# ON THE FUČIK SPECTRUM AND A RESONANCE PROBLEM FOR THE $p$-LAPLACIAN WITH A NONLINEAR BOUNDARY CONDITION

SANDRA R MARTINEZ AND JULIO D. ROSSI

Abstract. In this paper we prove that there exists a first curve of the Fučik spectrum of the problem $\Delta_p u = |u|^{p-2}u$ in $\Omega$ with a nonlinear boundary condition given by $|\nabla u|^{p-2}\partial u/\partial\nu = \alpha(u^+)^{p-1} - \beta(u^-)^{p-1}$ on the boundary of the domain. We also prove that there exists a sequence of curves of the Fučik spectrum which exist locally in the neighborhood of suitable eigenvalues of the $p$-laplacian with a nonlinear boundary condition. Finally, we study a resonance problem with respect to the Fučik spectrum.

## 1. Introduction.

This paper is devoted to find existence results for elliptic problems involving the $p-$laplacian with a nonlinear flux boundary condition. More precisely, we look for solutions of

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$$\begin{cases} \Delta_p u = A(x,u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = B(x,u) & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $1 < p < \infty$, $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-laplacian and $\frac{\partial}{\partial\nu}$ is the outer normal derivative. We assume that $A(x,u)$ and $B(x,u)$ behave like

$$A(x,u) = |u|^{p-2}u - f(x,u),$$

$$B(x,u) = \alpha(u^+)^{p-1} - \beta(u^-)^{p-1} + h(x,u),$$

where $f$ and $h$ are lower order terms. Hence we are assuming that we deal with a resonance problem with an asymmetric nonlinearity at the flux boundary condition.

To study this problem, as happens for the usual Dirichlet boundary conditions, we first have to make an analysis of the associated spectrum. The Fučik spectrum for the $p$-laplacian with a nonlinear boundary condition is defined as the set $\widetilde{\Sigma}_p$ of

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$(\alpha,\beta)\in\mathbb{R}^2$ such that

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$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega,\\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \alpha(u^+)^{p-1} - \beta(u^-)^{p-1} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

has a nontrivial solution.

If $\alpha=\beta=\lambda$ then we arrive to the following eigenvalue problem,

1.2
$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega,\\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda|u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

We say that $\lambda$ is an eigenvalue if (1.3) has non trivial solutions. The set of solutions for a given $\lambda$ (called eigenfunctions) will be denoted by $A_\lambda$. Let us briefly recall some properties of this spectrum. Problems of the form (1.3) appears in a natural way when one considers the Sobolev trace inequality. In fact, the immersion $W^{1,p}(\Omega)\hookrightarrow L^p(\partial\Omega)$ is compact, hence there exits a best constant $\lambda_1$ such that

$$\lambda_1^{1/p}\|u\|_{L^p(\partial\Omega)}\le \|u\|_{W^{1,p}(\Omega)}.$$

This Sobolev trace constant $\lambda_1$ can be characterized as

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$$\lambda_1 = \inf_{u\in W^{1,p}(\Omega)}\left\{\int_\Omega |\nabla u|^p + |u|^p\,dx, \qquad \int_{\partial\Omega}|u|^p = 1\right\}, \tag{1.4}$$

and is the first eigenvalue of (1.3) in the sense that $\lambda_1\le\lambda$ for any other eigenvalue $\lambda$. The extremals (functions where the constant is attained) are solutions of (1.2). The first eigenvalue is simple and isolated, and it has a first eigenfunction that is strictly positive in the whole $\overline{\Omega}$ (see [21]). In [15] it is proved that there exists a sequence of eigenvalues $\lambda_n$ of (1.3) such that $\lambda_n\to+\infty$ as $n\to+\infty$. In [22] is defined a new variational sequence of eigenvalues. As happens for the Dirichlet problem, it is not known if this sequence characterize completely the whole spectrum.

Let us go back to equation (1.2). We observe that if $\lambda$ is an eigenvalue for (1.3) then the point $(\lambda,\lambda)$ belongs to $\widetilde{\Sigma}_p$. Moreover, the lines $\mathbb{R}\times\{\lambda_1\}$ and $\{\lambda_1\}\times\mathbb{R}$ belong to $\widetilde{\Sigma}_p$, this follows from the fact that the first eigenfunction is positive.

**Obtained results.**

The main results in this paper can be summarized as follows: first we construct a first nontrivial curve in $\widetilde{\Sigma}_p$, then we find a sequence of curves locally around a suitable sequence of eigenvalues $(\lambda_k,\lambda_k)$ and finally we study problem (1.1) at resonance obtaining an existence results under adequate hypotheses. To obtain the mentioned results the main idea is to use variational techniques supported on the compactness given by the Sobolev trace immersion $W^{1,p}(\Omega)\hookrightarrow L^p(\partial\Omega)$.

**Organization of the paper.**

The paper is divided in six sections. In Section 2 we will use similar ideas to those used in [10] to construct a first nontrivial curve in $\widetilde{\Sigma}_p$, described as $(s+c(s),c(s))$.

In Section 3 we prove that the lines $\mathbb{R}\times\{\lambda_1\}$ and $\{\lambda_1\}\times\mathbb{R}$ are isolated in $\widetilde{\Sigma}_p$ and that the curve that we have constructed in Section 2 is the first nontrivial curve. This implies in particular that this curve passes through $(\lambda_2,\lambda_2)$ which

yields a variational characterization of the second eigenvalue $\lambda_2$ by a mountain-pass procedure.

In Section 4 we prove the monotonicity of the first curve and we also study its asymptotic behavior. We find that in the case $p \leq N$ its asymptotic behavior it is similar to the Dirichlet problem, $c(s) \to \lambda_1$ as $s \to \infty$, and in the case $p > N$ it is similar to the homogeneous Newmann problem, $c(s) \to \overline{\lambda} > \lambda_1$ as $s \to \infty$ (see [4]).

In Section 5 we study the Fučik spectrum locally in a neighborhood of a suitable sequence of eigenvalues. Following ideas from [25], for the case $p \geq 2$ we prove that there exist two curves emanating from these eigenvalues.

Finally, in Section 6 we study the resonance problem respect to the Fučik spectrum, problem (1.1) (here we will not need any restriction on $p$). We find that the problem has a solution if $(\alpha, \beta)$ belongs to some suitable defined curves in the Fučik spectrum and the lower order terms $f$ and $h$ verify some resonance conditions.

**Related work.**

The study of the eigenvalue problem when the nonlinear term is placed in the equation, that is when one consider a quasilinear problem of the form $-\Delta_p u = \lambda |u|^{p-2}u$ with Dirichlet boundary conditions, has received considerable attention, see for example [2], [3], etc.

The Fučik spectrum was introduced by Fučik in [17] for the Laplacian in one space dimension with periodic boundary conditions. Fučik showed that this spectrum consists of two families of curves emanating from the points $(\lambda_k, \lambda_k)$, where $\lambda_k$ are the eigenvalues of the problem. Many authors have studied the Fučik spectrum for the Dirichlet problem ($\Sigma_2$) in dimension greater than one, see for example, [4], [12], [24], [23] and [29]. In particular, in [1], [6], [18], [23], [20], [26], [30] and [31] the existence of one or two continuous, nondecreasing curves in $\Sigma_2$ crossing the point $(\lambda_k, \lambda_k)$ was proved. In [11] it is proved that the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ are isolated in $\Sigma_2$, and in [16] it is constructed a first curve emanating from the second eigenvalue.

The Fučik spectrum for the $p$-laplacian was studied by Drábeck in the case $p \neq 2$ and $N = 1$ in [14], where it is shown that $\Sigma_p$ have similar properties as in the linear case. The Fučik spectrum for the $p$-laplacian Dirichlet problem has been recently studied by Cuesta et al. in [10]. There it is characterized a first nontrivial curve emanating from $(\lambda_2, \lambda_2)$ and it is proved that the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ are isolated in the Fučik spectrum ($\Sigma_p$). They also studied the asymptotic behavior of the first nontrivial curve. Micheletti and Pistoia proved in [25] for $p \geq 2$ that there exists a sequence of curves in $\Sigma_p$ which exists locally in a neighborhood of a suitable sequence of eigenvalues. Perera characterize in [27] a sequence of curves in $\Sigma_p$ in the general case $1 < p < \infty$. In [28] it is studied a resonance problem respect to the Fučik spectrum for the Dirichlet problem. In [5] it is studied the Fučik spectrum for the $p$-laplacian with homogeneous Newmann boundary conditions.

Finally, let us note that nonlinear boundary conditions have only been considered in recent years. For reference purposes, we cite previous works. For the Laplace operator with nonlinear boundary conditions see for example [8], [9], [13]. For previous work for the $p-$Laplacian with nonlinear boundary conditions of different type see [15] and [21].

## 2. A Nontrivial Curve

In this section we construct the first nontrivial curve of the Fučik spectrum (1.2) and prove some properties of the curve following ideas from [10].

Let us fix $s \in \mathbb{R}^N$ and consider the functional

$$J_s(u) = \int_\Omega |\nabla u|^p + \int_\Omega |u|^p - s\int_{\partial\Omega} (u^+)^p$$

restricted to

$$S = \left\{ u \in W^{1,p}(\Omega) : T(u) = \int_{\partial\Omega} |u|^p = 1 \right\}.$$

We note $\widetilde{J}_s(u) = J_s|_S(u)$. If $u \in S$ is a critical point of $\widetilde{J}_s$ then there exists a Lagrange multiplier $t \in \mathbb{R}$ such that

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$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v + \int_\Omega |u|^{p-2}uv - s\int_{\partial\Omega} (u^+)^{p-1}v = t\int_{\partial\Omega} |u|^{p-2}uv, \tag{2.1}$$

for all $v \in W^{1,p}(\Omega)$. Hence $u$ is a nontrivial weak solution of

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = (s+t)(u^+)^{p-1} - t(u^-)^{p-1} & \text{on } \partial\Omega. \end{cases}$$

That means that $(t+s,t) \in \widetilde{\Sigma}_p$, taking $v = u$ in (2.1) we also have that the Lagrange multiplier $t$ is equal to $\widetilde{J}_s(u)$, so we have the following result,

**Lemma 2.1.** *The points in $\widetilde{\Sigma}_p$ of the form $(t+s,t)$ are exactly those points such that $t = \widetilde{J}_s(u)$ with $u$ a critical point of $\widetilde{J}_s$.*

Let us consider $\varphi_1 > 0$ the first eigenfunction for the problem (1.3) normalized with $\|\varphi_1\|^p_{L^p(\partial\Omega)} = 1$. Using that $\lambda_1$ is a minimizer for the trace immersion with best constant $\lambda_1$ we have that

$$\widetilde{J}_s(\varphi_1) = \lambda_1 - s, \ \widetilde{J}_s(-\varphi_1) = \lambda_1, \text{ and } \widetilde{J}_s(u) \geq \lambda_1 - s, \qquad \forall\ u \in S.$$

Therefore we get the following,

**Proposition 2.1.** *The first eigenfunction $\varphi_1$ normalized as $\|\varphi_1\|_{L^p(\partial\Omega)} = 1$ is a global minimum for $\widetilde{J}_s$ with $\widetilde{J}_s(\varphi_1) = \lambda_1 - s$. The corresponding point in $\widetilde{\Sigma}_p$ is $(\lambda_1, \lambda_1 - s)$.*

We have a second critical point at $-\varphi_1$,

**Proposition 2.2.** *The negative eigenfunction $-\varphi_1$ is a strict local minimum of $\widetilde{J}_s$ and $\widetilde{J}_s(-\varphi_1) = \lambda_1$. The corresponding point in $\widetilde{\Sigma}_p$ is $(\lambda_1 + s, \lambda_1)$.*

*Proof.* Let us suppose by contradiction that there exists a sequence $u_n \in S$, $u_n \neq -\varphi_1$ with $\widetilde{J}_s(u_n) \leq \lambda_1$, $u_n \to -\varphi_1$. Let us see that $u_n$ must change sign. If $u_n \leq 0$ on the boundary then,

$$\widetilde{J}_s(u_n) = \int_\Omega |u_n|^p + \int_\Omega |\nabla u|^p - s\int_{\partial\Omega} (u_n^+)^p = \int_{\partial\Omega} |u_n|^p + \int_\Omega |\nabla u_n|^p > \lambda_1,$$

a contradiction.

As $u_n \to -\varphi_1$ then $u_n < 0$ at some point on the boundary. Let us take

$$r_n := \frac{\displaystyle\int_\Omega |u_n^+|^p + \int_\Omega |\nabla u_n^+|^p}{\displaystyle\int_{\partial\Omega} (u_n^+)^p}$$

and

$$w_n := \frac{u_n^+}{\|u_n^+\|_{L^p(\partial\Omega)}},$$

then

$$r_n = \int_\Omega |w_n|^p + \int_\Omega |\nabla w_n|^p.$$

We are going to prove that $r_n \to \infty$. Let us suppose by contradiction that $r_n$ is bounded, then using the fact that the immersion $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ is compact we have that there exists a subsequence and $w \in W^{1,p}(\Omega)$ such that $w_n \to w$ in $L^p(\partial\Omega)$. Using that $\|w_n\|_{L^p(\partial\Omega)} = 1$ and that $w_n \geq 0$ we also have $\|w\|_{L^p(\partial\Omega)} = 1$ and $w \geq 0$ a.e. As $u_n \to -\varphi_1$ then $|u_n > 0|_{\partial\Omega} \to 0$ and therefore $|w_n > 0|_{\partial\Omega} \to 0$, but this contradicts the fact that $\|w\|_{L^p(\partial\Omega)} = 1$. Here and in what follows we use the notation $|A|_{\partial\Omega}$ to denote the $N-1$ dimensional measure of a subset of $\partial\Omega$.

On the other hand, we have,

$$\begin{aligned}
\widetilde{J}_s(u_n) &= \int_\Omega |u_n|^p + \int_\Omega |\nabla u_n|^p - s\int_{\partial\Omega} (u_n^+)^p \\
&= \int_\Omega |u_n^+|^p + \int_\Omega |\nabla u_n^+|^p + \int_\Omega |u_n^-|^p + \int_\Omega |\nabla u_n^-|^p - s\int_{\partial\Omega} (u_n^+)^p \\
&= (r_n - s)\int_{\partial\Omega} (u_n^+)^p + \int_\Omega |\nabla u_n^-|^p + \int_\Omega (u_n^-)^p \\
&\geq (r_n - s)\int_{\partial\Omega} (u_n^+)^p + \lambda_1 \int_{\partial\Omega} (u_n^-)^p.
\end{aligned}$$

As $u_n \in S$, we get

$$\widetilde{J}_s(u_n) \leq \lambda_1 = \lambda_1 \int_{\partial\Omega} (u_n^+)^p + \lambda_1 \int_{\partial\Omega} (u_n^-)^p.$$

Combining both inequalities we have,

$$(r_n - s)\int_{\partial\Omega} (u_n^+)^p + \lambda_1 \int_{\partial\Omega} (u_n^-)^p \leq \lambda_1 \int_{\partial\Omega} (u_n^+)^p + \lambda_1 \int_{\partial\Omega} (u_n^-)^p,$$

hence

$$(\lambda_1 - r_n + s)\int_{\partial\Omega} (u_n^+)^p \geq 0,$$

and this implies

$$r_n - s \leq \lambda_1,$$

which contradicts the fact that $r_n \to +\infty$. The result follows. □

To construct a first curve in the Fučik spectrum we will use a version of the mountain pass theorem on $C^1$ manifolds (see [10]). We define a norm of the derivative of the restriction $\widetilde{J}_s$ of $J_s$ at $u \in S$ as follows,

$$\|\widetilde{J}_s'(u_n)\|_* = \min\{t \in \mathbb{R} : \|J_s'(u) - tT'(u)\|_{W^{1,p}(\Omega)'}\}$$

**Definition 2.1.** *We say that $J_s$ satisfies the Palais-Smale condition on $S$ ((P.S) condition for short) if for any sequence $u_n \in S$ such that $J(u_n)$ is bounded and $\|\widetilde{J}_s'(u_n)\|_* \to 0$ there exists a subsequence that converges strongly in $W^{1,p}(\Omega)$.*

We will need some preliminary lemmas (the proofs are straightforward, see [15]). We denote by $W^{1,p}(\Omega)^*$ the dual space of $W^{1,p}(\Omega)$.

**Lemma 2.2.** *Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be given by*

$$A(u).v := \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega |u|^{p-2}uv,$$

*then $A$ is a continuous, odd, $(p-1)$- homogeneous and continuously invertible.*

**Lemma 2.3.** *Let $B : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be given by*

$$B(u).v := \int_{\partial\Omega} |u|^{p-2}uv.$$

*Then $B$ is a continuous, odd, $(p-1)$- homogeneous and compact.*

**Lemma 2.4.** *Let $B^+ : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be given by*

$$B^+(u).v := \int_{\partial\Omega} (u^+)^{p-1}v.$$

*Then $B^+$ is continuous and compact.*

*An analogous result holds for $B^-$ the operator obtained taking the negative part $u^-$ instead of $u^+$.*

Now we formulate here the version of the mountain pass theorem that will be used later, the proof follows from Theorem 3.2 in [19].

coma **Proposition 2.3.** *Let $E$ be a Banach space, $g, f \in C^1(E,\mathbb{R})$, $M = \{u \in E : g(u) = 0\}$, $u_0$, $u_1 \in M$, let $\varepsilon > 0$ such that $\|u_1 - u_0\| > \varepsilon$ and*

$$inf\{f(u) : u \in M \text{ and } \|u - u_0\|_E = \varepsilon\} > max\{f(u_0), f(u_1)\}.$$

*Assume that $f$ satisfies the (P.S) condition on $M$ and that*

$$\Gamma = \{\gamma \in C([-1,1], M) : \gamma(-1) = u_0 \text{ and } \gamma(1) = u_1\}$$

*is nonempty. Then*

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,1]} f(u)$$

*is a critical value of $f|_M$.*

To verify the involved hypotheses, we first need to prove the following.

ps **Lemma 2.5.** *$J_s$ satisfies the Palais-Smale condition on $S$.*

*Proof.* We want to prove that every sequence $u_n$ that verifies $\|u_n\|_{L^p(\partial\Omega)} = 1$, $\|\widetilde{J}'_s(u_n)\|_* \to 0$ and $\|\widetilde{J}_s(u_n)\|_{W^{1,p}(\Omega)} \leq k$ has a subsequence that converges strongly in $W^{1,p}(\Omega)$.

As $\|\widetilde{J}_s(u_n)\|_{W^{1,p}(\Omega)} \leq k$ and $\|u_n\|_{L^p(\partial\Omega)} = 1$, we have that $u_n$ is bounded in $W^{1,p}(\Omega)$, then we can assume that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, using the compactness of the immersion $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ we have that $u_n \to u$ in $L^p(\partial\Omega)$.

We also have that

$$J'_s(u)(v) = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega |u|^{p-2}uv - s\int_{\partial\Omega}(u^+)^{p-1}v$$

and

$$T'(u)(v) = \int_{\partial\Omega}|u|^{p-2}uv$$

then

$$\left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla v + \int_\Omega |u_n|^{p-2}u_n v - s\int_{\partial\Omega}(u_n^+)^{p-1}v - t_n\int_{\partial\Omega}|u_n|^{p-2}u_n v\right|$$

$$\leq \varepsilon_n\|v\|_{W^{1,p}(\Omega)}$$

where $\varepsilon_n \to 0$. Replacing in the last inequality by $v = u_n$ we have

$$t_n \leq \int_\Omega |\nabla u_n|^p + \int_\Omega |u_n|^p - s\int_{\partial\Omega}(u_n^+)^p + \varepsilon_n\|u_n\|_{W^{1,p}(\Omega)} \leq C.$$

Hence $t_n$ is bounded so it has a subsequence that converges to some $t$. We have that,

$$A(u_n)(v) = sB^+(u_n)(v) + t_nB(u_n)(v) + O(\varepsilon_n)$$

using the compactness of $B^+$ and $B$ and the continuity of $A^{-1}$ we have that $u_n \to A^{-1}(sB^+(u) + tB(u))$ in $W^{1,p}(\Omega)$. The result follows. □

Finally we prove,

min **Lemma 2.6.** *Let $\varepsilon_0 > 0$ such that*

$$\widetilde{J}_s(u) > \widetilde{J}_s(-\varphi_1)$$

*for all $u \in B(-\varphi_1, \varepsilon_0) \cap S$ with $u \neq -\varphi_1$, where the ball $B(-\varphi_1, \varepsilon_0)$ is taken in $W^{1,p}(\Omega)$-norm. Then for any $0 < \varepsilon < \varepsilon_0$,*

$$inf\{\widetilde{J}_s(u) : u \in\ S \text{ and } \|u - (-\varphi_1)\|_{W^{1,p}(\Omega)} = \varepsilon\} > \widetilde{J}_s(-\varphi_1).$$

*Proof.* Assume by contradiction that

$$inf\{\widetilde{J}_s(u) : u \in\ S, \|u - (-\varphi_1)\|_{W^{1,p}(\Omega)} = \varepsilon\} = \lambda_1.$$

Then there exists $u_n \in\ S$ such that $\|u_n - (-\varphi_1)\|_{W^{1,p}(\Omega)} = \varepsilon$ and $\widetilde{J}_s(u_n) \leq \lambda_1 + \frac{1}{2n^2}$. Consider the set,

$$C = \{u \in S : \varepsilon - \delta \leq \|u - (-\varphi_1)\|_{W^{1,p}(\Omega)} \leq \varepsilon + \delta\},$$

with $\delta$ such that $\varepsilon - \delta > 0$ and $\varepsilon + \delta < \varepsilon_0$, by our assumption $inf\{\widetilde{J}_s(u) : u \in C\} = \lambda_1$. We are going to use the well known Ekeland's principle that we formulate below,

**Ekeland's principle** *Let $X$ be a complete metric space and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function which is bounded below. Let $\varepsilon > 0$ and $\overline{u} \in X$ be given such that*

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$$\phi(\overline{u}) \leq inf_X \phi + \frac{\varepsilon}{2}. \tag{2.2}$$

*Then given $\lambda > 0$ there exists $u_\lambda \in X$ such that*

1
$$\phi(u_\lambda) \leq \phi(\overline{u}), \tag{2.3}$$

2
$$d(u_\lambda, \overline{u}) \leq \lambda, \tag{2.4}$$

3
$$\phi(u_\lambda) < \phi(u) + \frac{\varepsilon}{\lambda} d(u, u_\lambda) \qquad \forall u \neq u_\lambda. \tag{2.5}$$

Taking $X = C$, $\varepsilon = \frac{1}{n^2}$, $\lambda = \frac{1}{n}$ and using that $\widetilde{J}_s(u_n) \leq \lambda_1 + \frac{1}{2n^2}$, by the Ekeland principle, we have that there exists $v_n$ such that

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$$\widetilde{J}_s(v_n) \leq \widetilde{J}_s(u_n), \tag{2.6}$$

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$$\|u_n - v_n\|_{W^{1,p}(\Omega)} \leq \frac{1}{n}, \tag{2.7}$$

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$$\widetilde{J}_s(v_n) \leq \widetilde{J}_s(u) + \frac{1}{n}\|u - v_n\|_{W^{1,p}(\Omega)}, \quad \forall\ u \in\ C. \tag{2.8}$$

If we prove that $v_n$ is a Palais-Smale sequence then there exists $v \in\ W^{1,p}(\Omega)$ such that $v_n \to v$ in $W^{1,p}(\Omega)$ with

$$\|v - (-\varphi_1)\|_{W^{1,p}(\Omega)} = \varepsilon, \qquad \widetilde{J}_s(v) = \lambda_1.$$

This contradicts the fact that $-\varphi_1$ is a strict local minimum of $\widetilde{J}_s$.

Hence let us prove that $v_s$ is a Palais-Smale sequence. By (2.6) we have that $\widetilde{J}_s(v_n)$ is bounded. It remains to show that $\|\widetilde{J}_s'(v_n)\|_* \to 0$. Let $n > \frac{1}{\delta}$ and $w \in W^{1,p}(\Omega)$ such that

$$\int_{\partial\Omega} |v_n|^{p-2} v_n w = 0$$

and

$$u_t := \frac{v_n + tw}{\|v_n + tw\|_{L^p(\partial\Omega)}}.$$

Then

$$\lim_{t\to 0} \|u_t - (-\varphi_1)\|_{W^{1,p}(\Omega)} = \|v_n - (-\varphi_1)\|_{W^{1,p}(\Omega)}$$

$$\leq \|v_n - u_n\|_{W^{1,p}(\Omega)} + ||u_n - (-\varphi_1)\|_{W^{1,p}(\Omega)} \leq \frac{1}{n} + \varepsilon < \delta + \varepsilon,$$

and

$$\lim_{t\to 0} \|u_t - (-\varphi_1)\|_{W^{1,p}(\Omega)} = \|v_n - (-\varphi_1)\|_{W^{1,p}(\Omega)}$$

$$\geq ||u_n - (-\varphi_1) - \|v_n - u_n\|_{W^{1,p}(\Omega)}\|_{W^{1,p}(\Omega)} \geq \varepsilon - \frac{1}{n} > \varepsilon - \delta.$$

Hence for $t$ small enough $u_t \in C$ and replacing in (2.8) by $u_t$ we have,

$$\widetilde{J}_s(v_n) \leq \widetilde{J}_s(u_t) + \frac{1}{n}\|u_t - v_n\|_{W^{1,p}(\Omega)}$$

$$= \widetilde{J}_s\left(\frac{v_n + tw}{r(t)}\right) + \frac{1}{n}\left\|\frac{v_n + tw}{r(t)} - v_n\right\|_{W^{1,p}(\Omega)}$$

$$= \frac{\widetilde{J}_s(v_n + tw)}{r(t)^p} + \frac{1}{nr(t)}\|v_n(1 - r(t)) + tw\|_{W^{1,p}(\Omega)},$$

where $r(t) = \|v_n + tw\|_{L^p(\partial\Omega)}$.

Then

$$\frac{\widetilde{J}_s(v_n) - \widetilde{J}_s(v_n + tw)}{t} \leq \frac{1}{nr(t)t}\|v_n(1 - r(t)) + tw\|_{W^{1,p}(\Omega)}$$

$$+\frac{1}{t}\widetilde{J}_s(v_n + tw)\left(\frac{1}{r(t)^p} - 1\right).$$

As

$$\frac{d}{dt}r(t)^p|_{\{t=0\}} = p\int_{\partial\Omega}|v_n|^{p-2}v_n w = 0,$$

we obtain

$$\frac{r(t)^p - 1}{t} \to 0 \text{ as } t \to 0, \qquad 0 = \frac{d}{dt}r^p|_{t=0} = pr(0)^{p-1}\frac{d}{dt}r(0),$$

and then

$$\frac{r(t) - 1}{t} \to 0 \text{ as } t \to 0.$$

So we have,

$$\frac{\widetilde{J}_s(v_n) - \widetilde{J}_s(v_n + tw)}{t} \to 0 \text{ as } t \to 0,$$

that means

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$$|\langle \widetilde{J}'_s(v_n), w\rangle| \leq \frac{1}{n}\|w\|_{W^{1,p}(\Omega)}. \tag{2.9}$$

Let $z \in W^{1,p}(\Omega)$. Taking $\alpha_n = \int_{\partial\Omega}|v_n|^{p-2}v_n z$ and $w = z - \alpha_n v_n$ we have $\int_{\partial\Omega}|v_n|^{p-2}v_n w = 0$ and hence by (2.9)

$$\left|\langle \widetilde{J}'_s(v_n), z\rangle - \left(\int_{\partial\Omega}|v_n|^{p-2}v_n z\right)\langle \widetilde{J}'_s(v_n), v_n\rangle\right| \leq \frac{1}{n}\|z - \alpha_n v_n\|_{W^{1,p}(\Omega)},$$

Calling $t_n = \langle \widetilde{J}'_s(v_n), v_n\rangle$ by the last inequality we have,

$$\|\widetilde{J}'_s(v_n)\|_* \to 0,$$

as we wanted to prove. □

These lemmas lead us to prove the following Proposition.

**Proposition 2.4.** *Let $\varepsilon > 0$ such that $\|\varphi_1 - (-\varphi_1)\|_{W^{1,p}(\Omega)} > \varepsilon$ and*

$$inf\{\widetilde{J}_s(u) : u \in S \text{ and } \|u - (-\varphi_1)\|_{W^{1,p}(\Omega)} = \varepsilon\} > max\{\widetilde{J}_s(-\varphi_1), \widetilde{J}_s(\varphi_1)\},$$

*then*

$$\Gamma = \{\gamma \in C[-1,1], S) : \gamma(-1) = -\varphi_1 \text{ and } \gamma(1) = \varphi_1\}$$

*is nonempty and*

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$$c(s) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,1]} J_s(u) \tag{2.10}$$

*is a critical value of $\widetilde{J}_s$. Moreover $c(s) > \lambda_1$.*

*Proof.* The proposition follows from Proposition 2.3 and the previous results that guarantee the involved hypotheses. What we only have to prove is that $\Gamma$ is nonempty. To this end, just take $\varphi \in W^{1,p}(\Omega)$ with $\varphi \notin \mathbb{R}\varphi_1$, consider the path $t\varphi_1 + (1 - |t|)\varphi$ and normalize. Moreover the (P.S) condition and the geometric assumption are satisfied by the previous lemmas, then by Proposition 2.3 we have that $c(s)$ is a critical value of $\widetilde{J}_s$. Using the definition of $c(s)$ we have $c(s) > max\{\widetilde{J}_s(\varphi_1), \widetilde{J}_s(-\varphi_1)\} = \lambda_1$. □

We have proved the following

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**Theorem 2.1.** *For each $s \geq 0$, let $c(s) > \lambda_1$ be defined by the minimax formula (2.10), then the point $(s + c(s), c(s))$belongs to $\widetilde{\Sigma}_p$.*

It is a trivial fact that $\widetilde{\Sigma}_p$ is symmetric with respect to the diagonal. The whole curve, that we obtain using Theorem 2.1 and symmetrizing, is denoted by

$$\mathcal{C} := \{(s + c(s), c(s)), (c(s), s + c(s)) : s \geq 0\}.$$

## 3. The first nontrivial curve

We want to prove that the curve $\mathcal{C}$ constructed in the previous section is, in fact, the first nontrivial curve in the Fučik spectrum. In the first step of the proof of this fact we show that the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ are isolated in $\widetilde{\Sigma}_p$. In the second step we prove some topological properties of the functional $\widetilde{J}_s$, and we finally prove the above mentioned result.

**Proposition 3.1.** *There exists no sequence $(\alpha_n, \beta_n) \in \widetilde{\Sigma}_p$ with $\alpha_n > \lambda_1$ and $\beta_n > \lambda_1$ such that $(\alpha_n, \beta_n) \to (\alpha, \beta)$ with $\alpha = \lambda_1$ or $\beta = \lambda_1$.*

*Proof.* Suppose by contradiction that there exists a sequence $(\alpha_n, \beta_n)$ and $u_n$ such that $\alpha_n \to \lambda_1$ and

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$$\begin{cases} \Delta_p u_n = |u_n|^{p-2} u_n & \text{in } \Omega, \\ \\ |\nabla u_n|^{p-2} \dfrac{\partial u_n}{\partial \nu} = \alpha_n (u_n^+)^{p-1} - \beta_n (u_n^-)^{p-1} & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with $\|u_n\|_{L^p(\partial\Omega)} = 1$. We have

$$\int_\Omega |\nabla u_n|^p + \int_\Omega |u_n|^p = \alpha_n \int_{\partial\Omega} (u_n^+)^p - \beta_n \int_{\partial\Omega} (u_n^-)^p \leq \alpha_n \int_{\partial\Omega} (u_n^+)^p \leq \alpha_n,$$

then there exists $u \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\partial\Omega)$. Looking the weak form of (3.1) we have that, $\forall\ v \in\ W^{1,p}(\Omega)$

$$A(u_n)v = \alpha_n B^+(u_n)v - \beta_n B^-(u_n)v.$$

Hence, using the compactness of $B^+, B^-$ and the continuity of $A^{-1}$, we have,

$$u_n \to A^{-1}(\lambda_1 B^+(u) - \beta B^-(u))$$

strongly in $W^{1,p}(\Omega)$. Therefore the limit $u$ verifies

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$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega |u|^{p-2}uv = \lambda_1 \int_{\partial\Omega} (u^+)^{p-1}v - \beta \int_{\partial\Omega} (u^-)^{p-1}v, \tag{3.2}$$

that is, $u$ is a weak solution of

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$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda_1 (u^+)^{p-1} - \beta (u^-)^{p-1} & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Taking in (3.1) $v = u^+$ as test function we get,

$$\int_\Omega |\nabla u^+|^p + \int_\Omega |u^+|^p = \lambda_1 \int_{\partial\Omega} (u^+)^p,$$

so $u^+ = 0$ or $u^+ = \varphi_1$ (in other words $u \le 0$ or $u = \varphi_1$).

If $u^+ = 0$ by equation (3.3) we have,

$$\begin{cases} \Delta_p u^- = |u^-|^{p-2}u^- & \text{in } \Omega, \\ \\ |\nabla u^-|^{p-2}\dfrac{\partial u^-}{\partial \nu} = -\beta (u^-)^{p-1} & \text{on } \partial\Omega. \end{cases}$$

Then $u \le 0$ and is an eigenfunction. By [21] we have that the only eigenfunction that does not changes sing is the first one, so we must have $u = -\varphi_1$.

Hence $u_n \to (\varphi_1 \text{ or } -\varphi_1)$ in $L^p(\partial\Omega)$. Then

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$$|u_n < 0|_{\partial\Omega} \to 0 \quad \text{or} \quad |u_n > 0|_{\partial\Omega} \to 0. \tag{3.4}$$

On the other hand, taking in the weak form of equation (3.2) $v = u_n^+$ as test function, we obtain

$$\int_\Omega |\nabla u_n^+|^p + \int_\Omega |u_n^+|^p = \alpha_n \int_{\partial\Omega} (u_n^+)^p$$

$$= \alpha_n \int_{\partial\Omega \cap \{u_n > 0\}} (u_n^+)^p \le \alpha_n |\partial\Omega \cap \{u_n > 0\}|^{1-\frac{p}{q}} \|u_n^+\|^p_{L^q(\partial\Omega)}$$

$$\le C\alpha_n \int_{\partial\Omega \cap \{u_n > 0\}} (u_n^+)^p \le C\alpha_n |\partial\Omega \cap \{u_n > 0\}|^{1-\frac{p}{q}} \|u_n^+\|^p_{W^{1,p}(\Omega)}.$$

Where $p < q \le p^*$, and $C$ is the constant that appears in the trace immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$. Then we have,

$$|\partial\Omega \cap \{u_n > 0\}|^{1-\frac{p}{q}} \ge C^{-1}\alpha_n.$$

In an analogous way we obtain,

$$|\partial\Omega \cap \{u_n < 0\}|^{1-\frac{p}{q}} \ge C^{-1}\beta_n.$$

This contradicts (3.4) and ends the proof. □

local **Lemma 3.1.** *Let* $S = \left\{ u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u^p| = 1 \right\}$ *then,*

i. *$S$ is locally arcwise connected.*
ii. *Any open connected subset $C$ of $S$ is arcwise connected.*
iii. *If $C'$ is any connected component of an open set $C \subset S$ arcwise connected then $\partial C' \cap C = \emptyset$.*

*Proof.* i. The results follow using the fact that $S$ is a Banach manifold.

ii. Let us take

$$C_1 = \{u \in C : \text{ exists a continous path from } u_0 \text{ to } u \text{ in } C\}.$$

By i) $C_1$ is both open and close in $C$ then $C_1 = C$ and hence $C_1$ is arcwise connected.

iii. Let suppose that there exists $u_0 \in \partial C' \cap C$. As $u_0 \in C$ then there exists $\varepsilon > 0$ such that $B = B(u_0, \varepsilon)$ is arcwise connected, $B \subset C$ and $B \cap (C')^c \neq \emptyset$ then $C' \cup B$ is connected and this contradicts the fact that $C'$ is a connected component of $C$.

□

critico **Lemma 3.2.** *Let $C = \{u \in S : \widetilde{J}_s(u) < r\}$, then any connected component of $C$ has a critical point of $\widetilde{J}_s$.*

*Proof.* Let $C_1$ be any connected component of $C$, let $d = inf\{\widetilde{J}_s(u) : u \in \overline{C}_1\}$. Let us suppose that the infimum is archived at some $u_0$ , as $d \leq \widetilde{J}_s(u) < r$ for all $u \in C_1$, then $u_0 \in C$. By part iii) of Lemma 3.1 $u_0 \notin \partial C_1$ so $u_0 \in C_1$ and that means that $u_0$ is a critical point of $\widetilde{J}_s$. We only have to prove that the infimum $d$ is archived. Let us take $u_n$ such that $\widetilde{J}_s(u_n) \leq d + \frac{1}{2n^2}$, then applying Ekeland's principle to $\widetilde{J}_s$ in $\overline{C}_1$ with $\varepsilon = \frac{1}{n^2}$, $\lambda = \frac{1}{n}$ we have that there exists $v_n \in \overline{C}_1$ such that,

$$\widetilde{J}_s(v_n) \leq \widetilde{J}_s(u_n),$$

$$\|v_n - u_n\|_{W^{1,p}(\Omega)} \leq \tfrac{1}{n},$$

$$\widetilde{J}_s(v_n) \leq \widetilde{J}_s(v) + \tfrac{1}{n}\|v - v_n\|_{W^{1,p}(\Omega)}, \quad \forall\, v \in \overline{C}_1.$$

For $n$ large enough we have,

$$\widetilde{J}_s(v_n) \leq \widetilde{J}_s(u_n) \leq d + \frac{1}{2n^2} < r,$$

then $v_n \in C$. By the previous lemma we get that $v_n \notin \partial C_1$ so $v_n \in C_1$.

On the other hand, for $t$ small enough and $w$ such that $\int_{\partial\Omega} |v_n|^{p-2} v_n w = 0$, we have,

$$u_t := \frac{v_n + tw}{\|v_n + tw\|_{W^{1,p}(\Omega)}} \in \overline{C}_1.$$

Then

$$\widetilde{J}_s(v_n) \leq \widetilde{J}_s(u_t) + \frac{1}{n}\|u_t - v_n\|_{W^{1,p}(\Omega)}.$$

Following the same calculation as in Lemma 2.6 we have that $v_n$ is a (P.S) sequence, then $v_n \to v$ in $W^{1,p}(\Omega)$ and $u_n \to v$ in $W^{1,p}(\Omega)$. Hence $\widetilde{J}_s(v) = d$ and the result follows. □

Now we prove the main result of this section.

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**Theorem 3.1.** *Let $s \geq 0$ then the point $(s + c(s), c(s))$ is the first nontrivial point in the intersection between $\widetilde{\Sigma}_p$ and the line $(s, 0) + t(1, 1)$.*

*Proof.* Let us suppose by contradiction that there exists $\lambda_1 < \mu < c(s)$ such that $(s + \mu, \mu) \in \widetilde{\Sigma}_p$. Using the fact that the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ are isolated in $\widetilde{\Sigma}_p$ and that $\widetilde{\Sigma}_p$ is closed, we can chose $\mu$ the smaller with that property. If we find a continuous curve connecting $\varphi_1$ with $-\varphi_1$ such that $\widetilde{J}_s \leq \mu$ we contradict the definition of $c(s)$. Our purpose is to construct this curve.

Let $u \in S$ be a critical point of $\widetilde{J}_s$ at level $\mu$. Then $u$ satisfies,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega |u|^{p-2}uv = (s+\mu)\int_{\partial\Omega}(u^+)^{p-1}v - \mu\int_{\partial\Omega}(u^-)^{p-1}v,$$

for all $v \in W^{1,p}(\Omega)$. Replacing $v$ by $u^+$ we have,

$$\int_\Omega |\nabla u^+|^p + \int_\Omega |u^+|^p = (s+\mu)\int_{\partial\Omega}(u^+)^p,$$

and replacing $v$ by $u^-$,

$$\int_\Omega |\nabla u^-|^p + \int_\Omega |u^-|^p = \mu\int_{\partial\Omega}(u^-)^p.$$

Thus we obtain,

$$\widetilde{J}_s(u) = \widetilde{J}_s\left(\frac{u^+}{\|u^+\|_{L^p(\partial\Omega)}}\right) = \widetilde{J}_s\left(-\frac{u^-}{\|u^-\|_{L^p(\partial\Omega)}}\right) = \mu$$

and

$$\widetilde{J}_s\left(\frac{u^-}{\|u^-\|_{L^p(\partial\Omega)}}\right) = \mu - s.$$

Let

$$u_1(t) = \frac{tu + (1-t)u^+}{\|tu + (1-t)u^+\|_{L^p(\partial\Omega)}},$$

$$u_2(t) = \frac{tu^+ + (1-t)u^-}{\|tu^+ + (1-t)u^-\|_{L^p(\partial\Omega)}},$$

$$u_3(t) = \frac{-tu^- + (1-t)u}{\|-tu^- + (1-t)u\|_{L^p(\partial\Omega)}}.$$

We get $\widetilde{J}_s(u_1(t)) = \mu$, $\widetilde{J}_s(u_2(t)) \leq \mu$, $\widetilde{J}_s(u_3(t)) = \mu$. Let

$$C = \{v \in S : \widetilde{J}_s(v) < \mu - s\},$$

as we chose $\mu$ the minimum then the only critical points in $C$ are $\varphi_1$ and $-\varphi_1$.

Let us start to construct the path. First we go from $u$ until $\frac{u^+}{\|u^+\|_{L^p(\partial\Omega)}}$ by the path $u_1(t)$ and then from $\frac{u^+}{\|u^+\|_{L^p(\partial\Omega)}}$ until $\frac{u^-}{\|u^-\|_{L^p(\partial\Omega)}}$ by the path $u_2(t)$. As $\mu > \lambda_1$

then $\frac{u^-}{\|u^-\|_{L^p(\partial\Omega)}}$ is not a critical point of $\widetilde{J}_s$, thus we can move until a certain $v$ such that $\widetilde{J}_s(v) < \mu - s$ by a continuous path. Taking the connected component of $C$ containing $v$ and applying the previous lemma we have that either $\varphi_1$ or $-\varphi_1$ is in this component (suppose that is $\varphi_1$). Then we have constructed a path connecting $u$ with $\varphi_1$ which is at level less than $\mu$. If we call $u_4(t)$ at the part of the path that goes from $\frac{u^-}{\|u^-\|_{L^p(\partial\Omega)}}$ to $\varphi_1$, then the path $-u_4(t)$ connects $-\frac{u^-}{\|u^-\|_{L^p(\partial\Omega)}}$ with $-\varphi_1$ and

$$\widetilde{J}_s(-u_4(t)) \leq \widetilde{J}_s(u_4(t)) + s \leq \mu - s + s = \mu,$$

then this path it is also at level less than $\mu$. Taking all together we conclude that we have constructed a path $\gamma(t)$ going from $\varphi_1$ to $-\varphi_1$ such that $\widetilde{J}_s(\gamma(t)) \leq \mu$. The result follows. □

## 4. Properties of the first curve

In this section we will concentrate in the proof of some properties of the first curve. We will first prove the monotonicity of the curve constructed in Section 3, next we study its asymptotic behavior as $s$ goes to infinity. We will see that the asymptotic behavior of the first nontrivial curve in this problem is different from the Dirichlet problem. In the Dirichlet problem we have that for any $p$ $c(s) \to \lambda_1$ as $s \to \infty$. We will show in this section that for our problem we have that $c(s) \to \lambda_1$ for $1 < p \leq N$ and that $c(s) \to \overline{\lambda}$ with $\overline{\lambda} > \lambda_1$ in the case $p > N$, the difference between this two cases is due to the fact that when $p > N$ the embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact. This behavior of the curve is similar to the one that holds for the homogeneous Newmann problem (see [5]).

To prove the monotonicity of the curve we first need to prove a lemma.

**Lemma 4.1.** *Let $(\alpha, \beta) \in \mathcal{C}$, $\alpha(x), \beta(x) \in L^\infty(\partial\Omega)$ and $\lambda_1 \leq \alpha(x) \leq \alpha$, $\lambda_1 \leq \beta(x) \leq \beta$. Assume that $\lambda_1 < \alpha(x)$ and $\lambda_1 < \beta(x)$ in subsets of positive measure of $\partial\Omega$. Then any nontrivial solution of*

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$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \alpha(x)(u^+)^{p-1} - \beta(x)(u^-)^{p-1} & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

*changes sign on $\partial\Omega$ and*

$$\alpha(x) = \alpha \text{ a.e on } \{x \in \partial\Omega : u(x) > 0\}$$

*and*

$$\beta(x) = \beta \text{ a.e on } \{x \in \partial\Omega : u(x) < 0\}.$$

*Proof.* Let $u$ be a solution of (4.1) then $-u$ is solution of

$$\begin{cases} \Delta_p w = |w|^{p-2}w & \text{in } \Omega, \\ |\nabla w|^{p-2}\dfrac{\partial w}{\partial \nu} = \beta(x)(w^+)^{p-1} - \alpha(x)(w^-)^{p-1} & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

using that $\beta(x) \geq \beta$ and $\alpha(x) \leq \alpha$ we can suppose $\alpha \geq \beta$.

Let us assume, arguing by contradiction, that $u$ does not change sign on $\partial\Omega$ (for example, we may assume $u \geq 0$ a.e). Then $u$ is a solution of

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \alpha(x)u^{p-1} & \text{on } \partial\Omega. \end{cases}$$

That means that $u$ is an eigenfunction of the problem with weight $\alpha(x)$ corresponding to the eigenvalue equal to one. Using that the first eigenvalue of the problem with weight $\alpha(x) \geq \lambda_1 > 0$ can be characterized as

$$\lambda_1(\alpha(x)) = \inf_{u \in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^p + \int_\Omega |v|^p}{\displaystyle\int_{\partial\Omega} \alpha(x)|v|^p},$$

we have that if $u_1$ is the eigenfunction associated with the first eigenvalue $\lambda_1(\alpha(x))$ we can suppose $u_1 \geq 0$. Using the maximum principle and Hopf's lemma (see [32]), we have that $u_1 > 0$ in $\bar{\Omega}$ (for the details see [21]). Also, using similar arguments to those of [21], we can prove that $\lambda_1(\alpha(x))$ is simple. We also obtain that if $w$ is an eigenfunction corresponding to an eigenvalue $\lambda \neq \lambda_1$, $w$ changes sign on $\partial\Omega$ and moreover, we have that for some fixed constants $C$ and $\beta$

$$|\partial\Omega \cap \{w > 0\}| \geq C\lambda^{-\beta} \quad \text{and} \quad |\partial\Omega \cap \{w < 0\}| \geq C\lambda^{-\beta}.$$

Then we have that the only eigenfunction that does not change sign is the first one (for the details see [21]). Using that $u \geq 0$ and this last fact we get $\lambda_1(\alpha(x)) = 1$. So we have,

$$\inf_{u \in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^p + \int_\Omega |v|^p}{\displaystyle\int_{\partial\Omega} \alpha(x)|v|^p} = 1.$$

On the other hand, using that $\lambda_1 < \alpha(x)$ in a subset of positive measure of $\partial\Omega$, we have that the first eigenfunction verifies,

$$1 = \frac{\displaystyle\int_\Omega |\nabla\varphi_1|^p + |\varphi_1|^p}{\lambda_1} > \frac{\displaystyle\int_\Omega |\nabla\varphi_1|^p + |\varphi_1|^p}{\displaystyle\int_{\partial\Omega} \alpha(x)\varphi_1^p} \geq 1,$$

which is a contradiction. Hence $u$ changes sign on $\partial\Omega$.

Let us suppose by contradiction that either

$$|\{x \in \partial\Omega : \alpha(x) < \alpha \text{ and } u(x) > 0\}| > 0$$

or

$$|\{x \in \partial\Omega : \beta(x) < \beta \text{ and } u(x) < 0\}| > 0.$$

Let $s = \alpha - \beta \geq 0$, then $\beta = c(s)$, we will prove that there exists a continuous path $\gamma \in \Gamma$ such that,

max (4.3) $$\max_{u \in \gamma[-1,1]} \widetilde{J}_s(u) < \beta$$

but this contradicts the definition of $c(s)$, proving the last part of this lemma.

Assume that

$$|\{x \in \partial\Omega : \alpha(x) < \alpha \text{ and } u(x) > 0\}| > 0.$$

Let us prove that there exists $v \in W^{1,p}(\Omega)$ which changes sign and verifies

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$$\frac{\displaystyle\int_\Omega |\nabla v^+|^p + \int_\Omega |v^+|^p}{\displaystyle\int_{\partial\Omega} (v^+)^p} < \alpha \text{ and } \frac{\displaystyle\int_\Omega |\nabla v^-|^p + \int_\Omega |v^-|^p}{\displaystyle\int_{\partial\Omega} (v^-)^p} < \beta. \tag{4.4}$$

Let $x_1 \in \partial\Omega$ such that $u(x_1) > 0$. Let us take $\Gamma_1$ a connected component of the set $\{x \in \partial\Omega : u(x) > 0\}$ that contains $x_1$ and intersects the set $\{x \in \partial\Omega : \alpha(x) < \alpha\}$ in a set of positive measure. Now take $C_1$ a connected component of the set $\{x \in \Omega : u(x) > 0\}$ such that $\partial C_1 \cap \partial\Omega = \Gamma_1$.

Let us prove that if $x \in \partial C_1 \cap \Omega$ then $u(x) = 0$. Let us take $x \in \partial C_1 \cap \Omega$, then there exists $x_n \to x$ with $x_n \in C_1$, then $u(x_n) > 0$ and that implies $u(x) \geq 0$. If $u(x) > 0$ then there exists $r > 0$ such that $B_x(r) \subset C_x$ (where $C_x$ is the connected component containing $x$) as $C_1 \cap C_x \neq \emptyset$ then $C_x = C_1$, then $x \in C_1$ which contradicts the fact that $x \in \partial C_1$. Then we can define,

$$v(x) = \begin{cases} u(x) & x \in \overline{C}_1 \\ 0 & \text{otherwise .} \end{cases}$$

Then we have

$$\frac{\displaystyle\int_{C_1} |\nabla u|^p + \int_{C_1} |u|^p}{\displaystyle\int_{\partial C_1} |u|^p} < \frac{\displaystyle\int_{C_1} |\nabla u|^p + \int_{C_1} |u|^p}{\displaystyle\int_{\partial C_1} \frac{\alpha(x)}{\alpha} |u|^p} = \alpha \frac{\displaystyle\int_{C_1} |\nabla u|^p + \int_{C_1} |u|^p}{\displaystyle\int_{\partial C_1} \alpha(x)|u|^p} = \alpha,$$

here we are using that in $C_1$ the set where $\alpha(x) < \alpha$ has positive measure and we obtained the last identity replacing by $v$ as test function in the weak form of the equation.

Let us observe that the first eigenvalue of the problem

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$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } C_1, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda^* |u|^{p-2}u & \text{on } \Gamma_1, \\ u = 0 & \text{on } \partial C_1 \cap \Omega \end{cases} \tag{4.5}$$

can be characterized as

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$$\lambda_1^*(C_1) = \inf_{u \in A} \frac{\displaystyle\int_\Omega |\nabla v|^p + \int_\Omega |v|^p}{\displaystyle\int_{\partial\Omega} |v|^p}, \tag{4.6}$$

where $A_1 = \{u \in W^{1,p}(\Omega) : \ u = 0 \text{ on } \partial C_1 \cap \Omega\}$. Then we have that if $u_1$ is a minimizer for (4.6), we can suppose $u_1 \geq 0$. Using the maximum principle and Hopf's lemma (see [32]), we have that $u_1 > 0$ in $C_1 \cup \Gamma_1$ (for the details see [21]).

Also we can prove that $\lambda_1^*$ is simple and that if $w$ is an eigenfunction corresponding to an eigenvalue $\lambda^* \neq \lambda_1^*$ we have that for some fixed constants $C$ and $\beta$

$$|\Gamma_1 \cap \{w > 0\}| \geq C{\lambda^*}^{-\beta} \quad \text{and} \quad |\Gamma_1 \cap \{w < 0\}| \geq C{\lambda^*}^{-\beta}.$$

Then the only eigenfunction that does not change sign in $\Gamma_1$ is the first one. Therefore as $u > 0$ in $C_1 \cup \Gamma_1$ we conclude that $u$ is the first eigenfunction associated to the first eigenvalue, $\lambda_1^*(C_1)$, of the problem (4.5).

In an analogous way we let $x_2 \in \partial\Omega$ such that $u(x_2) < 0$ and take $\Gamma_2$ connected component containing $x_2$ of the set $\{x \in \partial\Omega : u(x) < 0\}$. Now take $C_2$ a connected component of

$$\{x \in \ \Omega : u(x) < 0\},$$

such that $\partial C_2 \cap \partial\Omega = \Gamma_2$. Then

$$\lambda_1^*(C_2) = \frac{\displaystyle\int_{C_2} |\nabla u|^p + \int_{C_2} |u|^p}{\displaystyle\int_{\partial C_2} |u|^p} \leq \beta \frac{\displaystyle\int_{C_2} |\nabla u|^p + \int_{C_2} |u|^p}{\displaystyle\int_{\partial C_2} \beta(x)|u|^p} = \beta.$$

Hence we have $\lambda_1^*(C_2) \leq \beta$, where $\lambda_1^*(C_2)$ is the first eigenvalue of the problem,

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } C_2, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \Gamma_2, \\ u = 0 & \text{on } \partial C_2 \cap \Omega. \end{cases} \tag{4.7}$$

Now we are going to construct two disjoint subsets of $C_1$ and $C_2$ such that $\lambda_1^*(\widetilde{C}_1) < \alpha$ and $\lambda_1^*(\widetilde{C}_2) < \beta$. Let

$$C_1(\nu) = \{x \in C_1 : dist(x, \partial C_1 \cap \Omega) > \nu\}$$

and

$$\lambda_1(C_1(\nu))) = \inf_{v \in A_\nu} \frac{\|v\|^p_{W^{1,p}(C_1(\nu))}}{\|v\|^p_{L^p(\partial C_1(\nu))}},$$

where $A_\nu = \{v \in W^{1,p}(\Omega) : \ v = 0 \text{ on } \partial C_1(\nu) \cap \Omega\}$. As any function in $W^{1,p}(C_1(\nu))$ such that $v = 0$ on $\partial C_1(\nu) \cap \Omega$ can be extended to a function in $W^{1,p}(C_1)$ then, using the fact that the first eigenfunction is strictly positive on $\partial\Omega$,

$$\lambda_1^*(C_1(\nu)) > \lambda_1^*(C_1)$$

and

$$\lambda_1^*(C_1(\nu)) \to \lambda_1^*(C_1)$$

as $\nu \to 0$, then there exists $\nu_0 > 0$ such that $\lambda_1^*(C_1(\nu)) < \alpha$ for $0 < \nu \leq \nu_0$. Let $x_3 \in \partial C_2 \cap \Omega$ (is not empty because $C_1 \cap C_2 = \emptyset$ and $\Omega$ is connected). Chose $0 < \nu < min\{\nu_0, dist(x_3, \Omega^c)\}$ and

$$\widetilde{C}_1 = C_1(\nu), \qquad \widetilde{C}_2 = C_2 \cup B\left(x_1, \frac{\nu}{2}\right).$$

Then $\widetilde{C}_1 \cap \widetilde{C}_2 = \emptyset$, and if we take

$$\lambda_1^*(C_2) = \inf_{v \in A_2} \frac{\|v\|^p_{W^{1,p}(C_2)}}{\|v\|^p_{L^p(\partial C_2)}}$$

$$\lambda_1^*(\widetilde{C}_2) = \inf_{v\in \widetilde{A}_2} \frac{\|v\|^p_{W^{1,p}(\widetilde{C}_2)}}{\|v\|^p_{L^p(\partial\widetilde{C}_2)}}$$

where $A_2 = \{v \in W^{1,p}(\Omega) : \ v = 0 \text{ on } \partial C_2 \cap \Omega\}$ and $\widetilde{A}_2 = \{v \in W^{1,p}(\Omega) : \ v = 0 \text{ on } \partial\widetilde{C}_2 \cap \Omega\}$, we have $\lambda_1^*(\widetilde{C}_2) \leq \lambda_1^*(C_2)$. Using that the function $w$ where the minimum $\lambda_1(\widetilde{C}_2)$ is achieved is strictly positive, we have that $w > 0$ in the set $\widetilde{C}_2 - C_2$ and by the definition of $\lambda_1^*$ we get that $\lambda_1^*(\widetilde{C}_2) < \lambda_1^*(C_2)$. Therefore

$$\lambda_1^*(\widetilde{C}_2) < \beta.$$

Finally let us take $v_1 > 0$ the minimizer for $\lambda_1^*(\widetilde{C}_1)$ and $v_2 > 0$ the minimizer for $\lambda_1^*(\widetilde{C}_2)$, then the function $v = v_1 - v_2$ satisfies (4.4).

Using that there exists $v \in W^{1,p}(\Omega)$ which changes sign and satisfies condition (4.4), we have,

$$\widetilde{J}_s\left(\frac{v}{\|v\|_{L^p(\partial\Omega)}}\right) = \int_\Omega \frac{|\nabla v|^p}{\|v\|^p_{L^p(\partial\Omega)}} + \int_\Omega \frac{|v|^p}{\|v\|^p_{L^p(\partial\Omega)}} - s\int_{\partial\Omega} \frac{(v^+)^p}{\|v\|^p_{L^p(\partial\Omega)}}$$

$$= \int_\Omega \frac{|\nabla v^+|^p + |v^+|^p}{\|v\|^p_{L^p(\partial\Omega)}} + \int_\Omega \frac{|\nabla v^-|^p + |v^-|^p}{\|v\|^p_{L^p(\partial\Omega)}} - s\int_{\partial\Omega} \frac{(v^+)^p}{\|v\|^p_{L^p(\partial\Omega)}}$$

$$= (\alpha - s)\frac{\displaystyle\int_{\partial\Omega}(v^+)^p}{\|v\|^p_{L^p(\partial\Omega)}} + \beta\frac{\displaystyle\int_{\partial\Omega}(v^-)^p}{\|v\|^p_{L^p(\partial\Omega)}} = \beta.$$

Then

$$\widetilde{J}_s\left(\frac{v}{\|v\|_{L^p(\partial\Omega)}}\right) < \beta,$$
$$\widetilde{J}_s\left(\frac{v^+}{\|v^+\|_{L^p(\partial\Omega)}}\right) < \alpha - s = \beta,$$
$$\widetilde{J}_s\left(\frac{v^-}{\|v^-\|_{L^p(\partial\Omega)}}\right) < \beta - s.$$

Using Lemma 3.2 we have that there exists a critical point in the connected component of the set $C = \{u \in S : \widetilde{J}_s(u) < \beta - s\}$. As the point $(\alpha, \beta) \in \mathcal{C}$ the only possible critical points in $C$ are $\varphi_1$ and $-\varphi_1$. Let suppose that the critical point is $\varphi_1$, then we can constructed a curve from $\frac{v}{\|v\|_{L^p(\partial\Omega)}}$ to $\varphi_1$ with energy level less than $\beta$. One then return from $\varphi_1$ to $-\frac{v^-}{\|v^-\|_{L^p(\partial\Omega)}}$ and finally to $\frac{v}{\|v\|_{L^p(\partial\Omega)}}$, exactly as in the proof of Theorem 3.1. Then we have constructed a path satisfying (4.3). The result follows. □

**Corollary 4.1.** *Let $(\alpha, \beta) \in \ \mathcal{C}$ and $\alpha(x),\ \beta(x)\ \in L^\infty(\partial\Omega)$ such that*

$$\lambda_1 \leq \alpha(x) \leq \alpha, \quad a.e$$
$$\lambda_1 \leq \beta(x) \leq \beta, \quad a.e$$

*and*

$$\lambda_1 < \alpha(x), \qquad \lambda_1 < \beta(x)$$

*on sets of positive measure on $\partial\Omega$. If $\alpha(x) < \alpha$ a.e of $\partial\Omega$ or $\beta(x) < \beta$ a.e of $\partial\Omega$ then the unique solution of (4.1) is the trivial one.*

*Proof.* By the previous lemma any non trivial solution of (4.1) changes sign and $\alpha(x) = \alpha$ on $\{x : u(x) > 0\}$ or $\beta(x) = \beta$ on $\{x : u(x) < 0\}$ then, by our hypotheses on $\alpha(x)$ and $\beta(x)$, (4.1) could only have the trivial solution. □

**Proposition 4.1.** *The curve $s \to (s+c(s), c(s))$, $s \in \mathbb{R}$ is Lipschitz continuous and strictly decreasing, that is, $s_1 < s_2$ implies $s_1 + c(s_1) < s_2 + c(s_2)$ and $c(s_1) > c(s_2)$.*

*Proof.* Let $s_1 < s_2$ then $\widetilde{J}_{s_1}(u) \geq \widetilde{J}_{s_2}(u)$ for all $u \in S$, and this implies $c(s_1) \geq c(s_2)$. We have that for all $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that

$$\max_{u \in \gamma[-1,1]} \widetilde{J}_{s_2}(u) \leq c(s_2) + \varepsilon,$$

then

$$0 \leq c(s_1) - c(s_2) \leq \max_{u \in \gamma[-1,1]} \widetilde{J}_{s_1}(u) - \max_{u \in \gamma[-1,1]} \widetilde{J}_{s_2}(u) + \varepsilon.$$

Let $u_0 \in \gamma[-1, 1]$, such that

$$\max_{u \in \gamma[-1,1]} \widetilde{J}_{s_1}(u) = \widetilde{J}_{s_1}(u_0),$$

then

$$0 \leq c(s_1) - c(s_2) \leq \widetilde{J}_{s_1}(u_0) - \widetilde{J}_{s_2}(u_0) + \varepsilon = s_1 - s_2 + \varepsilon$$

and that means that $c$ is Lipschitz.

Let us prove that $c$ is strictly decreasing. Let $s_1 < s_2$ and suppose by contradiction that

$$i)\ s_1 + c(s_1) \geq s_2 + c(s_2)$$

or

$$ii)\ c(s_1) \leq c(s_2).$$

If $i)$ holds

$$s_1 + c(s_1) \geq s_2 + c(s_2) > s_1 + c(s_2)$$

then $c(s_1) > c(s_2)$ if we take in the last corollary $(\alpha, \beta) = (s_1 + c(s_1), c(s_1))$ and $(\widetilde{\alpha}, \widetilde{\beta}) = (s_2 + c(s_2), c(s_2))$ then the only solution of (4.1) with $(\widetilde{\alpha}, \widetilde{\beta})$ is the trivial one which contradicts the fact that $(s_2 + c(s_2), c(s_2)) \in \widetilde{\Sigma}_p$.

If $ii)$ holds then

$$s_1 + c(s_1) \leq s_1 + c(s_2) < s_2 + c(s_2),$$

if we take in the last corollary $(\alpha, \beta) = (s_2 + c(s_2), c(s_2))$ and $(\widetilde{\alpha}, \widetilde{\beta}) = (s_1 + c(s_1), c(s_1))$ then the only solution of (4.1) with $(\widetilde{\alpha}, \widetilde{\beta})$ is the trivial one which contradicts the fact that $(s_1 + c(s_1), c(s_1)) \in \widetilde{\Sigma}_p$. The result follows. □

As $c(s)$ is deceasing and positive there exists $\lim_{s \to \infty} c(s)$, in the next theorem we find this asymptotic behavior of the first nontrivial curve.

**Theorem 4.1.** *The limit of $c(s)$ as $s \to +\infty$ is*

$$i)\ \lambda_1 \quad \text{if } p \leq N,$$

$$ii)\ \overline{\lambda} > \lambda_1 \quad \text{if } p > N,$$

*where*

$$\overline{\lambda} = \inf_{u\in L}\max_{r\in\mathbb{R}} \frac{\|r\varphi_1+u\|^p_{W^{1,p}(\Omega)}}{\|r\varphi_1+u\|^p_{L^p(\partial\Omega)}}$$

*and* $L = \{u \in W^{1,p}(\Omega)$ *such that* $u$ *vanishes somewhere on* $\partial\Omega\}$.

*Proof. i)* As $p \le N$ we can choose $\phi \in W^{1,p}(\Omega)$ such that there does not exist $r \in \mathbb{R}$ verifying $\varphi(x) \le r\varphi_1(x)$ a.e in $\partial\Omega$ (it suffices to take $\varphi$ a function in $W^{1,p}(\Omega)$ which is unbounded from above in a neighborhood of some $x \in \partial\Omega$). Suppose that the result is false, then there exists $\delta > 0$ such that, $\max_{u\in\gamma[-1,1]} \widetilde{J}_s(u) \ge \lambda_1 + \delta$ for all $\gamma \in \Gamma$ and all $s \ge 0$. Consider the path $\gamma \in \Gamma$ defined by

$$\gamma(t) = \frac{t\varphi_1 + (1-|t|)\varphi}{\|t\varphi_1 + (1-|t|)\varphi\|_{L^p(\partial\Omega)}} \quad t \in [-1,1].$$

The maximum of $\widetilde{J}_s$ is achieved at some $t_s$. Replacing by $v_s = t_s\varphi_1 + (1-|t_s|)\varphi$, we have

vs

$$\int_\Omega |\nabla v_s|^p + \int_\Omega |v_s|^p - s\int_{\partial\Omega}(v_s^+)^p \ge (\lambda_1+\delta)\int_{\partial\Omega}|v_s|^p. \tag{4.8}$$

Letting $s \to +\infty$, we can assume that $t_s \to \bar{t}$. Since $v_s$ remains bounded in $W^{1,p}(\Omega)$ looking at the right hand side of the last inequality we see that $\int_{\partial\Omega}(v_s^+)^p \to 0$ as $s \to +\infty$. Then we have

$$\int_{\partial\Omega}((\bar{t}\varphi_1 + (1-|\bar{t}|)\varphi)^+)^p = 0.$$

Hence

$$\bar{t}\varphi_1 + (1-|\bar{t}|)\varphi \le 0.$$

By our choice of $\varphi$, $\bar{t}$ must be equal to $-1$. Taking $s \to +\infty$ in (4.8) we get

$$\lambda_1\int_{\partial\Omega}|\varphi_1|^p = \int_\Omega |\nabla\varphi_1|^p + \int_\Omega |\varphi_1|^p \ge (\lambda_1+\delta)\int_{\partial\Omega}|\varphi_1|^p,$$

a contradiction.

*ii)* Let us first see that $\overline{\lambda} > \lambda_1$. We have that for all $u \in L$

$$\inf_{w\in L}\frac{\|w\|^p_{W^{1,p}(\Omega)}}{\|w\|^p_{L^p(\partial\Omega)}} \le \frac{\|u\|^p_{W^{1,p}(\Omega)}}{\|u\|^p_{L^p(\partial\Omega)}} \le \max_{r\in\mathbb{R}}\frac{\|u+r\varphi_1\|^p_{W^{1,p}(\Omega)}}{\|u+r\varphi_1\|^p_{L^p(\partial\Omega)}},$$

then

$$\inf_{w\in L}\frac{\|w\|^p_{W^{1,p}(\Omega)}}{\|w\|^p_{L^p(\partial\Omega)}} \le \overline{\lambda}.$$

Let us see that this last number is bigger than $\lambda_1$. Suppose by contradiction, that

$$\inf_{w\in L}\frac{\|w\|^p_{W^{1,p}(\Omega)}}{\|w\|^p_{L^p(\partial\Omega)}} = \lambda_1,$$

then there exists a sequence $u_n \in L$ such that

$$\frac{\|u_n\|^p_{W^{1,p}(\Omega)}}{\|u_n\|^p_{L^p(\partial\Omega)}} \to \lambda_1.$$

If we take $w_n = \dfrac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}}$, then we have

$$\frac{1}{\|w_n\|^p_{L^p(\partial\Omega)}} \to \lambda_1.$$

Using that $w_n$ is bounded in $W^{1,p}(\Omega)$, that the immersions $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ and $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ are compact, passing to a subsequence, we can suppose that $w_n \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$, $w_n \to w$ uniformly in $\overline{\Omega}$ and strongly in $L^p(\partial\Omega)$. As $w_n \in L$ the uniform convergence implies that $w \in L$, and using the $L^p(\partial\Omega)$ convergence we have that $\frac{1}{\|w\|^p_{L^p(\partial\Omega)}} = \lambda_1$ and that means that $w$ is an eigenfunction associated to $\lambda_1$, but this contradicts the fact that the first eigenfunction does not vanishes in $\overline{\Omega}$.

Let $u_n$ be a sequence such that

$$\max_{r\in\mathbb{R}} \frac{\|r\varphi_1 + u_n\|^p_{W^{1,p}(\Omega)}}{\|r\varphi_1 + u_n\|^p_{L^p(\partial\Omega)}} \to \overline{\lambda},$$

as

$$\frac{\|r\varphi_1 + u_n\|^p_{W^{1,p}(\Omega)}}{\|r\varphi_1 + u_n\|^p_{L^p(\partial\Omega)}} \to \lambda_1 \quad \text{when } r \to \infty$$

we can choose $r_n$ such that

$$\frac{\|r_n\varphi_1 + u_n\|^p_{W^{1,p}(\Omega)}}{\|r_n\varphi_1 + u_n\|^p_{L^p(\partial\Omega)}} \to \overline{\lambda}.$$

We first have that $r_n$ must be bounded, if not then there exists $r_{n_k} \to +\infty$ and then

$$\frac{\|r_{n_k}\varphi_1 + u_{n_k}\|^p_{W^{1,p}(\Omega)}}{\|r_{n_k}\varphi_1 + u_{n_k}\|^p_{L^p(\partial\Omega)}} \to \lambda_1$$

hence $\lambda_1 = \overline{\lambda}$ which is a contradiction . Consequently we can assume that $r_n \to \overline{r}$, and if we take $\|u_n\|_{W^{1,p}(\Omega)} = 1$ then there exists $u \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\partial\Omega)$. We have

$$\overline{\lambda} = \liminf \frac{\|r_n\varphi_1 + u_n\|^p_{W^{1,p}(\Omega)}}{\|r_n\varphi_1 + u_n\|^p_{L^p(\partial\Omega)}} \geq \frac{\|\overline{r}\varphi_1 + u_n\|^p_{W^{1,p}(\Omega)}}{\|\overline{r}\varphi_1 + u_n\|^p_{L^p(\partial\Omega)}}.$$

As the immersion $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact, then we have that $u_n \to u$ uniformly in $\overline{\Omega}$. We are assuming that $u_n \in L$, hence there exists $x_n \in \overline{\Omega}$ such that $u_n(x_n) = 0$ then passing to a subsequence we can suppose that $x_n \to x_0$ and hence $u(x_0) = 0$. Therefore we have that $u$ vanishes at some point, by the definition of $\overline{\lambda}$

$$\overline{\lambda} = \frac{\|\overline{r}\varphi_1 + u\|^p_{W^{1,p}(\Omega)}}{\|\overline{r}\varphi_1 + u\|^p_{L^p(\partial\Omega)}}.$$

Now, suppose by contradiction that *ii)* does not hold. Then there exists $\delta > 0$ such that for all $s \geq 0$ and $\gamma \in \Gamma$

$$\max_{v\in\gamma[-1,1]} \overline{J}_s(u) \geq \overline{\lambda} + \delta.$$

Take $\gamma(t) = t\varphi_1 + (1-|t|)u$, for every $s \geq 0$ there exists $v_s = t_s\varphi_1 + (1-|t_s|)u$, where the maximum is attained. Then $\overline{J}_s(v_s) \geq \overline{\lambda} + \delta$, that is

vs2
$$\int_\Omega |\nabla v_s|^p + \int_\Omega |v_s|^p - s\int_{\partial\Omega} (v_s^+)^p \geq (\overline{\lambda}+\delta)\int_{\partial\Omega} |v_s|^p. \tag{4.9}$$

As in the previous case we have that $\int_{\partial\Omega}(v_s^+)^p \to 0$ as $s \to +\infty$, and passing to a subsequence $t_s \to r$. Taking $s \to +\infty$ we obtain

$$r\varphi_1 \leq -(1-|r|)u.$$

Using the facts that $u$ vanishes somewhere and that $\varphi_1 >$ in $\overline{\Omega}$ we obtain $r \leq 0$. If $r \neq -1$ by inequality (4.9), passing to the limit we have

vss
$$\begin{aligned}\int_\Omega |\nabla(r\varphi_1 + (1-|r|)u)|^p + \int_\Omega |(r\varphi_1 + (1-|r|)u)|^p \\ \geq (\overline{\lambda}+\delta)\int_{\partial\Omega} |(r\varphi_1 + (1-|r|)u)|^p.\end{aligned} \tag{4.10}$$

Using that $r \leq 0$ we have

$$\frac{\displaystyle\int_\Omega \left|\nabla\left(\frac{r}{1+r}\varphi_1 + u\right)\right|^p + \int_\Omega \left|\frac{r}{1+r}\varphi_1 + u\right|^p}{\displaystyle\int_{\partial\Omega}\left|\frac{r}{1+r}\varphi_1 + u\right|^p} \geq \overline{\lambda} + \delta.$$

As the maximum is attained in $\overline{r}$ then

$$\overline{\lambda} = \frac{\|\overline{r}\varphi_1 + u\|^p_{W^{1,p}(\Omega)}}{\|\overline{r}\varphi_1 + u\|^p_{W^{1,p}(\Omega)}} \geq \frac{\displaystyle\int_\Omega \left|\nabla\left(\frac{r}{1+r}\varphi_1 + u\right)\right|^p + \int_\Omega \left|\frac{r}{1+r}\varphi_1 + u\right|^p}{\displaystyle\int_{\partial\Omega}\left|\frac{r}{1+r}\varphi_1 + u\right|^p} \geq \overline{\lambda} + \delta,$$

which is a contradiction. If $r = -1$ then by inequality (4.10) we have that $\lambda_1 \geq \overline{\lambda}+\delta$, which is again a contradiction. □

## 5. The Fučik spectrum emanating from $\lambda_k$

In this section we will look for curves in the Fučik spectrum in a neighborhood of a suitable sequence of eigenvalues. The idea, as in the first section, is to search for critical points of the functionals

$$J^\pm(\theta, u) = \int_\Omega |\nabla u|^p + \int_\Omega |u|^p - \theta\int_{\partial\Omega}(u^\pm)^p$$

constrained to the unit ball of $L^p(\partial\Omega)$. We can think that $J^\pm(\theta, u)$ is a perturbation of the functional

$$J^\pm(0, u) = I(u) = \int_\Omega |\nabla u|^p + \int_\Omega |u|^p.$$

We will start from a sequence of critical points $c_k$ of $I|_S$ (or $J^\pm(0,u)$) and for each $k$ and a neighborhood of 0 we found two curves $c_k^\pm(\theta)$ which are critical points of the functional $J^\pm(\theta,u)|_S$. These critical points provide us with two curves, $(c_k^\pm(\theta)+\theta, c_k^\pm(\theta))$ in $\widetilde{\Sigma}_p$ where for each $k$, the corresponding curves passe through the point $(c_k, c_k)$. As we do not know if this sequence of eigenvalues characterize completely the usual spectrum, we will nether know this fact for these curves in $\widetilde{\Sigma}_p$.

To clarify the exposition we will divide the argument in several steps.

Now we are going to state some definitions and enunciate some theorems that will be used to find these curves in the Fučik spectrum, for the proofs and references see [25].

Let $A$ and $B$ be compact sets of a Finsler manifold $M$ and let $C$ be a closed subset of $M$.

**Definition 5.1.** *We say that $C$ links $B$ via $A$ by homotopy in $M$ if $A \subset B$, $C \cap A = \emptyset$ and $\gamma(1, B) \cap C \neq \emptyset \ \forall \gamma \in \Gamma$, where the set $\Gamma = \Gamma_{A,B}$ is given by,*

$$\Gamma_{A,B} = \left\{\gamma \in C^0([0,1] \times M, M : \gamma(0,.) = Id,\ \gamma(t,.)|_A = Id \ \forall\ t \in [0,1]\right\}.$$

link.teo **Theorem 5.1.** *Let $M$ be a $C^1$ Finsler manifold and $f : M \to \mathbb{R}$ be a $C^1$ function. Assume that $C$ links $B$ via $A$ by homotopy in $M$ and*

$$\max_A f < \inf_C f \leq \max_B .$$

*Let $c = \inf_{\gamma \in \Gamma_{A,B}} \max_{u \in B} f(\gamma(1,u))$. If $f$ satisfies the Palais-Smale condition at level $c$, then $c$ is a critical value of $f$.*

stable **Definition 5.2.** *We say that $c$ is a topologically stable critical value of $f$ if there exist $A$, $B$, $C$ subsets of $M$ such that $C$ links $B$ via $A$ by homotopy in $M$,*

$$\max_A f < \inf_C f \leq \max_B$$

*and*

$$c = \inf_{\gamma \in \Gamma_{A,B}} \max_{u \in B} f(\gamma(1,u)).$$

theta **Theorem 5.2.** *Let $M$ be a $C^2$ Finsler manifold and let $I : \mathbb{R} \times M \to \mathbb{R}$ be a $C^{1,1}$ function. Assume*

*($H_1$) For any $\theta \in \mathbb{R}$, $I(\theta,.)$ satisfies the Palais-Smale condition in $M$.*

*($H_2$) There exist two constants $a$, $b$ such that*

$$\forall\ \theta \in \mathbb{R},\ \forall\ u \in M\ a \leq \frac{\partial}{\partial \theta} I(\theta, u) \leq b.$$

*($H_3$) $c$ is a topologically stable critical value of $I(0,.)$; i.e., there exist $A$, $B$, $C$ satisfying the conditions of Definition 5.2. Set*

$$\mu_0 = \frac{inf I(0,C) - max I(0,A)}{b-a}.$$

*Then for any $\theta \in [0, \mu_0)$, $I(\theta,.)$ has a critical value $c(\theta)$ such that*

$$c(\theta) \in [inf I(0,C) + a\theta, max I(0,B) + b\theta].$$

*Moreover, if*

*($H_4$) $\lim_{\|x\| \to +\infty} I(\theta, x) = +\infty$ uniformly with respect to $\theta$ in bounded sets,*

*($H_5$) $I(\theta, x)$ is Lipschitz continuous on bounded sets,*
*then $c(\theta)$ is Lipschitz continuous with respect to $\theta$ and*

$$\theta_1 > \theta_2 \implies c(\theta_1) - c(\theta_2) \leq b(\theta_1 - \theta_2).$$

Now that we have stated all the definitions and theorems that we need we are going to characterize a new sequence of eigenvalues $\{c_k\}$ of (1.3) that, by construction, will be a topologically stable sequence of critical values. Finally we will prove, using Theorem (5.2) the existence of infinitely many curves emanating from the sequence $\{c_k\}$.

Set, as in Section 2,

$$S = \left\{ u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^p = 1 \right\}.$$

**Remark 5.1.** *For $p > 1$, $S$ is a complete, connected $C^1$- Finsler manifold and the tangent space at the point $u \in S$ is*

$$T_u(S) = \left\{ v \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^{p-2}uv = 0 \right\}.$$

*For $p \geq 2$, $S$ is a $C^2$-manifold.*

We now use a variational characterization for a sequence of eigenvalues for the problem (1.3). Indeed, for solutions of (1.3) we can understand critical points of the associated energy functional

$$I(u) = \int_\Omega |\nabla u|^p + \int_\Omega |u|^p,$$

under the constraint $u \in S$. Remark that for $p > 1$, $I$ is a $C^1$- function and for $p \geq 2$, $I$ is a $C^2$ -function. We can find a sequence of variational eigenvalues with the characterization,

$$\lambda_k := \inf_{A \in C_k} \sup_{u \in A} I(u),$$

where

$$C_k := \{A \subset S : \text{there exists } h : S^{k-1} \to A \text{ continuous, odd and surjective } \}.$$

To prove that these $\lambda_k$ are critical values one first proves a Palais-Smale condition for the functional. Next, using a deformation argument, one can check that $\lambda_k$ is an eigenvalue (see [15] for the details). It is not know if these are all the eigenvalues of this problem. Now we are going to construct a new sequence of critical points of the functional $I$, that, by construction, will consists of topologically stable critical points.

link **Proposition 5.1.** *Suppose that $\lambda_{k+1} > \lambda_k$ for some $k \geq 1$. Then for all $\varepsilon > 0$ there exists $A, B$ and $C$ subspaces of $S$ such that $C$ links $B$ via $A$ and*

$$\max_{u \in A} I(u) \leq \lambda_k + \varepsilon < \lambda_{k+1} = \inf_{u \in C} I(u) \leq \max_{u \in B} I(u).$$

*Also there exists $c_{k+1}$ a stable critical point with*

$$\max_{u \in B} I(u) \geq c_{k+1} \geq \lambda_{k+1}.$$

*Proof.* By the definition of $\lambda_k$, we have that for all $\varepsilon > 0$ small enough there exists, $A \in C_k$ such that

$$\max_{u \in A} I(u) \leq \lambda_k + \varepsilon < \lambda_{k+1}.$$

Let

$$C := \left\{ u \in M : \int_\Omega |\nabla u|^p + \int_\Omega |u|^p \geq \lambda_{k+1} \int_{\partial\Omega} |u|^p \right\}.$$

Take $h$ such that $h(S^{k-1}) = A$. Chose $e \in S$ such that $e \notin A$ (then it also happens that $e \notin -A$). Define $h^* : S^k \to W^{1,p}(\Omega)$ as

$$h^*(v, x_{k+1}) = \begin{cases} \|v\| h\left(\frac{v}{\|v\|}\right) + x_{k+1} e, & \text{if } v \neq 0 \\ x_{k+1} e \ \text{ if } v = 0, \end{cases}$$

where $v = (x_1, ..., x_k)$ and $\|\cdot\| = \|\cdot\|_{L^p(\partial\Omega)}$. Let us see that $0 \notin h^*(S^k)$. If $x \in S^k$ and $v = 0$ then $x_{k+1} \neq 0$ and therefore $h^*(x) \neq 0$. If $x \in S^k$, $v \neq 0$ and $h^*(x) = 0$ then

$$\|v\| h\left(\frac{v}{\|v\|}\right) = -x_{k+1} e,$$

taking norm $L^p(\partial\Omega)$ in both sides we have that $\|v\| = |x_{k+1}|$ and then

$$h\left(\frac{v}{\|v\|}\right) = \pm e,$$

but this contradicts the fact that $\pm e \notin A$.

So we can define, $\overline{h} : S^k \to S$ as

$$\overline{h}(x) = \frac{h^*(x)}{\|h^*(x)\|},$$

as $\overline{h}$ is odd and continuous, we have that $\overline{h}(S^k) \in C^{k+1}$. We now chose $B \subset S$ as $B = \overline{h}(S^k)$. By definition of $\lambda_{k+1}$ we have

$$\lambda_{k+1} \leq \max_{u \in B} I(u).$$

Then

AA (5.1) $$\max_{u \in A} I(u) \leq \lambda_k + \varepsilon < \lambda_{k+1} = \inf_{u \in C} I(u) \leq \max_{u \in \overline{h}(S^k)} I(u) = \max_{u \in B} I(u).$$

As

$$\max_{u \in A} I(u) < \inf_{u \in C} I(u)$$

we obtain that $A \cap C = \emptyset$.

On the other hand, if $u \in A$ then there exists $x \in S^{k-1}$ such that $h^*(x, 0) = h(x) = u$. Therefore $A \subset h^*(S^{k-1}, 0) \subset B$.

We finally have to prove that, if $\gamma \in \Gamma$ then $\gamma(1, B) \cap C \neq \emptyset$. To this end, let us observe that if we prove that

$$\max_{u \in \gamma(1,B)} I(u) \geq \lambda_{k+1}$$

we are done. Take $g : S^k \to S$ as $g(x) = \gamma(1, \overline{h}(x))$ and define

$$\overline{g}(x) = \begin{cases} g(x) & \text{if } x_{k+1} \geq 0, \\ -g(-x) & \text{if } x_{k+1} < 0. \end{cases}$$

If $x_{k+1} > 0$ then $\overline{g}(-x) = -g(x) = -\overline{g}(x)$ and if $x_{k+1} < 0$ then $\overline{g}(-x) = g(-x) = -\overline{g}(x)$. If $x_{k+1} = 0$ then $\overline{g}(x) = g(x) = h(x) = -h(-x) = -g(-x) = -\overline{g}(-x)$.

Hence $\overline{g}(S^k) \in C^{k+1}$. As $I$ is even we get

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$$\max_{u\in g(S^k)} I(u) = \max_{u\in \overline{g}(S^k)} I(u) \geq \lambda_{k+1}. \tag{5.2}$$

We then have constructed subsets $A, B$ and $C$ of $S$ such that $C$ links $B$ via $A$ and satisfying (5.1). Then using Theorem 5.1 the point

$$c_{k+1} = \inf_{\gamma\in\Gamma} \max_{u\in B} I(\gamma(1,u))$$

is a stable critical point and by (5.2) $c_{k+1} \geq \lambda_{k+1}$. The result follows. □

**Theorem 5.3.** *There exists a sequence $(c_k)_{k\geq 1}$ of eigenvalues of problem (1.3) such that $\lim_{k\to\infty} c_k = +\infty$ with $c_k$ stable critical values of $I$ constrained to $M$.*

*Proof.* The result follows from the fact that $c_k \geq \lambda_k$ and $\lambda_k \to +\infty$. □

Now, we will need to improve the restriction anther $p$.

**Theorem 5.4.** *Let $p \geq 2$. Suppose that $\lambda_k < \lambda_{k+1}$ for some $k \geq 1$. Then there exist two decreasing curves $c_+, c_- : [0, \lambda_{k+1} - \lambda_k) \to \mathbb{R}$ with*

1. *$(\theta + c_+(\theta), c_+(\theta)) \in \widetilde{\Sigma}_p$ and $(c_-(\theta), c_-(\theta) + \theta) \in \widetilde{\Sigma}_p$,*
2. *$c_-(0) = c_+(0) = c_{k+1} \geq \lambda_{k+1}$,*
3. *$\lambda_{k+1} - \theta \leq c_+(\theta) \leq c_{k+1}$ and $\lambda_{k+1} - \theta \leq c_-(\theta) \leq c_{k+1}$,*
4. *the curves are Lipschitz continuous.*

*Proof.* Let us define

$$J_\pm(\theta, u) = \int_\Omega |\nabla u|^p + \int_\Omega |u|^p - \theta \int_{\partial\Omega} (u^\pm)^p.$$

In order to apply Theorem 5.2, observe that if $p \geq 2$, $S$ is of class $C^2$. Moreover, the functionals $J_\pm$ are of class $C^2$ if $p > 2$ and of class $C^{1,1}$ if $p = 2$. Now we are going to prove that the functionals $J_\pm(\theta, .)$ satisfied the hypothesis $H_1, H_2, H_3, H_4$ and $H_5$ of theorem mentioned above.

($H_1$) We have prove in section 2 that $J_+(\theta, .)$ satisfied the Palais-Smale condition. In a similar way one can prove it for $J_-(\theta, .)$.

($H_2$) We have

$$\frac{\partial}{\partial\theta} J_\pm(\theta, u) = -\int_{\partial\Omega} (u^\pm)^p,$$

hence

$$-1 \leq \frac{\partial}{\partial\theta} J_\pm(\theta, u) \leq 0.$$

Then we can take $a = -1$ and $b = 0$.

($H_3$) By Proposition 5.1 we have that there exist sets $A$, $B$ and $C$ such that

$$\max_A J_\pm(0, \cdot) < \inf_C J_\pm(0, \cdot) \leq \max_B J_\pm(0, \cdot).$$

As $H_1, H_2$ and $H_3$ holds there exists $c_\pm(\theta)$ critical values of $J_\pm(\theta, .)$ for $\theta \in [0, \mu_0)$ where

$$\mu_0 = \inf J_\pm(0, C) - \max(0, A) = \lambda_{k+1} - \max(0, A)$$

then

$$\lambda_{k+1} - \lambda_k - \varepsilon \leq \mu_0 \leq \lambda_{k+1} - \lambda_k.$$

That is, $c_\pm(\theta)$ is defined in $[0, \lambda_{k+1} - \lambda_k - \varepsilon)$. We also have

$$c_\pm(\theta) \in [\inf J_\pm(0, C) - \theta, \max J_\pm(0, B)]$$

then

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$$\lambda_{k+1} - \theta \leq c_\pm(\theta). \tag{5.3}$$

Let see what happens with the other hypothesis.

$(H_4)$ It is easy to see that $\lim_{|u|\to\infty} J_\pm(\theta, u) = \infty$ uniformly with respect to $\theta$ in bounded sets.

$(H_5)$ $J_\pm(\theta, u)$ is Lipschitz continuous on bounded sets.

The last two hypothesis imply that $c_\pm(\theta)$ are Lipschitz continuous and if

$$\theta_1 > \theta_2 \Longrightarrow c_\pm(\theta_1) - c_\pm(\theta_2) \leq b(\theta_1 - \theta_2) = 0,$$

that means that the curves $c_\pm$ are decreasing. Then by (5.3) and the monotonicity we have,

$$\lambda_{k+1} - \theta \leq c_\pm(\theta) \leq c_\pm(0) = c_{k+1}.$$

This completes the proof. □

**Theorem 5.5.** *The points* $(c_+(\theta) + \theta, c_+(\theta))$ *and* $(c_-(\theta), \theta + c_-(\theta))$ *belong to* $\widetilde{\Sigma}_p$.

*Proof.* As for each $\theta$, $c_+(\theta)$ is a critical point of $J_+(\theta, .)$ constrained to $S$ then

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega |u|^{p-2}uv - \theta \int_{\partial\Omega} (u^+)^{p-1}v - c_+(\theta) \int_{\partial\Omega} |u|^{p-2}uv = 0$$

for all $v \in W^{1,p}(\Omega)$. Hence

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega |u|^{p-2}uv - (\theta + c_+(\theta)) \int_{\partial\Omega} (u^+)^{p-1}v - c_+(\theta) \int_{\partial\Omega} (u^-)^{p-1}v = 0$$

for all $v \in W^{1,p}(\Omega)$ and looking at the weak form of equation (1.2) the result follows. In an analogous way we prove that $(c_-(\theta), \theta + c_-(\theta)) \in \widetilde{\Sigma}_p$. □

**Corollary 5.1.** *If* $\lim_{k\to+\infty} \lambda_{k+1} - \lambda_k = \sigma > 0$, *then for* $\mu \in [0, \sigma)$ *the set*

$$\widetilde{\Sigma}_p \cap \{(\alpha, \beta) \in \mathbb{R}^2 : |\alpha - \beta| = \mu\}$$

*contains infinitely many points.*

*Proof.* As $c_k \geq \lambda_k$ then $\lim_{k\to\infty} c_k = \infty$. On the other hand as $\lim_{k\to\infty} \lambda_{k+1} - \lambda_k = \sigma > 0$, then for all $\varepsilon > 0$ there exists $k_0$ such that for all $k \geq k_0$

$$\sigma - \varepsilon < \lambda_{k+1} - \lambda_k < \sigma + \varepsilon.$$

Then we have that for all $k \geq k_0$ there exist two curves $c_+^k, c_-^k : [0, \lambda_{k+1} - \lambda_k) \to \mathbb{R}$ such that for all $0 \leq \theta < \sigma - \epsilon$

$$(\theta + c_+^k(\theta), c_+^k(\theta)) \in \widetilde{\Sigma}_p$$

and

$$(c_-^k(\theta), \theta + c_-^k(\theta)) \in \widetilde{\Sigma}_p.$$

Then for any $\theta \in [0, \sigma - \varepsilon)$ we have infinitely many points $(\alpha, \beta) \in \widetilde{\Sigma}_p$ such that $|\alpha - \beta| = \theta$. □

## 6. The resonance problem

In this section, we look for solutions of

reso.6

$$\begin{cases} \Delta_p u = |u|^{p-2}u - f(x,u) & \text{in } \Omega, \\ \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \alpha(u^+)^{p-1} - \beta(u^-)^{p-1} + h(x,u) & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Here $f$ is a caratheodory function on $\Omega \times \mathbb{R}$ and $h$ is a caratheodory function on $\partial\Omega \times \mathbb{R}$ satisfying the conditions,

$$|f(x,t)| \leq V_1(x)^{p-q}|t|^{q-1} + W_1(x)^{p-1},$$

$$|h(x,t)| \leq V_2(x)^{p-q}|t|^{q-1} + W_2(x)^{p-1}$$

where $1 \leq q < p$, $V_1, W_1 \in L^p(\Omega)$ and $V_2, W_2 \in L^p(\partial\Omega)$. To find solutions of (6.1) we first construct a new sequence of curves emanating from the variational eigenvalues $\lambda_k$ that we will call it $\mathcal{C}_l^{\pm}$. As we do not know if the sequences $\{\lambda_k\}$ and $\{c_k\}$ are equal, we will do neither know if this new sequence of curves coincides with the one previously defined in Section 5.

Let $F(x,t) := \displaystyle\int_0^t f(x,s)ds$ and $H(x,t) := \displaystyle\int_0^t h(x,s)ds$. The result that we get is the following,

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**Theorem 6.1.** *Problem* (6.1) *has a solution if*

*i)* $(\alpha, \beta) \in \mathcal{C}_l^+$ *and*

$$p\int_\Omega F(x,u_j) - \int_\Omega f(x,u_j) + p\int_{\partial\Omega} H(x,u_j) - \int_{\partial\Omega} u_j h(x,u_j) \to +\infty$$

*ii)*$(\alpha, \beta) \in \mathcal{C}_l^-$ *and*

$$p\int_\Omega F(x,u_j) - \int_\Omega f(x,u_j) + p\int_{\partial\Omega} H(x,u_j) - \int_{\partial\Omega} u_j h(x,u_j) \to -\infty$$

*for any sequence* $u_j \in W^{1,p}(\Omega)$ *such that* $\|u_j\| \to \infty$ *and* $\frac{u_j}{\|u_j\|}$ *converges to some element of* $N$*, where* $N = N(\alpha, \beta)$ *is the set of eigenfunctions associated to* $(\alpha, \beta) \in \widetilde{\Sigma}_p$*, that is,* $N$ *is the set of solutions of* (1.2)*.*

To prove Theorem 6.1 we follow ideas from [28]. We will need to find new characterizations of two curves emanating from the variational eigenvalues $\lambda_k$, but here we will not prove any result related with the properties of these curves. After defining these curves we will finally prove the general resonance result, Theorem 6.1.

**Remark 6.1.** *For the construction of this curve here we will not need the restriction* $p \geq 2$*. As* $S$ *is a* $C^1$ *manifold and for* $p < 2$ *it is not* $C^{1,1}$*, we will use the critical point theory on* $C^1-$*Finsler manifold developed by Ghoussoub in* [19].

Let $\{\lambda_k\}$ be the variational sequence of eigenvalues defined in the previous section. Suppose that $\lambda_l - \lambda_{l-1} > \varepsilon > 0$, using our definition, we have that there exists $A^{l-2} \in C_{l-1}$ such that

$$\max_{u \in A^{l-2}} I(u) < \lambda_{l-1} + \varepsilon.$$

Let $h_{l-2} : S^{l-2} \to A^{l-2}$ be the corresponding function of the definition of $C_{l-1}$. Let us define:

$$C_l^+ := \{A_+ \subset S / \ \exists\ h : S_+^{l-1} \to A_+ \text{ continous, surjective and } h|_{S^{l-2}} = h_{l-2}\},$$

here $S_+^{l-1}$ is the upper hemisphere of $S^{l-1}$ with boundary $S^{l-2}$.

We have that $C_l^+$ is a homotopy-stable family of compact subsets of $S$ with close boundary $A^{l-2}$, i.e;

i) every $A_+ \subset C_l^+$ contains $A^{l-2}$.

ii) For any $A_+ \subset C_l^+$ and any $\eta \in C([0,1] \times S; S)$ satisfying $\eta(t,u) = u$ for all $(t,u) \in (\{0\} \times S) \cup ([0,1] \times A^{l-2})$ we have that $\eta(\{1\} \times A_+) \in C_l^+$.

Let us see that ii) holds. Let $A_+ \in C_l^+$ and $h : S_+^{l-1} \to A_+$ the corresponding function of the definition. Take $\eta$ such that $\eta(t,u) = u \ \forall\ u \in A^{l-2}$. If $x \in S^{l-2}$ then $\widetilde{h}(x) := \eta(1, h(x)) = \eta(1, h_{l-2}(x)) = h_{l-2}(x)$, and therefore $\eta(1, A_+) \in C_l^+$ which implies that ii) holds.

Let us recall that we are looking for critical points of $J_s^{\pm}$. Now we are going to define for each point $s \in [0, \lambda_l - \lambda_{l-1} - \varepsilon]$

$$c_l^{\pm}(s) = \inf_{A_+ \in C_l^+} \max_{u \in A_+} J_s^{\pm}(u).$$

If $c_l^{\pm}(s) < \lambda_l - s$, then there exists $A_+ \in C_l^+$ such that

$$\max_{u \in A_+} J_s^{\pm}(u) < \lambda_l - s.$$

Let $A = A_+ \cup (-A_+)$ then

$$\max_{u \in A} I(u) = \max_{u \in A_+} I(u) \leq \max_{u \in A_+} J_s^{\pm}(u) + s < \lambda_l,$$

which is a contradiction. Therefore

$$c_l^{\pm}(s) \geq \lambda_l - s \geq \lambda_{l-1} + \varepsilon > \max_{u \in A^{l-2}} I(u) \geq \max_{u \in A^{l-2}} J_s^{\pm}(u).$$

Using Theorem 3.2 of [19] we get that $c_l^{\pm}$ is a critical value of $J_s^{\pm}$.

Now that we have constructed these new curves we are going to prove Theorem 6.1.

*Proof of Theorem 6.1.* Observe that solutions of problem (6.1) are critical points of the functional defined as follows, for $u \in W^{1,p}(\Omega)$

$$\Phi(u) = \int_\Omega |\nabla u|^p + \int_\Omega |u|^p - \alpha \int_{\partial\Omega} (u^+)^p - \beta \int_{\partial\Omega} (u^-)^p - p \int_\Omega F(x,u) - p \int_{\partial\Omega} H(x,u).$$

Let us take $(\alpha,\beta)=(s+c_l^+(s),c_l^+(s))$, $s\geq 0$ and

$$\Phi_j(u)=\Phi(u)+\frac{1}{j}\int_{\partial\Omega}|u|^p+\frac{1}{j}\int_\Omega|u|^p$$

$$=\int_\Omega|\nabla u|^p+\int_\Omega|u|^p-s\int_{\partial\Omega}(u^+)^p-\left(c_l^+(s)-\frac{1}{j}\right)\int_{\partial\Omega}|u|^p$$

$$+\frac{1}{j}\int_\Omega|u|^p-p\int_\Omega F(x,u)-p\int_\Omega H(x,u).$$

We are going to use a deformation argument by Cerami [7] to find a sequence $u_j$ which converges strongly to a critical point of $\Phi$. More precisely, we are going to prove that for each $j$ there exists $u_j\in W^{1,p}(\Omega)$ such that

cerami (6.2) $$\|u_j\|\|\Phi_j(u_j)\|\to 0,\quad inf\Phi_j(u_j)>-\infty.$$

This sequence $u_j$ will converge to a certain $u\in W^{1,p}(\Omega)$ and this will be our desired critical point of $\Phi$.

As

$$\max_{u\in A^{l-2}}J_s^+(u)<c_l^+(s),$$

then there exists $\alpha>0$ such that

$$\int_\Omega|\nabla u|^p+\int_\Omega|u|^p-s\int_{\partial\Omega}(u^+)^p-c_l^+(s)\int_{\partial\Omega}|u|^p\leq-\alpha\quad\forall\ u\in A^{l-2}.$$

On the other hand, we have that $A^{l-2}$ is bounded in $W^{1,p}(\Omega)$ (because it is compact in $W^{1,p}(\Omega)$), then for $j$ sufficiently large enough and for any $u\in A^{l-2}$ we have that

jota (6.3) $$\int_\Omega|\nabla u|^p+\int_\Omega|u|^p-s\int_{\partial\Omega}(u^+)^p-c_l^+(s)\int_{\partial\Omega}|u|^p\leq-\frac{2}{j}\int_{\partial\Omega}|u|^p-\frac{1}{j}\int_\Omega|u|^p.$$

Let us take $u\in A^{l-2}$ and $R>0$, then by (6.3)

$$\begin{aligned}\Phi_j(Ru)=&\ R^p\left[J_s^+(u)-\left(c_l^+(s)-\frac{1}{j}\right)\right]+\frac{R^p}{j}\int_\Omega|u|^p\\&-p\int_\Omega F(x,Ru)-p\int_{\partial\Omega}H(x,Ru)\\\leq&\ -\frac{R^p}{j}\left(\int_{\partial\Omega}|u|^p+\int_\Omega|u|^p\right)-p\left(\int_\Omega F(x,Ru)-\int_{\partial\Omega}H(x,Ru)\right)\\\leq&\ -\frac{R^p}{j}+p\left(\|V_1\|_{L^p(\Omega)}^{p-q}R^q+\|W_1\|_{L^p(\Omega)}^{p-1}R\right.\\&\left.+\|V_2\|_{L^p(\partial\Omega)}^{p-q}R^q+\|W_2\|_{L^p(\partial\Omega)}^{p-1}R\right).\end{aligned}$$

Here we are using our hypothesis over $f$ and $h$. Then we have that

RR (6.4) $$\max_{u\in A^{l-2}}\Phi_j(Ru)\to-\infty\quad\text{as } R\to\infty.$$

Let us define

$$F = \left\{ u \in W^{1,p}(\Omega) : J_s^+(u) \geq c_l^+(s) \int_{\partial\Omega} |u|^p \right\}.$$

For $u \in F$,

$$\Phi_j(u) \geq \frac{1}{j}\int_{\partial\Omega}|u|^p + \frac{1}{j}\int_\Omega |u|^p - p\int_\Omega F(x,u) - p\int_{\partial\Omega} H(x,u)$$

$$\geq \frac{1}{j}\int_\Omega |u|^p - p\left(\|V_1\|^{p-q}_{L^p(\Omega)}\|u\|^q_{L^p(\Omega)} + \|W_1\|^{p-1}_{L^p(\Omega)}\|u\|_{L^p(\Omega)}\right)$$

$$+\frac{1}{j}\int_{\partial\Omega}|u|^p - p\left(\|V_2\|^{p-q}_{L^p(\partial\Omega)}\|u\|^q_{L^p(\partial\Omega)} + \|W_2\|^{p-1}_{L^p(\partial\Omega)}\|u\|_{L^p(\partial\Omega)}\right).$$

Then

$$\inf_{u\in F}\Phi_j(u) \geq \quad C := \min_{r,t\geq 0}\left[\frac{r^p}{j} - p\left(\|V_1\|^{p-q}_{L^p(\Omega)}r^q + \|W_1\|^{p-1}_{L^p(\Omega)}r\right.\right.$$

$$\left.\left.+\frac{t^p}{j} - \|V_2\|^{p-q}_{L^p(\partial\Omega)}t^q + \|W_2\|^{p-1}_{L^p(\partial\Omega)}t\right)\right].$$

By (6.4) we can take $R > 0$ large enough such that

$$max\Phi_j(B) < C,$$

where $B = \left\{Ru : \ u \in A^{l-2}\right\}$. To prove (6.2) we are going to use a deformation argument of Cerami. Let us consider the homotopy-stable family of compact subsets of $S$ with boundary $B$, given by

$$\mathcal{F} = \quad \{A \subset S : \text{ there is a continous surjection } h : S_+^{l-1} \to A \\ \text{such that } h|_{S^{l-2}} = Rh_{l-2}\}$$

We claim that

$$F \cap B = \emptyset, \quad F \cap A \neq \emptyset, \ \ \forall\ A \in \mathcal{F}.$$

It is clear by the definition of $C$, that $F \cap B = \emptyset$. Now, let $A \in \mathcal{F}$, if $0 \in A$, then we are done. If $0 \notin A$, we can take the radial projection onto $S$, $\pi(A) \in C_l^+$ and hence

$$\max_{u\in\pi(A)} J_s^+(u) \geq c_l^+(s)$$

then $F \cap \pi(A) \neq \emptyset$. This implies $F \cap A \neq \emptyset$. By the deformation argument of Cerami [7] we have that there exists $u_j \in W^{1,p}(\Omega)$ such that

$$\|u_j\|\|\Phi_j'(u_j)\| \to 0 \quad \text{and } |\Phi_j(u_j) - c_j| \to 0,$$

where

$$c_j = \inf_{A\in\mathcal{F}}\max_{u\in A}\Phi_j(u) \geq C.$$

What we want to prove now is that this sequence $u_j$ converges to a solution of (6.1). This is easy to see if $u_j$ is bounded. To check this fact, let us suppose that $\|u_j\|_{W^{1,p}(\Omega)} \to \infty$. Let us call $\rho_j = \|u_j\|_{W^{1,p}(\Omega)}$ and $v_j = u_j/\rho_j$, then $v_j$ is bounded.

Therefore there exists $v \in W^{1,p}(\Omega)$ such that $v_j$ converges weakly to $v$ in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and in $L^p(\partial\Omega)$. Hence

$$\int_\Omega |\nabla v_j|^{p-2}\nabla v_j \nabla(v_j - v) + \int_\Omega |v_j|^{p-2}v_j(v_j - v)$$

$$= \frac{\langle \Phi_j'(u_j), v_j - v\rangle}{p\rho_j^{p-1}} + \int_{\partial\Omega}(\alpha - \frac{1}{j})(v_j^+)^{p-1}(v_j - v) - \int_{\partial\Omega}(\beta - \frac{1}{j})(v_j^+)^{p-1}(v_j - v)$$

$$-\frac{1}{j}\int_\Omega |v_j|^{p-2}v_j(v_j - v) + \int_\Omega \frac{f(x,u_j)}{\rho_j^{p-1}}(v_j - v) + \int_{\partial\Omega}\frac{h(x,u_j)}{\rho_j^{p-1}}(v_j - v) \to 0$$

then we have that $v_j \to v$ strongly in $W^{1,p}(\Omega)$ (by the continuity of the operator $A$ defined in the Section 2). We also have that $\|v\|_{W^{1,p}(\Omega)} = 1$ and then $v \neq 0$. Let us take $w \in W^{1,p}(\Omega)$ then we have

$$\frac{\langle \Phi_j'(u_j), w\rangle}{p\rho_j^{p-1}} = \int_\Omega |\nabla v_j|^{p-2}\nabla v_j \nabla w + \int_\Omega |v_j|^{p-2}v_j w - \int_{\partial\Omega}(\alpha - \frac{1}{j})(v_j^+)^{p-1}w$$

$$-\int_{\partial\Omega}(\beta - \frac{1}{j})(v_j^+)^{p-1}w - \frac{1}{j}\int_\Omega |v_j|^{p-2}v_j w + \int_\Omega \frac{f(x,u_j)}{\rho_j^{p-1}}w + \int_{\partial\Omega}\frac{h(x,u_j)}{\rho_j^{p-1}}w,$$

passing to the limit we have that

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla w + \int_\Omega |v|^{p-2}vw = \int_{\partial\Omega}\alpha(v^+)^{p-1}w - \int_{\partial\Omega}\beta(v^+)^{p-1}w$$

which means that $v \in N$. Then $\{u_j\}$ is a sequence which satisfies $\|u_j\|_{W^{1,p}(\Omega)} \to \infty$ and $\frac{u_j}{\|u_j\|_{W^{1,p}(\Omega)}} \to v \in N$. Therefore by our hypothesis under $h$ and $f$ we have,

ultima

$$\frac{\langle \Phi_j'(u_j), u_j\rangle}{p} - \Phi_j(u_j) = p\int_\Omega F(x,u_j) - \int_\Omega u_j f(x,u_j)$$

$$+p\int_{\partial\Omega} H(x,u_j) - \int_{\partial\Omega} u_j h(x,u_j) \to +\infty. \tag{6.5}$$

On the other hand we have that

$$|\langle \Phi_j'(u_j), u_j\rangle| \leq \|\Phi_j'(u_j)\|\|u_j\| \to 0$$

and $\inf \Phi_j(u_j) > -\infty$, which contradicts (6.5). The first part of the theorem follows. In a similar way we can prove part ii) of the theorem. □

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Departamento de Matemática, FCEyN
UBA (1428) Buenos Aires, Argentina.

*E-mail address*: srmartin@bigua.dm.uba.ar

Departamento de Matemática, Universidad Católica de Chile,
Casilla 306, Correo 22, Santiago, Chile.

*E-mail address*: jrossi@mat.puc.cl