n} \in \Gamma_{\lambda_n}$ converging to $\tilde\gamma$ (which exist according to (H3')), passing to the limit in $A(\gamma_{\lambda_n}^*, 0) \leq A(\tilde\gamma_{\lambda_n}, 0)$ gives $A(\gamma_\lambda, 0) \leq A(\tilde\gamma, 0)$ for any $\tilde\gamma \in \Gamma_\lambda$. In view of (H4') we must have $\gamma_\lambda = \gamma_\lambda^*$. Thus $\gamma_{\lambda_n}^* \to \gamma_\lambda^*$ for any sequence $\lambda_n \to \lambda$. □

We can now end the proof of theorem 5. Recall that $\gamma^* : [0,1] \to \bar U$ is the unique constant-speed curve such that $\mathrm{diam}(U) = \mathrm{Long}(\gamma^*)$.

Step 5.2. *If for any $(x,y) \in K$, there exists an unique curve $\gamma \in \Gamma(x,y)$ such that $d(x,y) = Long(\gamma)$, then $t \to diam(U_t)$ is differentiable at $t = 0$ with derivative*

$$\frac{d}{dt} diam(U_t)_{|t=0} = \frac{1}{diam(U)} \int_0^1 (DV(\gamma^*(s))\gamma^{*'}(s), \gamma^{*'}(s))\, ds. \tag{42}$$

Proof. We apply lemma 5 with $\lambda = (x,y) \in K$, $\Gamma_\lambda = \Gamma(x,y)$ which is compact for the C^1-convergence, and, from (37),

$$A(\gamma, t) = \mathrm{Long}(\phi_t \circ \gamma), \quad A_1(\gamma) = \int_0^1 (DV(\gamma(s))\gamma'(s), \gamma'(s))\, \frac{ds}{|\gamma'(s)|}.$$

Then according to (37) and (39), we have

$$A(\gamma, t) = A(\gamma, 0) + A_1(\gamma) + o(t)$$

where the remainder $o(t)$ is uniform in $\gamma \in \Gamma_\lambda$, $\lambda \in K$. In particular (H1') holds. Moreover (H2'), (H3'), (H6') hold, and (H4'), (H5') hold by assumption. Thus

$$\frac{d}{dt}\mathrm{diam}(U_t)_{|t=0} = A_1(\gamma^*)$$

which is (42) recalling that $|{\gamma^*}'| = \mathrm{diam}(U)$. □

6. Proof of Proposition 1.1

The proof is a slight adaptation of the results in [19]. First a change of variable in the definition of $\lambda_p(U_t)$ gives

$$\lambda_p(U_t) = \inf_{u \in W^{1,p}(U)} \frac{\displaystyle\int_U A(x, \nabla u, t)\, dx}{\displaystyle\int_U B(x, u, t)\, dx}$$

with $A(x, \nabla u, t) = |D(x,t)\nabla u|^p C(x,t)$ and $B(x,u,t) = |u(x)|^p C(x,t)$ where

$$C(x,t) = |\det(D\phi_t(x)| = 1 + t\mathrm{div}V(x) + o(t),$$
$$D(x,t) = (D\phi_t(x))^{-1} = Id - tDV(x) + o(t).$$

Assuming that $\lambda_p(U)$ is simple, and denoting by u_p an extremal for λ_p normalized by $\|u_p\|_p = 1$, [19][thm 2] yields

(43)
$$\begin{aligned}\frac{d}{dt}\lambda_p(U_t)_{|t=0} &= \int_U \partial_t A(x, \nabla u_p, 0) - \lambda_p(U)\partial_t B(x, u_p, 0)\, dx\\ &= \int_U (\mathrm{div}\ V)|\nabla u_p|^p - p|\nabla u_p|^{p-2}(DV.\nabla u_p, \nabla u_p) - \lambda_p(U)|u_p|^p \mathrm{div}\ V\, dx.\end{aligned}$$

Up to approximate u_p by the smooth solution $u_{p,\varepsilon}$ of the equation (6) with $-\mathrm{div}((\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u)$ instead of $\Delta_p u$. Taking at the end the limit as $\varepsilon \to 0$, we will perform the computations without worrying for the lack of regularity of u_p. Multiplying (6) by $V\nabla u_p$ and integrating by parts gives

$$\begin{aligned}&\int_U -p|\nabla u_p|^{p-2}(DV.\nabla u_p, \nabla u_p) - \lambda_p(U)|u_p|^p \mathrm{div}\ V\, dx\\ &= p\int_U |\nabla u_p|^{p-2}(D^2 u_p V, \nabla u_p)\, dx - \lambda_p(U)\int_{\partial U}(V,\nu)|u_p|^p\, d\sigma\\ &= \int_U \mathrm{div}\ (|\nabla u_p|^p V) - |\nabla u_p|^p \mathrm{div}\ V\, dx - \lambda_p(U)\int_{\partial U}(V\nu)|u_p|^p\, d\sigma\\ &= \int_{\partial U}(|\nabla u_p|^p - \lambda_p(U)|u_p|^p)(V,\nu)\, d\sigma - \int_U |\nabla u_p|^p \mathrm{div}\ V\, dx\end{aligned}$$

Plugging this equality in (43) gives (18).

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