

OPTIMAL MATCHING PROBLEMS WITH COSTS GIVEN BY FINSLER DISTANCES

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ABSTRACT. In this paper we deal with an optimal matching problem, that is, we want to transport two commodities (modeled by two measures that encode the spacial distribution of each commodity) to a given location, where they will match, minimizing the total transport cost that in our case is given by the sum of the two different Finsler distances that the two measures are transported. We perform a method to approximate the matching measure and the pair of Kantorovich potentials associated with this problem taking limit as $p \rightarrow \infty$ in a variational system of p -Laplacian type.

1. Introduction. In this paper we continue the study of the optimal matching problem that we performed in [16]. An optimal matching problem (see [5], [6]) consists in transporting optimally two commodities to a prescribed location in such a way that they match there. The optimality criteria consists in minimizing the total cost of the operation measured in terms of the two Finsler distances that the commodities are transported. We deal with two general Finsler distances that are not necessarily symmetric, therefore the problem requires that we tackle some extra difficulties that are not present when the cost is given by the sum of two Euclidean distances.

By improving the tools developed in [16] and [17] we approach the problem by taking limit as $p \rightarrow \infty$ in a system of PDEs of p -Laplacian type, which allows us to

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give an approximation method to get a matching measure (that encodes the location where the matching takes place) and the Kantorovich potentials for the involved transports. This procedure to approximate mass transport problems (taking limit as $p \rightarrow \infty$ in a p -Laplacian equation) was first introduced by Evans and Gangbo in [11] and reveals quite fruitful, see [1], [3], [14], [15], [16], [17]. We have to remark that this limit procedure requires some care since here the involved PDE system is nontrivially coupled and therefore the estimates for one component are related to the ones for the other.

Optimal matching problems for uniformly convex costs were analyzed in [4], [5], [6] and have implications in economic theory (hedonic markets and equilibria), see also [7], [8], [9] and references therein. For the case in which costs are given by the Euclidean distance see [16].

1.1. Optimal transport problems. Optimal matching problems are closely related to optimal mass transport problems. For notations, concepts and results from the Monge-Kantorovich Mass Transport Theory we refer to [1], [10], [18] and [19]. Below, for the reader's convenience, we just briefly introduce the usual terminology of optimal mass transport theory that we will use in the rest of the paper.

Let us define $\pi_t(x, y) := (1 - t)x + ty$. Given a Radon measure γ in $\Omega \times \Omega$, its marginals are defined by $proj_x(\gamma) := \pi_0 \# \gamma$ and $proj_y(\gamma) := \pi_1 \# \gamma$.

The Monge-Kantorovich problem. Fix $\mu, \nu \in \mathcal{M}^+(\Omega)$ satisfying the mass balance condition

$$\mu(\Omega) = \nu(\Omega). \quad (1)$$

Given a cost function $c : \Omega \times \Omega \rightarrow [0, \infty)$, the Monge-Kantorovich problem is the minimization problem

$$\min \left\{ \int_{\Omega \times \Omega} c(x, y) d\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\},$$

where $\Pi(\mu, \nu) := \{\text{Radon measures } \gamma \text{ in } \Omega \times \Omega : \pi_0 \# \gamma = \mu, \pi_1 \# \gamma = \nu\}$. The elements $\gamma \in \Pi(\mu, \nu)$ are called transport plans between μ and ν , and a minimizer γ^* an optimal transport plan. Assuming c is lower semicontinuous,

$$\exists \gamma^* = \arg \min_{\gamma \in \Pi(\mu, \nu)} \int_{\Omega \times \Omega} c(x, y) d\gamma(x, y). \quad (2)$$

The Monge-Kantorovich problem has a dual formulation that can be stated in this case as follows (see for instance [18, Theorem 1.14]).

Kantorovich-Rubinstein's Theorem. Let $\mu, \nu \in \mathcal{M}(\Omega)$ be two measures satisfying the mass balance condition (1). Assume cost c is lower semicontinuous and satisfies the triangular inequality. Then,

$$\min \left\{ \int_{\Omega \times \Omega} c(x, y) d\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\} = \sup \left\{ \int_{\Omega} u d(\nu - \mu) : u \in K_c(X) \right\},$$

where $K_c(\Omega) := \{u : \Omega \mapsto \mathbb{R} : u(y) - u(x) \leq c(x, y)\}$. Moreover, there exists $u \in K_c(\Omega)$ such that

$$\int_{\Omega} u(\nu - \mu) = \sup \left\{ \int_{\Omega} v(\nu - \mu) : v \in K_c(\Omega) \right\}.$$

Such maximizers are called Kantorovich potentials.

For two measures $\mu, \nu \in \mathcal{M}^+(\Omega)$ satisfying the mass balance condition (1), the *Kantorovich-Rubinstein distance* between μ and ν is defined as

$$W_c(\mu, \nu) := \inf \left\{ \int_{\Omega \times \Omega} c(x, y) d\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\}.$$

1.2. Finsler distances as cost functions. Now, to state our problem in mathematical terms, we need to introduce the definition of a Finsler distance, for extra details and some properties of such functions we refer to Section 2.

A Finsler function Φ in \mathbb{R}^N is a function that is non-negative, continuous, convex, positively homogeneous of degree 1, that is,

$$\Phi(t\xi) = t\Phi(\xi) \quad \text{for any } t \geq 0, \xi \in \mathbb{R}^N,$$

and vanishes only at 0. The *dual function (or polar function)* of a Finsler function Φ is defined as

$$\Phi^*(\xi^*) := \sup\{\langle \xi^*, \xi \rangle : \Phi(\xi) \leq 1\} \quad \text{for } \xi^* \in \mathbb{R}^N.$$

It is immediate to verify that Φ^* is also a Finsler function.

A *Finsler structure* F on Ω is a measurable function $F : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}_+$ such that for any $x \in \Omega$, $F(x, \cdot)$ a Finsler function in \mathbb{R}^N . For a Finsler structure F on Ω , we define the dual structure $F^* : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}_+$ by

$$F^*(x, \xi) := \sup\{\langle \eta, \xi \rangle : F(x, \eta) \leq 1\}.$$

Important examples of Finsler structures on Ω are those of the form $\Phi(B(x)\xi)$, being Φ a Finsler function and $B(x)$ a symmetric $N \times N$ matrix, positive definite. Such kind of Finsler structures are known as *deformations of Minkowski norms*.

Let us introduce the cost functions we will handle. Given a Finsler structure F on Ω , we define the following cost function c_F :

$$c_F(x, y) := \inf_{\sigma \in \Gamma_{x,y}^\Omega} \int_0^1 F((\sigma(t)), \sigma'(t)) dt, \quad (3)$$

where, for $x, y \in \Omega$, the set $\Gamma_{x,y}^\Omega$ is given by

$$\Gamma_{x,y}^\Omega := \{\sigma \in C^1([0, 1], \Omega), \sigma(0) = x, \sigma(1) = y\}.$$

We have that c_F is a *Finsler distance*. We make emphasis on the fact that c_F is not necessary symmetric (i.e., $c_F(x, y) \neq c_F(y, x)$ may happen) because F is merely positively homogeneous. This fact creates new difficulties in the optimal mass transport problem compared with the case in which the cost is given by a norm (that is symmetric).

1.3. The optimal matching problem. We fix two non-negative compactly supported functions $f^+, f^- \in L^\infty$, with supports X_+, X_- , respectively, satisfying the mass balance condition

$$M_0 := \int_{X_+} f^+ = \int_{X_-} f^-.$$

We also consider a compact set D (the target set). Then we take a large bounded domain Ω that contains all the relevant sets, the supports of f_+ and f_- , X_+, X_- and the target set D . For simplicity we will assume that Ω is a convex C^2 bounded open set. We also assume that the resulting configuration verifies

$$X_+ \cap X_- = \emptyset, \quad (X_+ \cup X_-) \cap D = \emptyset \quad \text{and} \quad (X_+ \cup X_-) \cup D \subset\subset \Omega.$$

Now, we are given two continuous Finsler structures F and G and, associated to them, two Finsler distances, c_F , c_G , given by (3). Let us consider the set of measures

$$\Pi_D(f^+, f^-) = \left\{ \begin{array}{l} (\gamma_+, \gamma_-) \in \mathcal{M}^+(\Omega \times \Omega)^2 : \pi_0 \# \gamma_{\pm} = f^{\pm}, \\ \pi_1 \# \gamma_+ = \pi_1 \# \gamma_-, \text{ supp}(\pi_1 \# \gamma_{\pm}) \subset D \end{array} \right\}.$$

The *optimal matching problem* is the minimization problem

$$\min_{(\gamma_+, \gamma_-) \in \Pi_D(f^+, f^-)} \left\{ \int_{\Omega \times \Omega} c_F(x, y) d\gamma_+ + \int_{\Omega \times \Omega} c_G(x, y) d\gamma_- \right\}. \quad (4)$$

If $(\gamma_+, \gamma_-) \in \Pi_D(f^+, f^-)$ is a minimizer of the optimal matching problem (4), we shall call the measure $\mu^* = \pi_1 \# \gamma_+ = \pi_1 \# \gamma_-$ an *optimal matching measure*.

Let us denote by

$$\mathcal{M}(D, M_0) := \{\mu \in \mathcal{M}^+(\Omega) : \text{supp}(\mu) \subset D, \mu(\Omega) = M_0\}$$

the set of all possible matching measures. We have

$$\begin{aligned} W_{f^{\pm}}^D &:= \inf_{(\gamma_+, \gamma_-) \in \Pi_D(f^+, f^-)} \left\{ \int_{\Omega \times \Omega} c_F(x, y) d\gamma_+ + \int_{\Omega \times \Omega} c_G(x, y) d\gamma_- \right\} \\ &= \inf_{\mu \in \mathcal{M}(D, M_0)} \inf_{(\gamma_+, \gamma_-) \in \Pi(f^+, f^-, \mu)} \left\{ \int_{\Omega \times \Omega} c_F(x, y) d\gamma_+ + \int_{\Omega \times \Omega} c_G(x, y) d\gamma_- \right\} \\ &= \inf_{\mu \in \mathcal{M}(D, M_0)} \left\{ W_{c_F}(f_+, \mu) + W_{c_G}(f_-, \mu) \right\}, \end{aligned} \quad (5)$$

where

$$\Pi(f^+, f^-, \mu) := \{(\gamma_+, \gamma_-) \in \mathcal{M}^+(\Omega \times \Omega)^2 : \gamma_+ \in \Pi(f^+, \mu), \gamma_- \in \Pi(f^-, \mu)\}.$$

Note that on the right-hand side of (5) we are considering all possible measures supported in D with total mass M_0 and then we minimize the total transport cost. This is probably the most natural way of looking at the optimal matching problem and, as shown above, it is equivalent to our previous formulation.

The following result shows that there exist measures $(\gamma_+^*, \gamma_-^*) \in \Pi_D(f^+, f^-)$ such that

$$W_{f^{\pm}}^D = \int_{\Omega \times \Omega} c_F(x, y) d\gamma_+^* + \int_{\Omega \times \Omega} c_G(x, y) d\gamma_-^*.$$

Consequently, we have existence of optimal matching measures.

Theorem 1.1. *The optimal matching problem (4) has a solution, that is, there exist measures $(\gamma_+^*, \gamma_-^*) \in \Pi_D(f^+, f^-)$ such that*

$$W_{f^{\pm}}^D = \int_{\Omega \times \Omega} c_F(x, y) d\gamma_+^* + \int_{\Omega \times \Omega} c_G(x, y) d\gamma_-^*.$$

Proof. Take in (5) a minimizing sequence $\mu_n \in \mathcal{M}(D, M_0)$, then by the weak compactness of $\mathcal{M}(D, M_0)$ there exist a subsequence, denoted equal, that converges weakly in the sense of measures to a $\mu_{\infty} \in \mathcal{M}(D, M_0)$. Hence, by the weakly lower semi-continuity of the functions $\nu \mapsto W_{c_F}(\mu, \nu)$ and $\nu \mapsto W_{c_G}(\mu, \nu)$, we have

$$W_{c_F}(f_+, \mu_{\infty}) + W_{c_G}(f_-, \mu_{\infty}) \leq \liminf_n (W_{c_F}(f_+, \mu_n) + W_{c_G}(f_-, \mu_n)) = W_{f^{\pm}}^D.$$

Therefore,

$$W_{c_F}(f_+, \mu_{\infty}) + W_{c_G}(f_-, \mu_{\infty}) = W_{f^{\pm}}^D,$$

and thanks to (2) we conclude. \square

As we have mentioned, our main goal here is to find a pair of Kantorovich potentials and a matching measures taking the limit as $p \rightarrow \infty$ in a system of p -Laplacian type equations. Let us briefly introduce this. Consider the variational problem

$$\min_{\substack{(v, w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \\ v + w \geq 0 \text{ in } D}} \frac{1}{p} \int_{\Omega} (F^*(x, Dv))^p + \frac{1}{p} \int_{\Omega} (G^*(x, Dw))^p - \int_{\Omega} v f^+ - \int_{\Omega} w f^-, \quad (6)$$

where F^* and G^* are the dual Finsler structures associated to F and G . Under adequate differentiability conditions on F and G , given a minimizer (u_p, v_p) of (6), there exists a positive Radon measure h_p supported in D such that

$$\begin{cases} -\operatorname{div} \left([F^*(x, Dv_p(x))]^{p-1} \frac{\partial F^*}{\partial \xi}(x, Dv_p(x)) \right) = h_p - f^+ & \text{in } \Omega, \\ -\operatorname{div} \left([G^*(x, Dw_p(x))]^{p-1} \frac{\partial G^*}{\partial \xi}(x, Dw_p(x)) \right) = h_p - f^- & \text{in } \Omega, \end{cases}$$

together with Neumann boundary conditions on $\partial\Omega$. Then, our main result reads as follows:

Theorem 1.2. *Assume that $F^*(x, \cdot)$ and $G^*(x, \cdot)$ are $C^1(\mathbb{R}^N \setminus \{0\})$. Up to a subsequence,*

$$\lim_{p \rightarrow \infty} (v_p, w_p) = (v_{\infty}, w_{\infty}) \quad \text{uniformly}$$

and

$$h_p \rightharpoonup h_{\infty} \quad \text{as } p \rightarrow \infty, \text{ weakly}^* \text{ as measures,}$$

with h_{∞} a nonnegative Radon measure supported in $\{x \in D : v_{\infty}(x) = w_{\infty}(x)\}$.

These limit functions and the limit measure provide a solution to the optimal matching problem in the sense that they satisfy:

$$h_{\infty} \quad \text{is an optimal matching measure,}$$

$$v_{\infty} \text{ is a Kantorovich potential for the transport of } f^+ \text{ to } h_{\infty}, \text{ and}$$

$$w_{\infty} \text{ is a Kantorovich potential for the transport of } f^- \text{ to } h_{\infty}.$$

Remark 1.3. In the case that both costs are given by the Euclidean distance we want to point out the following: First, the corresponding Monge transport problems have a solution and moreover (see [16])

$$W_{f^{\pm}}^D = \min_{(T_+, T_-) \in \mathcal{A}_D(f^+, f^-)} \int_{\Omega} |x - T_+(x)| f^+(x) dx + \int_{\Omega} |y - T_-(y)| f^-(y) dy,$$

where $\mathcal{A}_D(f^+, f^-)$ is the set

$$\{(T_+, T_-) : T_{\pm} : \Omega \rightarrow \Omega \text{ are Borel functions, } T_{\pm}(X_{\pm}) \subset D, T_+ \# f^+ = T_- \# f^-\}.$$

In addition, in [16] we also showed that there exist optimal matching measures supported on the boundary of the target set. In this general setting of costs given by Finsler structures this is not always true, as the following simple example shows (for simplicity we consider discontinuous Finsler structures, but the same example can be adapted easily to provide a continuous example). Let $\Omega = (-1, 6)$, $f_+ = \chi_{[4,5]}$,

$f_- = \chi_{[0,1]}$ and $D = [2, 3]$. For $0 < \epsilon < 1$, we consider continuous Finsler structures, F_ϵ and G_ϵ , defined as

$$F_\epsilon(x, \xi) := \begin{cases} |\xi| & \text{if } x \in \Omega \setminus (2 - \epsilon, 2 + \epsilon), \\ \frac{1}{\epsilon^2} |\xi| & \text{if } x \in (2 - \epsilon, 2 + \epsilon), \end{cases}$$

and

$$G_\epsilon(x, \xi) := \begin{cases} |\xi| & \text{if } x \in \Omega \setminus (3 - \epsilon, 3 + \epsilon), \\ \frac{1}{\epsilon^2} |\xi| & \text{if } x \in (3 - \epsilon, 3 + \epsilon). \end{cases}$$

Hence, for $x \in [4, 5]$, if $y \in (2, 2 + \epsilon)$ then

$$c_{F_\epsilon}(x, y) = \int_0^1 F_\epsilon(x + t(y - x), y - x) dt = x - 2 - \epsilon + \frac{1}{\epsilon^2}(2 + \epsilon - y),$$

and if $y \in (2 + \epsilon, 3)$, $c_{F_\epsilon}(x, y) = |x - y|$. Similarly, for $x \in [0, 1]$, if $y \in (3 - \epsilon, 3)$ then

$$c_{G_\epsilon}(x, y) = 3 - \epsilon - x + \frac{1}{\epsilon^2}(y - 3 - \epsilon),$$

and if $y \in (2, 3 - \epsilon)$, $c_{G_\epsilon}(x, y) = |x - y|$. Then it is clear that, when ϵ is small enough, any optimal matching measure will be supported in the set $[2 + \frac{1}{2}\epsilon, 3 - \frac{1}{2}\epsilon]$, since otherwise we have to pay something of order $1/\epsilon$ per unit of mass for one of the two transports, and in the set $(2 + \epsilon, 3 - \epsilon)$ we have to pay at most 3 for every unit of mass transported there.

Finally, let us remark that in the particular case in which c_F and c_G are geodesic distances on a geodesically complete, connected Riemannian manifold (see [17] for several examples), by the results in [13], it is known that the corresponding Monge transport problems has a solution, that is, there exists Borel functions $T_\pm^* : \Omega \rightarrow \Omega$ such that $T_+ \# f^+ = \mu_\infty = T_- \# f^-$, such that

$$\int_{\Omega \times \Omega} c_F(x, y) d\gamma_+^* = \int_{\Omega} c_F(x, T_+^*(x)) f^+(x) dx$$

and

$$\int_{\Omega \times \Omega} c_G(x, y) d\gamma_-^* = \int_{\Omega} c_G(x, T_-^*(x)) f^-(x) dx.$$

Therefore,

$$W_{f^\pm}^D = \int_{\Omega} c_F(x, T_+^*(x)) f^+(x) dx + \int_{\Omega} c_G(x, T_-^*(x)) f^-(x) dx.$$

Moreover, we also have

$$W_{f^\pm}^D = \min_{(T_+, T_-) \in \mathcal{A}_D(f^+, f^-)} \int_{\Omega} c_F(x, T_+(x)) f^+(x) dx + \int_{\Omega} c_G(y, T_-(y)) f^-(y) dy.$$

These are the contents of the paper: in Section 2 we collect some properties of Finsler functions that will be used in the core of the paper; Section 3 contains our main results, we deal with the p -Laplacian system and show that we can pass to the limit as $p \rightarrow \infty$ obtaining the optimal matching results, in addition we find transport densities for the transport problems involved.

2. Preliminaries on Finsler functions. In this section we collect some properties of Finsler functions in \mathbb{R}^N that will be used in the sequel. Recall from the introduction that a Finsler function Φ is a non-negative continuous convex function, positively homogeneous of degree 1,

$$\Phi(t\xi) = t\Phi(\xi) \quad \text{for any } t \geq 0, \xi \in \mathbb{R}^N,$$

that vanishes only at 0. Observe that Φ satisfies

$$\alpha|\xi| \leq \Phi(\xi) \leq \beta|\xi| \quad \text{for any } \xi \in \mathbb{R}^N,$$

for some positive constants α, β .

Note that Finsler functions are extensions of norms. In fact, any norm in \mathbb{R}^N is a Finsler function, and any symmetric Finsler function is a norm. Moreover, for any Finsler function, convexity is equivalent to the triangular inequality.

Let

$$B_\Phi := \{\xi \in \mathbb{R}^N : \Phi(\xi) \leq 1\}.$$

This set B_Φ is a closed, bounded and convex set with $0 \in \text{int}(B)$. It is symmetric around the origin if Φ is a norm. Conversely, for any closed bounded convex set K with $0 \in \text{int}(K)$, $\phi_K(\xi) := \inf\{\alpha > 0 : \xi \in \alpha K\}$ is a Finsler function with $B_{\phi_K} = K$; when K is centrally symmetric, we have a norm. In the literature the Finsler functions are also denominated as Minkowski norms.

The *dual function* (or *polar function*) of a Finsler function Φ is defined as

$$\Phi^*(\xi^*) := \sup\{\langle \xi^*; \xi \rangle : \xi \in B_\Phi\} \quad \text{for } \xi^* \in \mathbb{R}^N.$$

It is immediate to verify that Φ^* is also a Finsler function; and a norm when Φ is a norm. We also have

$$\Phi^*(\xi^*) = \sup_{\xi \neq 0} \frac{\langle \xi^*; \xi \rangle}{\Phi(\xi)}.$$

Therefore, the following inequality of Cauchy-Schwarz type holds,

$$\langle \xi^*; \xi \rangle \leq \Phi(\xi)\Phi^*(\xi^*). \quad (1)$$

If Φ is a norm, we have

$$|\langle \xi^*; \xi \rangle| \leq \Phi(\xi)\Phi^*(\xi^*). \quad (2)$$

Now, for general Finsler functions the inequality (2) is not true. An example of a Finsler function that is not a norm in \mathbb{R} is given by $\Phi(\xi) := a\xi^- + b\xi^+$, with $0 < a < b$.

It is not difficult to see that

$$\Phi^{**}(\xi) = \Phi(\xi), \quad \forall \xi \in \mathbb{R}^N.$$

Hence,

$$\Phi(\xi) = \sup_{\xi^* \neq 0} \frac{\langle \xi; \xi^* \rangle}{\Phi^*(\xi^*)}.$$

If we assume that the Finsler function Φ is differentiable at ξ , then by Euler's Theorem,

$$\Phi(\xi) = \langle D\Phi(\xi); \xi \rangle. \quad (3)$$

Moreover, if we assume Φ is differentiable in $K \subset \mathbb{R}^N$, since Φ is convex and satisfies the triangle inequality, we have

$$\langle D\Phi(\xi); \eta \rangle \leq \Phi(\eta) \quad \forall \xi, \eta \in K. \quad (4)$$

If we assume Φ is differentiable in $\mathbb{R}^N \setminus \{0\}$, by Lagrange multipliers, from

$$\Phi^*(\xi^*) = \sup_{\Phi(\xi)=1} \langle \xi; \xi^* \rangle,$$

we get that:

$$\begin{aligned} & \text{if } \Phi(\xi) = 1 \quad \text{and} \quad \Phi^*(\xi^*) = \langle \xi; \xi^* \rangle \\ & \text{then there exists } \lambda \in \mathbb{R} \text{ such that } \xi^* = \lambda D\Phi(\xi). \end{aligned} \quad (5)$$

From (3) and (4), we also have

$$\Phi^*(D\Phi(\xi)) = 1 \quad \forall \xi \neq 0. \quad (6)$$

3. The limit as $p \rightarrow \infty$ in a p -Laplacian system. In this section we show that we can follow the ideas of Evans and Gangbo in [11] to get the matching measure and the Kantorovich potentials at the same time.

From now on we will assume that F, G are continuous Finsler structures on Ω satisfying

$$\alpha_1 |\xi| \leq F^*(x, \xi) \leq \beta_1 |\xi| \quad \text{for any } \xi \in \mathbb{R}^N \text{ and } x \in \Omega, \quad (7)$$

$$\alpha_2 |\xi| \leq G^*(x, \xi) \leq \beta_2 |\xi| \quad \text{for any } \xi \in \mathbb{R}^N \text{ and } x \in \Omega, \quad (8)$$

being α_i, β_i positive constants. We can take w.l.g. $\alpha_i = \alpha$ and $\beta_i = \beta$.

In [17] we proved the following result that will be used later on.

Lemma 3.1. *$u \in W^{1,\infty}(\Omega)$ if and only if $Lip(u, c_F) < \infty$, where*

$$Lip(u, c_F) := \sup \left\{ \frac{u(y) - u(x)}{c_F(x, y)} : x, y \in \Omega, x \neq y \right\};$$

and

$$esssup_{x \in \Omega} F^*(x, Du(x)) = Lip(u, c_F).$$

As consequence of Lemma 3.1, we have that the set of functions

$$K_{c_F}(\Omega) = \{u : \Omega \mapsto \mathbb{R} : u(y) - u(x) \leq c_F(x, y)\}.$$

coincides with the set

$$K_F^*(\Omega) := \{u : \Omega \mapsto \mathbb{R} : esssup_{x \in \Omega} F^*(x, Du(x)) \leq 1\}$$

3.1. The limit procedure. Take $p > N$ from now on, and recall that, for simplicity, we assumed that Ω is a convex C^2 bounded open set. We will use the following result whose proof follows standard Functional Analysis arguments.

Lemma 3.2 (A Poincaré's type inequality). *There exists a constant $C > 0$ such that*

$$\|(f, g)\|_p \leq C \left(\|(\nabla f, \nabla g)\|_p + \left| \int_{\Omega} (f + g) \right| \right)$$

for all $(f, g) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega)$, $f(x_0) = g(x_0)$ for some $x_0 \in D$.

The constants that appear in the previous inequality may not be uniform in p . It is not our aim here to make this dependence precise, then we are not allowed to use these results in the passage to the limit as $p \rightarrow \infty$, they are used only to show existence of a solution of the variational problem under consideration.

Let us consider the following variational problem

$$\min_{\substack{(v, w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \\ v + w \geq 0 \text{ in } D}} \frac{1}{p} \int_{\Omega} (F^*(x, Dv))^p + \frac{1}{p} \int_{\Omega} (G^*(x, Dw))^p - \int_{\Omega} v f^+ - \int_{\Omega} w f^-. \quad (9)$$

The next result deals with existence and uniqueness of solutions for the variational problem (9).

Theorem 3.3. *There exists a minimizer (v_p, w_p) of (9). Moreover, when F^* and G^* are strictly convex we have uniqueness of minimizers up to an additive constant, that is, if $(\tilde{v}_p, \tilde{w}_p)$ is another minimizer then there exists a constant c such that $(\tilde{v}_p, \tilde{w}_p) = (v_p - c, w_p + c)$.*

Proof. Set

$$\Psi_p(v, w) := \frac{1}{p} \int_{\Omega} (F^*(x, Dv))^p + \frac{1}{p} \int_{\Omega} (G^*(x, Dw))^p - \int_{\Omega} v f^+ - \int_{\Omega} w f^-.$$

Let us begin by observing that, since the functions in $W^{1,p}(\Omega)$ ($p > N$) are continuous, it is easy to see that

$$\min_{\substack{(v, w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \\ v + w \geq 0 \text{ in } D}} \Psi_p(v, w) = \min_{\substack{(v, w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \\ v + w \geq 0 \text{ in } D \\ \exists x_0 \in D, v(x_0) + w(x_0) = 0}} \Psi_p(v, w). \quad (10)$$

Moreover, since $\Psi_p(v, w) = \Psi_p(v - c, w + c)$ for any constant c , by taking

$$c = \frac{1}{2} \left(\frac{1}{|\Omega|} \int_{\Omega} v - \frac{1}{|\Omega|} \int_{\Omega} w \right),$$

we can minimize $\Psi_p(v, w)$ between functions (v, w) with $\int_{\Omega} v = \int_{\Omega} w$.

Now, by Lemma 3.2, and having in mind (7) and (8), $\Psi_p(v, w)$ is a finite lower semicontinuous and coercive convex functional for the closed convex subset of $W^{1,p}(\Omega) \times W^{1,p}(\Omega)$, \mathcal{B} , given by

$$\left\{ \begin{array}{l} (v, w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) : v + w \geq 0 \text{ in } D, v(x_0) + w(x_0) = 0 \\ \text{for some } x_0 \in D, \int_{\Omega} v = \int_{\Omega} w \end{array} \right\}.$$

Then, by [2, Corollary 3.23], Ψ_p attains its infimum on \mathcal{B} , which is equivalent to say that

$$\inf_{\substack{(v, w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \\ v + w \geq 0 \text{ in } D}} \Psi_p(v, w)$$

is attained.

Uniqueness for strictly convex Finsler structures follows as in [16]. \square

Now we prove that we can pass to the limit as $p \rightarrow \infty$ in a subsequence of minimizer functions.

Theorem 3.4. *Let (v_p, w_p) be minimizer functions of (9). Then, there exists a subsequence $p_i \rightarrow +\infty$ such that*

$$\lim_{i \rightarrow \infty} (v_{p_i}, w_{p_i}) = (v_{\infty}, w_{\infty}) \quad \text{uniformly,}$$

where (v_{∞}, w_{∞}) is a solution of the variational problem

$$\max_{\substack{v, w \in W^{1,\infty}(\Omega) \\ F^*(x, Dv(x)), G^*(x, Dw(x)) \leq 1 \text{ a.e.} \\ v + w \geq 0 \text{ in } D}} \int_{\Omega} -v f^+ - w f^-. \quad (11)$$

Proof of Theorem 3.4. Let us take $(v_p, w_p) \in \mathcal{B}$ a minimizer of (9). For $(v, w) \in W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$, with $F^*(x, Dv) \leq 1$ and $G^*(x, Dw) \leq 1$ for a.e. $x \in \Omega$, and $v + w \geq 0$ in D , we have that

$$\begin{aligned} \int_{\Omega} v_p f^+ + \int_{\Omega} w_p f^- &\leq \frac{1}{p} \int_{\Omega} (F^*(x, Dv_p))^p \\ &\quad + \frac{1}{p} \int_{\Omega} (G^*(x, Dw_p))^p + \int_{\Omega} v_p f^+ + \int_{\Omega} w_p f^- \\ &\leq \frac{1}{p} \int_{\Omega} (F^*(x, Dv))^p + \frac{1}{p} \int_{\Omega} (G^*(x, Dw))^p \\ &\quad + \int_{\Omega} v f^+ + \int_{\Omega} w f^- \\ &\leq 2 \frac{|\Omega|}{p} + \int_{\Omega} v f^+ + \int_{\Omega} w f^-. \end{aligned} \quad (12)$$

Now, by (10), we can assume that there exists $x_p \in D$ such that $v_p(x_p) + w_p(x_p) = 0$. We can also assume that $v_p(z_{\infty}) = 0$ for all p , for any $z_{\infty} \in \Omega$. Hence, as $p > N$, we have,

$$\|v_p\|_{\infty} \leq C_1 \|Dv_p\|_p, \quad (13)$$

and

$$\|w_p\|_{\infty} \leq C_1 (\|Dw_p\|_p + \|Dv_p\|_p), \quad (14)$$

with C_1 not depending on p . See [16] for the details.

From (12), (7), (8), (13) and (14) and using Hölder's inequality, we get

$$\left(\int_{\Omega} [F^*(x, Dv_p(x))]^p \right)^{1/p}, \left(\int_{\Omega} [G^*(x, Dw_p(x))]^p \right)^{1/p} \leq (C_2 p)^{\frac{1}{p-1}} \quad \forall p > N,$$

and

$$\|Dv_p\|_{L^p(\Omega)}, \|Dw_p\|_{L^p(\Omega)} \leq C_3 \quad \forall p > N,$$

with C_i independent of p .

Therefore, by Morrey's inequality and Arzela-Ascoli's compactness criterion, there exists a subsequence such that

$$v_{p_i} \rightarrow v_{\infty} \quad \text{uniformly in } \bar{\Omega} \quad \text{and} \quad w_{p_i} \rightarrow w_{\infty} \quad \text{uniformly in } \bar{\Omega},$$

and, so, $v_{\infty} + w_{\infty} \geq 0$ in D . Moreover, we get

$$\|Dv_{\infty}\|_{\infty}, \|Dw_{\infty}\|_{\infty} \leq C \quad \text{and} \quad \|F^*(\cdot, Dv_{\infty})\|_{\infty}, \|G^*(\cdot, Dw_{\infty})\|_{\infty} \leq 1.$$

Finally, passing to the limit in (12), we get

$$\int_{\Omega} -v_{\infty} f^+ - w_{\infty} f^- = \sup_{\substack{v, w \in W^{1,\infty}(\Omega) \\ F^*(x, Dv(x)), G^*(x, Dw(x)) \leq 1 \text{ a.e.} \\ v + w \geq 0 \text{ in } D}} \int_{\Omega} -v f^+ - w f^-.$$

This ends the proof. \square

Remark 3.5. Note that the convergence as $p \rightarrow \infty$ is only along a subsequence. The main content of our result is that there is enough compactness to pass to the limit along subsequences and moreover that all possible limits are solutions to the maximization limit problem (11).

We now prove some properties of the minimizers that allow to show that (v_∞, w_∞) are Kantorovich potentials. Moreover, we will see that this limit procedure gives much more since it allows us to identify the optimal matching measure.

Lemma 3.6. *Assume that $F^*(x, \cdot)$ and $G^*(x, \cdot)$ are $C^1(\mathbb{R}^N \setminus \{0\})$. Let (v_p, w_p) be minimizer functions of problem (9). Then, there exists a positive Radon measure h_p of mass M_0 such that*

1.

$$\begin{cases} -\operatorname{div} \left([F^*(x, Dv_p(x))]^{p-1} \frac{\partial F^*}{\partial \xi}(x, Dv_p(x)) \right) = h_p - f^+ & \text{in } \Omega, \\ [F^*(x, Dv_p(x))]^{p-1} \left\langle \frac{\partial F^*}{\partial \xi}(x, Dv_p(x)); \eta \right\rangle = 0 & \text{on } \partial\Omega. \end{cases} \quad (15)$$

$$\begin{cases} -\operatorname{div} \left([G^*(x, Dw_p(x))]^{p-1} \frac{\partial G^*}{\partial \xi}(x, Dw_p(x)) \right) = h_p - f^- & \text{in } \Omega, \\ [G^*(x, Dw_p(x))]^{p-1} \left\langle \frac{\partial G^*}{\partial \xi}(x, Dw_p(x)); \eta \right\rangle = 0 & \text{on } \partial\Omega. \end{cases} \quad (16)$$

Here η is the exterior normal vector on $\partial\Omega$, and $\frac{\partial F^*}{\partial \xi}$ is the gradient of $F^*(x, \xi)$ with respect the second variable ξ , similarly for G^* .

2. The positive measure h_p is supported on $\{x \in D : v_p(x) + w_p(x) = 0\}$.

Proof. In this proof, for shortness, we will avoid to write the x dependence of F^* and G^* , that is, we write $F^*(Dv_p)$ and $G^*(Dw_p)$. Recall that since $p > N$, we have $W^{1,p}(\Omega) \subset C(\bar{\Omega})$. For any $\varphi, \psi \in W^{1,p}(\Omega)$ such that $\varphi + \psi = 0$ in D , since (v_p, w_p) is a minimizer of Ψ in the set

$$\{(v, w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) : v + w \geq 0 \text{ in } D\},$$

the function

$$I_1(t) := \Psi(v_p + t\varphi, w_p + t\psi)$$

has a minimum at $t = 0$. Therefore, $I_1'(0) = 0$, from where it follows that

$$\begin{aligned} & \int_{\Omega} [F^*(Dv_p)]^{p-1} \frac{\partial F^*}{\partial \xi}(Dv_p) D\varphi + \int_{\Omega} [G^*(Dw_p)]^{p-1} \frac{\partial G^*}{\partial \xi}(Dw_p) D\psi \\ &= - \int_{\Omega} f^+ \varphi - \int_{\Omega} f^- \psi. \end{aligned} \quad (17)$$

Observe that, taking $\psi = -\varphi$ in (17), we get that

$$\begin{cases} -\operatorname{div} \left([F^*(Dv_p)]^{p-1} \frac{\partial F^*}{\partial \xi}(Dv_p) \right) + \operatorname{div} \left([G^*(Dw_p)]^{p-1} \frac{\partial G^*}{\partial \xi}(Dw_p) \right) \\ \qquad \qquad \qquad = f^- - f^+ & \text{in } \Omega, \\ [F^*(Dv_p)]^{p-1} \left\langle \frac{\partial F^*}{\partial \xi}(Dv_p); \eta \right\rangle + [G^*(Dw_p)]^{p-1} \left\langle \frac{\partial G^*}{\partial \xi}(Dw_p); \eta \right\rangle = 0 & \text{on } \partial\Omega. \end{cases} \quad (18)$$

Similarly, for any $\varphi \in W^{1,p}(\Omega)$, $\varphi \geq 0$, and any $t > 0$, we have

$$I_2(t) := \Psi(v_p + t\varphi, w_p) - \Psi(v_p, w_p) \geq 0$$

and

$$I_3(t) := \Psi(v_p, w_p + t\varphi) - \Psi(v_p, w_p) \geq 0.$$

Then, by taking limits in $\frac{I_i(t)}{t}$ $i = 2, 3$, as $t \rightarrow 0$, we get

$$\begin{cases} -\operatorname{div} \left([F^*(Dv_p)]^{p-1} \frac{\partial F^*}{\partial \xi}(Dv_p) \right) + f^+ \geq 0 & \text{in } \Omega, \\ -\operatorname{div} \left([G^*(Dw_p)]^{p-1} \frac{\partial G^*}{\partial \xi}(Dw_p) \right) + f^- \geq 0 & \text{in } \Omega. \end{cases}$$

Therefore, thanks to (18)

$$\begin{aligned} h_p &:= -\operatorname{div} \left([F^*(Dv_p)]^{p-1} \frac{\partial F^*}{\partial \xi}(Dv_p) \right) + f^+ \\ &= -\operatorname{div} \left([G^*(Dw_p)]^{p-1} \frac{\partial G^*}{\partial \xi}(Dw_p) \right) + f^- \end{aligned}$$

defines a positive measure. Moreover, h_p is supported on $\{x \in D : v_p(x) + w_p(x) = 0\}$. Indeed, for $\varphi \in \mathcal{D}(\Omega)$ supported on $\Omega \setminus \{x \in D : v_p(x) + w_p(x) = 0\}$ and $t \neq 0$ small enough,

$$I_4(t) := \Psi(v_p + t\varphi, w_p) - \Psi(v_p, w_p) \geq 0.$$

then, taking limits in $\frac{I_4(t)}{t}$ as $t \rightarrow 0$, we conclude.

Given now $\varphi \in \mathcal{D}(\mathbb{R}^N)$, if we take $\psi \in \mathcal{D}(\Omega)$ such that $\varphi + \psi = 0$ en D , (17) says that

$$\begin{aligned} &\int_{\Omega} [F^*(Dv_p)]^{p-1} \frac{\partial F^*}{\partial \xi}(Dv_p) D\varphi + \int_{\Omega} [G^*(Dw_p)]^{p-1} \frac{\partial G^*}{\partial \xi}(Dw_p) D\psi \\ &= - \int_{\Omega} f^+ \varphi - \int_{\Omega} f^- \psi. \end{aligned}$$

On the other hand, since $\psi \in \mathcal{D}(\Omega)$ and $\operatorname{supp}(h_p) \subset D$, we have

$$\int_{\Omega} [G^*(Dw_p)]^{p-1} \frac{\partial G^*}{\partial \xi}(Dw_p) D\psi = \int_{\Omega} \psi dh_p - \int_{\Omega} f^- \psi = - \int_{\Omega} \varphi dh_p - \int_{\Omega} f^- \psi.$$

Then, from the two above expressions, by density we obtain that

$$\int_{\Omega} [F^*(Dv_p)]^{p-1} \frac{\partial F^*}{\partial \xi}(Dv_p) D\varphi = \int_{\Omega} \varphi dh_p - \int_{\Omega} f^+ \varphi, \quad \forall \varphi \in W^{1,p}(\Omega), \quad (19)$$

which shows the first statement in (1) for the first problem. Similarly, we obtain the second one.

Finally, taking $\varphi = 1$ in (19), we get

$$\int_{\Omega} dh_p = M_0,$$

and the proof is finished. \square

3.2. The optimal matching problem. Let us begin with the following proposition.

Proposition 3.7.

$$W_{f^{\pm}}^D := \sup_{\substack{v, w \in W^{1,\infty}(\Omega) \\ F^*(x, Dv(x)), G^*(x, Dw(x)) \leq 1 \text{ a.e.} \\ v + w \geq 0 \text{ in } D}} \int_{\Omega} -v f^+ - w f^-.$$

Proof. For a fixed $\mu \in \mathcal{M}(D, M_0)$, by Kantorovich-Rubinstein's Theorem and Lemma 3.1, there exist Kantorovich potentials $u_{\pm, \mu} \in W^{1, \infty}(\Omega)$, depending on μ , with

$$\text{esssup } F^*(x, Du_{+, \mu}) \leq 1 \quad \text{and} \quad \text{esssup } G^*(x, Du_{-, \mu}) \leq 1,$$

such that

$$\int_{\Omega} u_{+, \mu}(\mu - f^+) = \sup \left\{ \int_{\Omega} v(\mu - f^+) : v \in K_{c_F}(\Omega) \right\} = \min_{\gamma \in \Pi(f^+, \mu)} \int_{\Omega \times \Omega} c_F(x, y) d\gamma$$

and

$$\int_{\Omega} u_{-, \mu}(\mu - f^-) = \sup \left\{ \int_{\Omega} v(\mu - f^-) : v \in K_{c_G}(\Omega) \right\} = \min_{\gamma \in \Pi(f^-, \mu)} \int_{\Omega \times \Omega} c_G(x, y) d\gamma.$$

Therefore,

$$\begin{aligned} & \sup_{\substack{v, w \in W^{1, \infty}(\Omega) \\ F^*(x, Dv(x)), G^*(x, Dw(x)) \leq 1 \text{ a.e.}}} \int_{\Omega} -vf^+ - wf^- + (v+w)\mu = \\ & = \inf_{(\gamma_+, \gamma_-) \in \Pi(f^+, f^-, \mu)} \int_{\Omega \times \Omega} c_F(x, y) d\gamma_+ + \int_{\Omega \times \Omega} c_G(x, y) d\gamma_-. \end{aligned}$$

Since

$$\begin{aligned} & \int_{\Omega} -vf^+ - wf^- + (v+w)\mu \\ & = \int_{\Omega} -vf^+ - (w - \min_D(w+v))f^- + (w - \min_D(w+v) + v)\mu \\ & \leq \sup_{\substack{\tilde{v}, \tilde{w} \in W^{1, \infty}(\Omega) \\ F^*(x, D\tilde{v}(x)), G^*(x, D\tilde{w}(x)) \leq 1 \\ \tilde{v} + \tilde{w} \geq 0 \text{ in } D}} \int_{\Omega} -\tilde{v}f^+ - \tilde{w}f^- + (\tilde{w} + \tilde{v})\mu, \end{aligned}$$

we have

$$\begin{aligned} & \sup_{\substack{v, w \in W^{1, \infty}(\Omega) \\ F^*(x, Dv(x)), G^*(x, Dw(x)) \leq 1 \text{ a.e.} \\ v+w \geq 0 \text{ in } D}} \int_{\Omega} -vf^+ - wf^- + (v+w)\mu = \\ & = \inf_{(\gamma_+, \gamma_-) \in \Pi(f^+, f^-, \mu)} \int_{\Omega \times \Omega} c_F(x, y) d\gamma_+ + \int_{\Omega \times \Omega} c_G(x, y) d\gamma_-. \end{aligned}$$

Hence, from (5), we get

$$\begin{aligned} & \inf_{\mu \in \mathcal{M}(D, M_0)} \sup_{\substack{v, w \in W^{1, \infty}(\Omega) \\ F^*(x, Dv(x)), G^*(x, Dw(x)) \leq 1 \text{ a.e.} \\ v+w \geq 0 \text{ in } D}} \int_{\Omega} -vf^+ - wf^- + (v+w)\mu \\ & = \inf_{(\gamma_+, \gamma_-) \in \Pi_D(f^+, f^-)} \int_{\Omega \times \Omega} c_F(x, y) d\gamma_+ + \int_{\Omega \times \Omega} c_G(x, y) d\gamma_-. \end{aligned}$$

Now, by Fan's Minimax Theorem ([12]), we can interchange inf sup by sup inf in the first part of the above expression and, since

$$\begin{aligned} & \sup_{\substack{v, w \in W^{1, \infty}(\Omega) \\ F^*(x, Dv(x)), G^*(x, Dw(x)) \leq 1 \text{ a.e.} \\ v+w \geq 0 \text{ in } D}} \min_{\mu \in \mathcal{M}(D, M_0)} \int_{\Omega} -vf^+ - wf^- + (v+w)\mu \\ & = \sup_{\substack{v, w \in W^{1, \infty}(\Omega) \\ F^*(x, Dv(x)), G^*(x, Dw(x)) \leq 1 \text{ a.e.} \\ v+w \geq 0 \text{ in } D}} \int_{\Omega} -vf^+ - wf^-, \end{aligned}$$

we get the desired conclusion. \square

Theorem 3.8. *Under the same assumptions of Lemma 3.6, up to a subsequence,*

$$h_p \rightharpoonup h_\infty \quad \text{as } p \rightarrow \infty, \text{ weakly* as measures,}$$

with h_∞ a positive Radon measure of mass M_0 supported on $\{x \in D : v_\infty(x) + w_\infty(x) = 0\}$. And the limit (v_∞, w_∞) obtained in Theorem 3.4 satisfies:

v_∞ is a Kantorovich potential for the transport of f^+ to h_∞ ,

w_∞ is a Kantorovich potential for the transport of f^- to h_∞ ,

with respect to c_F and c_G respectively, begin these cost functions the Finsler distances associated to F and G .

Proof. We will write again $F^*(Dv_p)$ in the proof for shortness, and the same for G^* . From the last equality in the proof of the previous lemma,

$$\int_{\Omega} dh_p = M_0,$$

we can assume that there exists a positive Radon measure h_∞ of mass M_0 such that, up to an increasing subsequence of the sequence given in Theorem 3.4,

$$h_{p_i} \rightharpoonup h_\infty.$$

Let $\varphi \in \mathcal{D}(\Omega)$ be supported on $\Omega \setminus \{x \in D : v_\infty(x) + w_\infty(x) = 0\}$. Then, since

$$\lim_i (v_{p_i}, w_{p_i}) = (v_\infty, w_\infty) \quad \text{uniformly,}$$

there exists $p_0 > N$ such that φ is supported on $\Omega \setminus \{x \in D : v_{p_i}(x) + w_{p_i}(x) = 0\}$ for all $p_i \geq p_0$. Therefore,

$$\int_{\Omega} \varphi dh_\infty = \lim_{i \rightarrow \infty} \int_{\Omega} \varphi dh_{p_i} = 0.$$

Consequently, h_∞ is supported on $\{x \in D : v_\infty(x) + w_\infty(x) = 0\}$.

Let us write p instead of p_i from now on.

Since $(F^*(\xi))^p - (F^*(\eta))^p \leq p(F^*(\xi))^{p-1} \frac{\partial F}{\partial \xi}(\xi)(\xi - \eta)$ for any $\xi, \eta \in \mathbb{R}^N$, we have

$$\begin{aligned} & \frac{1}{p} \int_{\Omega} (F^*(Dv_p))^p - \int_{\Omega} (dh_p - f^+)v_p \\ & + \int_{\Omega} (F^*(Dv_p))^{p-1} \frac{\partial F^*}{\partial \xi}(Dv_p) \cdot (D\varphi - Dv_p) - \int_{\Omega} (dh_p - f^+)(\varphi - v_p) \\ & \leq \frac{1}{p} \int_{\Omega} (F^*(D\varphi))^p - \int_{\Omega} (dh_p - f^+)\varphi \end{aligned}$$

for every $\varphi \in W^{1,p}(\Omega)$. Then, having in mind (19), we have

$$\frac{1}{p} \int_{\Omega} (F^*(Dv_p))^p - \int_{\Omega} (dh_p - f^+)v_p \leq \frac{1}{p} \int_{\Omega} (F^*(D\varphi))^p - \int_{\Omega} (dh_p - f^+)\varphi,$$

for every $\varphi \in W^{1,p}(\Omega)$. Therefore, for any $v \in W^{1,\infty}(\Omega)$, $F^*(x, Dv(x)) \leq 1$ a.e.,

$$- \int_{\Omega} (dh_p - f^+)v_p \leq \frac{1}{p} |\Omega| - \int_{\Omega} (f^+ - dh_p)v,$$

and, taking limits in the last inequality, we get

$$\int_{\Omega} (dh_\infty - f^+)v \leq \int_{\Omega} (dh_\infty - f^+)v_\infty,$$

from where it follows that

$$\int_{\Omega} (dh_{\infty} - f^+) v_{\infty} = \sup_{\substack{v \in W^{1,\infty}(\Omega) \\ F^*(x, Dv(x)) \leq 1 \text{ a.e.}}} \int_{\Omega} v (dh_{\infty} - f^+),$$

and consequently, v_{∞} is a Kantorovich potential for the transport of f^+ to h_{∞} , with respect to c_F . The proof for w_{∞} is similar. \square

Theorem 3.9. *The measure h_{∞} is a matching measure to the optimal matching problem.*

Proof. From Theorem 3.8 and Kantorovich-Rubinstein's Theorem, we have

$$\int_{\Omega} v_{\infty} (dh_{\infty} - f^+) = \min_{\mu \in \Pi(f^+, h_{\infty})} \int c_F(x, y) d\mu = \int c_F(x, y) d\mu_0,$$

$\mu_0 \in \Pi(f^+, h_{\infty})$ and

$$\int_{\Omega} w_{\infty} (dh_{\infty} - f^-) = \min_{\nu \in \Pi(f^-, h_{\infty})} \int c_G(x, y) d\nu = \int c_G(x, y) d\nu_0,$$

$\nu_0 \in \Pi(f^-, h_{\infty})$. Then, by adding the above inequalities and since $\int_{\Omega} (w_{\infty} + v_{\infty}) h_{\infty} = 0$, by Proposition 3.7, we get

$$\begin{aligned} & \int_{\Omega} -v_{\infty} f^+ - w_{\infty} f^- = \int c_F(x, y) d\mu_0 + \int c_G(x, y) d\nu_0 \\ & \geq \inf_{(\gamma_+, \gamma_-) \in \Pi_D(f^+, f^-)} \left\{ \int_{\Omega \times \Omega} c_F(x, y) d\gamma_+ + \int_{\Omega \times \Omega} c_G(x, y) d\gamma_- \right\} \\ & = \sup_{\substack{v, w \in W^{1,\infty}(\Omega) \\ F^*(x, Dv(x)), G^*(x, Dw(x)) \leq 1 \text{ a.e.} \\ v + w \geq 0 \text{ in } D}} \int_{\Omega} -v f^+ - w f^-. \end{aligned} \quad (20)$$

Now, by Theorem 3.4, all the inequalities in (20) are equalities. This ends the proof. \square

Theorem 1.2 is now proved. Let us see that we can also describe the transport densities of the transport problems involved. Remember we are assuming that F^* and G^* are continuous and that $F^*(x, \cdot), G^*(x, \cdot) \in C^1(\mathbb{R}^N \setminus \{0\})$.

Theorem 3.10. *Consider v_{∞}, w_{∞} the Kantorovich potentials obtained in Theorem 3.8 and h_{∞} the matching measure of Theorem 3.9. Then, there exists two nonnegative L^1 functions a and b such that*

1. *for almost every x , $a(x) > 0$ implies $F^*(x, Dv_{\infty}(x)) = 1$, and*

$$\int_{\Omega} a(x) \left\langle \frac{\partial F^*}{\partial \xi}(x, Dv_{\infty}(x)); Dv(x) \right\rangle dx = \int_{\Omega} (h_{\infty}(x) - f^+(x)) v(x) dx$$

for all $v \in W^{1,\infty}(\Omega)$; hence

$$-\operatorname{div} \left(a \frac{\partial F^*}{\partial \xi}(\cdot, Dv_{\infty}) \right) = (h_{\infty} - f^+) \quad \text{in the sense of distributions,}$$

and, in addition,

$$\int_{\Omega} a(x) dx = \int_{\Omega} v_{\infty}(x) (h_{\infty}(x) - f^+(x)) dx; \quad (21)$$

2. for almost every x , $b(x) > 0$ implies $G^*(x, Dw_\infty(x)) = 1$, and

$$\int_{\Omega} b(x) \left\langle \frac{\partial G^*}{\partial \xi}(x, Dw_\infty(x)); Dv(x) \right\rangle dx = \int_{\Omega} (h_\infty(x) - f^-(x))v(x) dx$$

for all $v \in W^{1,\infty}(\Omega)$; hence

$$-\operatorname{div} \left(b \frac{\partial G^*}{\partial \xi}(\cdot, Dw_\infty) \right) = (h_\infty - f^-) \quad \text{in the sense of distributions,}$$

and, in addition,

$$\int_{\Omega} b(x) dx = \int_{\Omega} w_\infty(x)(h_\infty(x) - f^-(x)) dx. \quad (22)$$

3.

$$\int_{\Omega} a(x) dx + \int_{\Omega} b(x) dx = - \int_{\Omega} (v_\infty(x)f^+(x) + w_\infty(x)f^-(x)) dx$$

gives the total transport cost.

Proof. We will write $F^*(Dv_p)$ in the proof for shortness, and the same for G^* . Since v_p is a weak solution of problem (15), if we define

$$\mathcal{X}_p := [F^*(Dv_p(x))]^{p-1} \frac{\partial F^*}{\partial \xi}(Dv_p(x)),$$

then

$$\int_{\Omega} \langle \mathcal{X}_p; Dv \rangle = \int_{\Omega} (h_p - f^+)v \quad \forall v \in W^{1,p}(\Omega). \quad (23)$$

Let us see that

$$\{\mathcal{X}_p : p \geq N\} \quad \text{is weakly relatively compact in } L^1(\Omega, \mathbb{R}^N). \quad (24)$$

In fact, first, taking v_p as test function in (23) and having in mind (3), we have

$$\int_{\Omega} [F^*(Dv_p(x))]^p dx \leq C_1, \quad \forall p > N. \quad (25)$$

Then, by Hölder's inequality, we have

$$\int_{\Omega} [F^*(Dv_p(x))]^{p-1} dx \leq C_2, \quad \forall p > N. \quad (26)$$

On the other hand, given $\varphi \in L^\infty(\Omega, \mathbb{R}^N)$, from (4), (7) and (26), we have

$$\begin{aligned} \left| \int_{\Omega} \langle \mathcal{X}_p; \varphi \rangle \right| &\leq \int_{\Omega} [F^*(Dv_p(x))]^{p-1} \left| \left\langle \frac{\partial F^*}{\partial \xi}(Dv_p(x)); \varphi(x) \right\rangle \right| dx \\ &\leq \int_{\Omega} [F^*(Dv_p(x))]^{p-1} F^*(\varphi(x)) dx \\ &\leq \|F^*(\varphi)\|_\infty \int_{\Omega} [F^*(Dv_p(x))]^{p-1} dx \\ &\leq C_2 M \|\varphi\|_\infty, \end{aligned}$$

from where it follows that $\{\mathcal{X}_p : p \geq N\}$ is bounded in $L^1(\Omega, \mathbb{R}^N)$. Moreover, by (4),

$$\left| \frac{\partial F^*}{\partial \xi}(Dv_p(x)) \right| \leq \sup_{|\eta| \leq 1} F^*(\eta) \leq \beta \quad \text{uniformly in } \Omega.$$

Hence, for any measurable subset $E \subset \Omega$, applying Hölder's inequality and having in mind (25), we have

$$\int_E |\mathcal{X}_p| \leq \beta \int_E [F^*(Dv_p(x))]^{p-1} dx \leq \beta \left(\int_\Omega [F^*(Dv_p(x))]^p dx \right)^{\frac{p-1}{p}} |E|^{\frac{1}{p}}.$$

Therefore, $\{\mathcal{X}_p : p \geq N\}$, being bounded and equi-integrable in $L^1(\Omega, \mathbb{R}^N)$ is weakly relatively compact in $L^1(\Omega, \mathbb{R}^N)$, and (24) holds. Consequently, we may assume there exists a sequence v_{p_i} with $p_i \rightarrow +\infty$, converging uniformly to u_∞ , such that

$$\mathcal{X}_{p_i} \rightarrow \mathcal{X}_\infty \quad \text{weakly in } L^1(\Omega, \mathbb{R}^N).$$

Thus, for any $v \in W^{1,\infty}(\Omega)$, having in mind (23), we get

$$\int_\Omega (h_p - f^+)v = - \int_\Omega \operatorname{div}(\mathcal{X}_{p_i})v = \int_\Omega \langle \mathcal{X}_{p_i}; Dv \rangle \rightarrow \int_\Omega \langle \mathcal{X}_\infty; Dv \rangle.$$

Hence,

$$\int_\Omega (h_p - f^+)v = \int_\Omega \langle \mathcal{X}_\infty; Dv \rangle \quad \forall v \in W^{1,\infty}(\Omega). \quad (27)$$

In addition by (23) and (27),

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_\Omega \langle \mathcal{X}_{p_i}; Dv_{p_i} \rangle &= \lim_{i \rightarrow \infty} \int_\Omega (h_p - f^+)v_{p_i} \\ &= \int_\Omega (h_p - f^+)v_\infty \\ &= \int_\Omega \langle \mathcal{X}_\infty; Du_\infty \rangle. \end{aligned} \quad (28)$$

Applying Hölder's inequality, (6), (3) and (28), we get

$$\begin{aligned} \int_\Omega F(\mathcal{X}_\infty(x))dx &\leq \liminf_{i \rightarrow \infty} \int_\Omega F(\mathcal{X}_{p_i}(x))dx \\ &= \liminf_{i \rightarrow \infty} \int_\Omega [F^*(Dv_{p_i}(x))]^{p_i-1} F\left(\frac{\partial F^*}{\partial \xi}(Dv_{p_i}(x))\right) dx \\ &\leq \liminf_{i \rightarrow \infty} \left(\int_\Omega [F^*(Dv_{p_i}(x))]^{p_i} dx \right)^{\frac{p_i-1}{p_i}} \\ &= \liminf_{i \rightarrow \infty} \left(\int_\Omega [F^*(Dv_{p_i}(x))]^{p_i-1} \left\langle \frac{\partial F^*}{\partial \xi}(Dv_{p_i}(x)); Dv_{p_i}(x) \right\rangle dx \right)^{\frac{p_i-1}{p_i}} \\ &= \liminf_{i \rightarrow \infty} \left(\int_\Omega \langle \mathcal{X}_{p_i}; Dv_{p_i} \rangle \right)^{\frac{p_i-1}{p_i}} = \int_\Omega \langle \mathcal{X}_\infty; Dv_\infty \rangle. \end{aligned}$$

Now, since $u_\infty \in K_F^*(\Omega)$,

$$\|F^*(Dv_\infty)(x)\|_{L^\infty(\Omega)} \leq 1.$$

Moreover, by (1),

$$\langle \mathcal{X}_\infty; Dv_\infty \rangle \leq F^*(Dv_\infty(x))F(\mathcal{X}_\infty(x)).$$

Then, we have

$$F(\mathcal{X}_\infty(x)) = \langle \mathcal{X}_\infty(x); Dv_\infty(x) \rangle \quad (29)$$

and

$$F^*(Dv_\infty(x)) = 1 \quad \text{a.e. on } \{F(\mathcal{X}_\infty(x)) > 0\}. \quad (30)$$

Now, by (5), for a.e. $x \in \Omega$ there exists $a(x)$ such that

$$\mathcal{X}_\infty(x) = a(x) \frac{\partial F^*}{\partial \xi}(Dv_\infty(x)). \quad (31)$$

Moreover, from (29), (30), (31) and (3), we have

$$0 \leq F(\mathcal{X}_\infty(x)) = a(x) \left\langle \frac{\partial F^*}{\partial \xi}(Dv_\infty(x)); Dv_\infty \right\rangle = aF^*(Dv_\infty(x)) = a \quad a.e.,$$

hence $a = F(\mathcal{X}_\infty(x))$ is a nonnegative L^1 function, and by (30), we get that

$$\text{for almost every } x, a(x) > 0 \text{ implies } F^*(Dv_\infty(x)) = 1. \quad (32)$$

Now, by (27) we have

$$\int_\Omega a(x) \left\langle \frac{\partial F^*}{\partial \xi}(Dv_\infty(x)); Dv(x) \right\rangle dx = \int_\Omega (h_\infty(x) - f^+(x))v(x) dx \quad (33)$$

for all $v \in W^{1,\infty}(\Omega)$, and in particular

$$-\text{div} \left(a \frac{\partial F^*}{\partial \xi}(Dv_\infty) \right) = h_\infty - f^+$$

in the sense of distributions.

Finally, by (33) and (32) and using (3), we have

$$\begin{aligned} \int_\Omega u_\infty(x)(h_\infty(x) - f^+(x))dx &= \int_\Omega a(x) \left\langle \frac{\partial F^*}{\partial \xi}(Dv_\infty(x)); Du_\infty(x) \right\rangle dx \\ &= \int_\Omega a(x)F^*(Dv_\infty(x))dx = \int_\Omega a(x)dx. \end{aligned}$$

We can do the same for w_∞ , and then we have shown that this procedure gives also transport densities for the optimal transport problems involved between f^- and h_∞ . Finally, (21)+(22) gives (3). \square

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