Bifurcation curves of a diffusive logistic equation when harvesting is orthogonal to the first eigenfunction*

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Abstract

We study the global bifurcation curves of a diffusive logistic equation, when harvesting is orthogonal to the first eigenfunction of the Laplacian, for values of the linear growth up to $\lambda_2 + \delta$, examining in detail their behavior as the linear growth rate crosses the first two eigenvalues. We observe some new behavior with regard to earlier works concerning this equation. Namely, the bifurcation curves suffer a transformation at $\lambda_1$, they are compact above $\lambda_1$, there are precisely two families of degenerate solutions with Morse index equal to zero, and the whole set of solutions below $\lambda_2$ is not a two dimensional manifold.

1 Introduction

This paper concerns the study of logistic equations of the form

$$-\Delta u = au - f(u) - ch,$$  \hspace{1cm} (1)

in a smooth bounded domain $\Omega \in \mathbb{R}^N$, with $N \geq 1$. We are interested in weak solutions belonging to the space

$$\mathcal{H} = \{u \in W^{2,p}(\Omega) : u = 0 \text{ on } \partial \Omega\},$$

for some fixed $p > N$. Let $\lambda_1$ and $\lambda_2$ be the first and second eigenvalues of the Dirichlet Laplacian on $\Omega$, respectively. We denote by $\phi$ the first eigenfunction satisfying $\max_\Omega \phi = 1$. We assume that $\lambda_2$ is simple, with eigenspace spanned by $\psi$, and we also normalize the second eigenfunction so $\max_\Omega \psi = 1$.

The competition term $f$ is assumed to satisfy the following hypotheses:

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(i) $f \in C^2(\mathbb{R})$.

(ii) $f(u) = 0$ for $u \leq M$, and $f(u) > 0$ for $u > M$; throughout $M \geq 0$ is fixed.

(iii) $f''(u) \geq 0$.

(iv) $\lim_{u \to +\infty} \frac{f(u)}{u} = +\infty$.

In [8], the authors obtained global bifurcation curves, of positive solutions to (1), for values of the parameter $a$ in a right neighborhood of $\lambda_1$, when $f(u) = u^2$ and $h$ is a positive function. The proof uses the fact that the competition term is quadratic. Therefore, it is not applicable to more general nonlinearities whose second derivative vanishes at the origin.

In [6], the first author generalized the results of [8] to competition terms satisfying (i)-(iv), and studied the bifurcation curves, of sign changing solutions, for $a$ up to $\lambda_2 + \delta$, for some $\delta > 0$. This was done under the assumption that $h$ was positive a.e. in $\Omega$, a hypothesis which was used in the proof, although, as noted in [6], in a right neighborhood of $\lambda_1$, one may relax the requirement on $h$ to $\int_{\Omega} h\phi \neq 0$.

In this paper, we analyze the situation when the harvesting function $h$, which in our case might be more appropriately called harvesting and plantation function, is orthogonal to the first eigenfunction of the Laplacian. Our motivation for doing so is that in this case one is forced to provide new arguments, and we suspected the geometry of the problem would be different from the one in [6]. Indeed, it turns out that the bifurcation curves suffer a complete transformation when the parameter $a$ crosses the first eigenvalue.

We examine in detail the way in which this change occurs. When seen in the $(a, u, c)$ space, the set of solutions of (1) between $\lambda_1$ and $\lambda_2$ has the shape of a piece of a paraboloid, with a flat bottom at $a = \lambda_1$. A 2-dimensional space of solutions is attached to this bottom at $a = \lambda_1$, along a segment, and lies in the region $a \leq \lambda_1$. The whole set of solutions below $\lambda_2$ is not a two-dimensional manifold. Therefore we find a richer behavior regarding this equation than in the earlier works. Also, in contrast to the bifurcation curves obtained in the previous papers, our curves turn out to be compact above $\lambda_1$, and, instead of one, we get two families of degenerate solutions with Morse index equal to zero above $\lambda_1$.

Specifically, we assume:

(a) $h \in L^\infty(\Omega)$.

(b) $\int_{\Omega} h\phi = 0$. 

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(c) \( \int_{\Omega} h \psi \neq 0. \)

Hypothesis (c) also appears in [6]. Our main results are Theorems 3.2, 4.8, 4.11, 4.12, 5.1 and 5.2. The proofs involve bifurcation methods ([3, 4]), a blow up argument, the Morse indices, and a careful choice of coordinates at each step. In particular, around \( \lambda_1 \) we decompose the space as in [1].

For other works related to logistic equations with harvesting we refer the reader to [2, 7, 9].

This paper is organized as follows: We treat successively the cases where the linear growth parameter \( a \) is equal to \( \lambda_1 \) (Section 2), below \( \lambda_1 \) (Section 3), between \( \lambda_1 \) and \( \lambda_2 \) (Section 4), and greater than or equal to \( \lambda_2 \) (Section 5).

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2 Linear growth \( a \) equal to \( \lambda_1 \)

For \( a = \lambda_1 \), the solutions of (1) are the solutions of the linear problem

\[-\Delta u = \lambda_1 u - c h,\]

i.e. are of the form \( u = t \phi + c(\Delta + \lambda_1)^{-1} h \) for

\[(t, c) \in \Lambda := \{(t, c) \in \mathbb{R}^2 : t \phi + c(\Delta + \lambda_1)^{-1} h \leq M\}.\] (2)

The set \( \Lambda \) is closed and convex. Let

\[T := \sup\{t : \text{there exists } c \text{ such that } (t, c) \in \Lambda\}.\] (3)

Clearly, \( M \leq T < +\infty \). The value \( T \) is a maximum. We define two functions, \( c_{-\lambda_1} \) and \( c_{+\lambda_1} \), in the interval \( ] - \infty, T[ \), by

\[c_{-\lambda_1}(t) := \min_{(t, c) \in \Lambda} c \quad \text{and} \quad c_{+\lambda_1}(t) := \max_{(t, c) \in \Lambda} c.\] (4)

Notice that

\[\lim_{t \to -\infty} c_{-\lambda_1}(t) = -\infty \quad \text{and} \quad \lim_{t \to -\infty} c_{+\lambda_1}(t) = +\infty,\] (5)

since, when \( t \to -\infty \), denoting by \( \nu \) the outward normal to \( \Omega \), we have

\[\frac{\partial}{\partial \nu} (M - t \phi) = -t \frac{\partial \phi}{\partial \nu} \geq -t \min_{\partial \Omega} \frac{\partial \phi}{\partial \nu} \to +\infty, \quad \text{and} \quad M - t \phi \to +\infty \text{ uniformly in each compact subset of } \Omega.\]

From (5) and the fact that \( \Lambda \) is convex, it follows that \( c_{-\lambda_1} \) is convex, continuous and strictly increasing. Similarly, \( c_{+\lambda_1} \) is concave, continuous and strictly decreasing.
If $M = 0$, then $T = 0$ and $c_{\lambda_1}^{-}(0) = c_{\lambda_1}^{+}(0) = 0$, because the function $(\Delta + \lambda_1)^{-1}h$ changes sign. One can check with specific examples, it might happen that $T = M$. In such a situation $c_{\lambda_1}^{-}(0) \leq 0 \leq c_{\lambda_1}^{+}(0)$. On the other hand, if $T > M$, then either $c_{\lambda_1}^{-}(M) = 0$ or $c_{\lambda_1}^{+}(M) = 0$. Indeed, take $M < t < T$ and $(t, c) \in \Lambda$. We have $c(\Delta + \lambda_1)^{-1}h \leq M - t\phi$. The function $M - t\phi$ is negative in an open subset of $\Omega$. Therefore, $\{c : (t, c) \in \Lambda\} \subset \mathbb{R}^{-}$ or $\{c : (t, c) \in \Lambda\} \subset \mathbb{R}^{+}$, as the function $(\Delta + \lambda_1)^{-1}h$ cannot vanish at any point of that open subset of $\Omega$. This shows that $c_{\lambda_1}^{-}(t) > 0$ or $c_{\lambda_1}^{+}(t) < 0$, for $M < t < T$. Since $(M, 0) \in \Lambda$, we have $c_{\lambda_1}^{-}(M) = 0$ or $c_{\lambda_1}^{+}(M) = 0$, as claimed. In any of the three possible cases, $M = T = 0$, $0 < M = T$ and $0 < M < T$, we have

$$c_{*,\lambda_1}^{-} := c_{\lambda_1}^{-}(0) \leq 0 \quad \text{and} \quad c_{*,\lambda_1}^{+} := c_{\lambda_1}^{+}(0) \geq 0. \quad (6)$$

We can describe the solutions $(a, u, c)$ of (1) in a neighborhood of $(\lambda_1, t_0\phi + c_0(\Delta + \lambda_1)^{-1}h, c_0)$. We define

$$\mathcal{R} := \{y \in \mathcal{H} : \int y\phi = 0\}.$$ 

When the region of integration is omitted it is understood to be $\Omega$.

**Lemma 2.1.** Let $(t_0, c_0) \in \Lambda$ with $t_0 \neq 0$. There exists a neighborhood $U \subset \mathbb{R} \times \mathcal{H} \times \mathbb{R}$ of $(\lambda_1, t_0\phi + c_0(\Delta + \lambda_1)^{-1}h, c_0)$ such that the solutions of (1) in $U$ are a $C^1$ manifold that can be parametrized, in a neighborhood $V$ of $(t_0, c_0)$, by $(t, c) \mapsto (a(t, c), t\phi + y(t, c), c)$. Here $y \in \mathcal{R}$. For $t > 0$ we have $a(t, c) \geq \lambda_1$, whereas for $t < 0$ we have $a(t, c) \leq \lambda_1$.

**Proof.** This lemma is a direct consequence of the Implicit Function Theorem applied to the function $g : \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \mathbb{R} \to L^p(\Omega)$, defined by $g(a, t, y, c) = \Delta(t\phi + y) + a(t\phi + y) - f(t\phi + y) - ch$, at the point $(t_0, c_0)$ with $t_0 \neq 0$. The statement about the sign of $a$ follows from the equality

$$(a - \lambda_1)t \int \phi^2 = \int f(u)\phi,$$  

where $u = t\phi + y$. \qed

For use in the next sections, where we consider values of $a$ different from $\lambda_1$, we make the following

**Remark 2.2.** Let $(a_n, u_n, c_n)$ be a sequence of solutions of (1) with $(a_n)$ and $(c_n)$ bounded. Then $(u_n)$ is uniformly bounded above. The same conclusion follows if, instead of assuming $(c_n)$ bounded, we suppose $(\frac{c_n}{\max u_n})$ is bounded.
Indeed, denoting by \( x_n \) a point of maximum for \( u_n \), clearly
\[
a_n u_n(x_n) - f(u_n(x_n)) - c_n h(x_n) \geq 0. \tag{8}
\]
Admitting that \( u_n(x_n) \to +\infty \), from
\[
\frac{f(u_n(x_n))}{u_n(x_n)} \leq a_n - \frac{c_n}{u_n(x_n)} h(x_n),
\]
whose left hand side is bounded, we contradict hypothesis (iv).

## 3 Linear growth \( a \) below \( \lambda_1 \)

In this section we are going to analyze the case \( a < \lambda_1 \). We observe that, for \( c \) fixed, there exists a unique solution of (1). This solution is nondegenerate. Thus, for a fixed \( a \) the set of solutions of (1) is a one dimensional \( C^1 \) manifold in \( \mathcal{H} \), that can be parametrized by \( c \mapsto u_a(c) \). The component in \( \phi \) of a solution \((a, u, c)\), \( t := \frac{\int u\phi}{\int \phi^2} \), is nonpositive due to (7). We wish to examine the behavior of the solutions as \( a \) increases to \( \lambda_1 \).

**Lemma 3.1.** Let \( c^- < c^{-, \lambda_1} \), \( c^+ > c^{+, \lambda_1} \) and \( \dot{t} < 0 \). There exists \( \delta > 0 \) such that for all \( \lambda_1 - \delta < a < \lambda_1 \) and \((a, u, c)\) solution of (1), with \( c^- \leq c \leq c^+ \), we have \( t > \tau_\dot{t}(c) \) with
\[
\tau_\dot{t}(c) := \begin{cases} 
\min\{(c^{-, \lambda_1})^{-1}(c), \dot{t}\} & \text{if } c^- \leq c \leq c^{-, \lambda_1}, \\
\dot{t} & \text{if } c^{-, \lambda_1} \leq c \leq c^{+, \lambda_1}, \\
\min\{\dot{t}, (c^{+, \lambda_1})^{-1}(c)\} & \text{if } c^{+, \lambda_1} \leq c \leq c^+.
\end{cases} \tag{9}
\]

**Proof.** We argue by contradiction. Let \( a_n \not\to \lambda_1 \) such that \((a_n, u_n, c_n)\) is a solution of (1), with \( c^- \leq c_n \leq c^+ \) and \( t_n := \frac{\int u_n\phi}{\int \phi^2} \leq \tau_\dot{t}(c_n) \). As remarked, for each fixed \( a < \lambda_1 \), the map \( c \mapsto u_a(c) \) is, in particular, continuous, and \( u_a(0) = 0 \). Thus, without loss of generality, by changing \( u_n \) and \( c_n \), we may assume
\[
t_n = \tau_\dot{t}(c_n). \tag{10}
\]
In addition, we may suppose \( c_n \to c_0 \) and \( t_n \to t_0 \).

Let \( y_n = u_n - t_n\phi \). Multiplying (1) by \( y_n \) and integrating over \( \Omega \),
\[
\int |\nabla y_n|^2 = a_n \int y_n^2 - \int f(t_n\phi + y_n)y_n - c_n \int h y_n.
\]
We observe that the second term in the right hand side is nonpositive since
\[
- \int f(t_n\phi + y_n)y_n = - \int f(t_n\phi + y_n)(t_n\phi + y_n) + t_n \int f(t_n\phi + y_n)\phi. \tag{11}
\]
Thus, we have
\[ \int |\nabla y_n|^2 \leq a_n \int y_n^2 - c_n \int h y_n, \]
which, together with
\[ \int |\nabla y_n|^2 \geq \lambda_2 \int y_n^2, \]
implies that \((y_n)\) is bounded in \(H^1_0(\Omega)\). We may assume \(y_n \to y_0\) in \(L^2(\Omega)\) and a.e. in \(\Omega\). By Remark 2.2, \((\text{ess sup} \, u_n)\) is uniformly bounded. Letting \(u_0 = t_0 \phi + y_0\), it follows \(f(u_n) \to f(u_0)\) in \(L^p(\Omega)\). Using equation (1) and elliptic regularity theory (see [5]), \(y_n \to y_0\) in \(H^\ast\).

From the previous paragraph, the limit \((\lambda_1, u_0, c_0)\) satisfies equation (1) and, from (10), \(t_0 = \tau_\ell(c_0)\). We use Lemma 2.1 at the point \((t_0, c_0)\). The solutions of (1) in a neighborhood of \((t_0, c_0)\) can be parametrized by \((t, c) \mapsto (a(t, c), t \phi + y(t, c), c)\). From the choice of \((a_n, u_n, c_n)\), we have \(a(\tau_\ell(c_n), c_n) = a(t_n, c_n) < \lambda_1\).

We claim that \(a(\tau_\ell(c), c) = \lambda_1\) for any \(c \in [c^-, c^+]\). Indeed, from (2) and (4), for any \(t \leq T\), \((\lambda_1, t \phi + \hat{c}_{\lambda_1}(t)(\Delta + \lambda_1)^{-1} h, \hat{c}_{\lambda_1}(t))\) and \((\lambda_1, t \phi + c_\lambda^+(t)(\Delta + \lambda_1)^{-1} h, c_\lambda^+(t))\) are solutions of (1). Take \(c\) in (9) equal to \(c_\lambda(t)\) with \(t < 0\). If the minimum in (9) is not equal to \(\hat{t}\), then \(\tau_\ell(c) = t\). So \((\lambda_1, t \phi + \hat{c}_{\lambda_1}(t)(\Delta + \lambda_1)^{-1} h, \hat{c}_{\lambda_1}(t)) = (\lambda_1, \tau_\ell(c) \phi + c(\Delta + \lambda_1)^{-1} h, c)\), which means \(a(\tau_\ell(c), c) = \lambda_1\). Similarly if \(c = c_\lambda^+(t)\) with \(t < 0\) and the minimum in (9) is not equal to \(\hat{t}\). In the case \(c\) is such that \(\tau_\ell(c) = \hat{t}\), \((\lambda_1, \hat{t} \phi + c(\Delta + \lambda_1)^{-1} h, c)\) is a solution of (1), because \(c_{\lambda_1}(\hat{t}) \leq c \leq c_{\lambda_1}^+(\hat{t})\), as \(c_{\lambda_1}\) is increasing and \(c_{\lambda_1}^+\) is decreasing. Therefore \(a(\hat{t}, c) = \lambda_1\). Since \(\hat{t} = \tau_\ell(c)\), also in this situation \(a(\tau_\ell(c), c) = \lambda_1\).

In conclusion, on the one hand \(a(\tau_\ell(c_n), c_n) < \lambda_1\) and on the other hand \(a(\tau_\ell(c), c) = \lambda_1\) for any \(c \in [c^-, c^+]\). We reached the desired contradiction. The lemma is proved.

We extend the definition of \(\tau\) in (9) to zero,
\[ \tau_0(c) := \begin{cases} (c_{\lambda})^{-1}(c) & \text{if } c^- \leq c \leq c_{\lambda_1}, \\ 0 & \text{if } c_{\lambda_1} \leq c < c_{\lambda_1}^+, \\ (c_{\lambda_1}^+)^{-1}(c) & \text{if } c_{\lambda_1}^+ \leq c \leq c^+. \end{cases} \]
In the next theorem, we prove that, as \(a \nearrow \lambda_1\), the solutions \((a, u_a(c), c)\) converge to
\[ (\lambda_1, u_{\lambda_1}(c), c) := (\lambda_1, \tau_0(c) \phi + c(\Delta + \lambda_1)^{-1} h, c). \]

**Theorem 3.2.** Let \(-\infty < c^- < c_{\lambda_1}^- \leq c_{\lambda_1}^+ < c^+ < \infty\). As \(a \nearrow \lambda_1\), the solutions \((a, u_a(c), c)\) converge uniformly in \(H\) to \((\lambda_1, u_{\lambda_1}(c), c)\) for \(c \in [c^-, c^+]\).
Proof. Again the proof is by contradiction. We admit there exists $\delta > 0$, $a_n \not\rightarrow \lambda_1$ and $c_n \in [c^-, c^+]$ such that the corresponding solutions $(a_n, u_{a_n}(c_n), c_n)$ satisfy
\[ \|u_{a_n}(c_n) - u_{\lambda_1}(c_n)\|_{\mathcal{H}} \geq \delta. \] (12)
We may assume $c_n \rightarrow c_0$. Let $\hat{t} < 0$. By Lemma 3.1, for large $n$, $\tau_{\hat{t}}(c_n) \leq t_n \leq 0$. Modulo a subsequence, $t_n \rightarrow t_0$, where $\tau_{\hat{t}}(c_0) \leq t_0$. Since $\tau_{\hat{t}}$ is arbitrary, $\tau_0(\hat{c}) \leq t_0$. Again arguing as in the proof of Lemma 3.1, $y_n = u_{a_n}(c_n) - \frac{c_n}{t_n} \phi \rightarrow y_0$ in $\mathcal{H}$ and $(\lambda_1, u_0, c_0) := (\lambda_1, t_0\phi + y_0, c_0)$ is a solution of (1). Using $\tau_0(c_0) \leq t_0$, we conclude $t_0 = \tau_0(c_0)$. Therefore $u_0 = u_{\lambda_1}(c_0)$. However, passing to the limit in both sides of (12),
\[ \|u_0 - u_{\lambda_1}(c_0)\|_{\mathcal{H}} \geq \delta. \]
This contradiction proves the theorem. \qed

4 Linear growth $a$ between $\lambda_1$ and $\lambda_2$

In the case $c = 0$, equation (1) reduces to
\[ -\Delta u = au - f(u). \] (13)

From [6], we know

Lemma 4.1. Suppose $f$ satisfies (i)-(iv). The set of positive solutions $(a, u)$ of (13) is a connected one dimensional manifold $\mathcal{C}_1$ of class $C^1$ in $\mathbb{R} \times \mathcal{H}$. The manifold is the union of the segment $\{(\lambda_1, t\phi) : t \in [0, M]\}$ with a graph $\{(a, u_1(a)) : a \in [\lambda_1, +\infty]\}$. The solutions are strictly increasing along $\mathcal{C}_1$. For $a > \lambda_1$ every positive solution is stable and, at each $a$, equation (13) has no other stable solution besides $u_1(a)$.

We turn to the case $c \neq 0$. Recall that by Remark 2.2, for $a$ and $c$ bounded, solutions are bounded above. On the other hand, for $c$ unbounded, it holds

Lemma 4.2. Let $I$ be any compact interval in $\mathbb{R}$. There exists $K > 0$ such that, for all $(a, u, c)$ solution of (1) with $a \in I$ and $|c| > K$, it holds $u \leq |c|$.

Proof. By contradiction, suppose that $(a_n, u_n, c_n)$, with $a_n \in I$ and $|c_n| \rightarrow \infty$, is a sequence of solutions of (1), verifying $\max u_n > |c_n|$. From Remark 2.2, $(u_n)$ is uniformly bounded above, which is a contradiction. \qed

Taking into account inequality (8), we immediately obtain
Corollary 4.3. Under the conditions of the previous lemma, there exists a constant $C$ such that $f(u) \leq C|c|$.

In the following proposition we show that for large values of $|c|$ equation (1) has no solutions.

Proposition 4.4. Let $\lambda_2 < a_2 < \lambda_3$ and $J = \{\lambda_1, a_2\}$. There exists $\bar{c} > 0$ such that for all $a \in J$ and $(a, u, c)$ solution of (1), we have $|c| \leq \bar{c}$.

Proof. We argue by contradiction. Suppose that $(a_n, u_n, c_n)$ is a solution to (1) with $a_n \in J$, and $c_n \to +\infty$ or $c_n \to -\infty$. We define $s = +1$ in the first case and $s = -1$ in the second case. Without loss of generality, we assume that $a_n \to a$. Define $v_n = \frac{u_n}{sc_n}$. The function $v_n$ satisfies

$$\Delta v_n + a_n v_n - \frac{f(u_n)}{sc_n} - sh = 0. \quad (14)$$

From Corollary 4.3, we know

$$\frac{f(u_n)}{sc_n} \leq C. \quad (15)$$

Recall that we assumed that $\lambda_2$ is simple and decompose

$$v_n = t_n \phi + \eta_n \psi + w_n,$$

where $w_n$ denotes the component of $v_n$ orthogonal to both $\phi$ and $\psi$. We prove successively $w_n, t_n$ and $\eta_n$ are bounded. By Corollary 4.3, the function $\frac{f(u_n)}{sc_n} + sh$ is bounded in $L^\infty(\Omega)$ and so is its component orthogonal to the first two eigenfunctions. Since $a_n$ is bounded away from $\lambda_3$, $(w_n)$ is uniformly bounded in $L^2(\Omega)$. Similarly to (7), the values $t_n$ are given by

$$(a_n - \lambda_1)t_n \int \phi^2 = \int \frac{f(u_n)}{sc_n} \phi$$

and hence are nonnegative. An upper bound for $t_n$ follows from $v_n \leq 1$, due to Lemma 4.2, which gives $t_n \leq \int \phi / \int \phi^2$. Suppose $|\eta_n| \to \infty$. The sequence $\eta_n \psi$ is not bounded above or below in $L^\infty(\Omega)$. This contradicts $v_n \leq 1$. We conclude $(\eta_n)$ is bounded and $(v_n)$ is uniformly bounded in $L^2(\Omega)$.

From (14), $(v_n)$ is uniformly bounded in $H^1_0(\Omega)$. We may assume $v_n \to v$ in $H^1_0(\Omega)$, $v_n \to v$ in $L^2(\Omega)$ and $v_n \to v$ a.e. in $\Omega$. We claim $v \leq 0$ a.e. in $\Omega$. Suppose there exists a point $x \in \Omega$, in the set where $v_n \to v$, such that $v(x) > 0$. Then $u_n(x) \to +\infty$. So,

$$\frac{f(u_n(x))}{sc_n} = \frac{f(u_n(x)) u_n(x)}{u_n(x) sc_n} \to +\infty.$$
This contradicts (15) a.e. and shows \( v \leq 0 \) a.e. in \( \Omega \). In order to pass to the limit in (14), we observe \( f(u_n) \to f_\infty \) in \( L^2(\Omega) \), where \( f_\infty \geq 0 \). The limit equation is

\[
\Delta v + av - f_\infty - s h = 0.
\]

Multiplying both sides by \( \phi \) and integrating over \( \Omega \),

\[
(a - \lambda_1) \int v \phi = \int f_\infty \phi.
\]

The left hand side is nonpositive and the right hand side is nonnegative. This implies \( v \) and \( f_\infty \) are both identically equal to zero. We have reached a contradiction because \( h \) is nontrivial.

The conclusion of the previous proposition also holds for the case where \( \lambda_2 \) has multiplicity greater than one, since if a linear combination of second eigenfunctions is bounded, then each of the coefficients of that linear combination is bounded.

In the next lemma, we give a condition which ensures that a sequence of solutions of (1) converges, modulo a subsequence.

**Lemma 4.5.** Let \((a_n, u_n, c_n)\) be a sequence of solutions of (1) with \( a_n \in J \), \( a_n \to a \) and \( c_n \to c \). Then, modulo a subsequence, \((u_n)\) converges in \( H \).

**Proof.** Using an argument similar to, but simpler than, the one in the proof of the previous proposition, one can show \((u_n)\) is uniformly bounded in \( H^1_0(\Omega) \). By elliptic regularity theory, \((u_n)\) is uniformly bounded in \( H \). Subtracting the equations for \( u_m \) and \( u_n \), one can prove that \((u_n)\) converges in \( H \). \( \square \)

Let \( a > \lambda_1 \). Starting at the stable solution \((a, u_1(a), 0)\), and keeping \( a \) fixed, we can use the Implicit Function Theorem to follow a branch of solutions, taking \( c \) as parameter. Lemma 4.5 guarantees the branch will not go to infinity. Since, from Proposition 4.4, solutions do not exist for large \(|c|\), there must exist at least two degenerate solutions with Morse index equal to zero, \((a, u^-_c(a), c^-(a))\) and \((a, u^+_c(a), c^+_c(a))\), the first corresponding to a negative value of \( c \) and the second corresponding to a positive value of \( c \). We recall the Morse index of a solution is the number of negative eigenvalues of the linearized problem at the solution, and we recall the solution is said to be degenerate if one of the eigenvalues of the linearized problem is equal to zero. In the next lemma we examine the behavior of the branch of solutions around a degenerate solution with Morse index equal to zero.

**Lemma 4.6 ([4, Theorem 3.2], [8, p. 3613]).** Let \( a > \lambda_1 \) be fixed and \( p_* = (u_*, c_*) \) be a degenerate solution with Morse index equal to zero, with \( c_* > 0 \)
(respectively \(c_* < 0\)). There exists a neighborhood of \(p_*\) in \(\mathcal{H} \times \mathbb{R}\) such that the set of solutions of (1) in the neighborhood is a \(C^1\) manifold. This manifold is \(m^2 \cup \{p_*\} \cup m^*\). Here

- \(m^2\) is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph \(\{(u^2(c), c) : c \in ]c_* - \varepsilon_*, c_*]\}\) \(\{(u^2(c), c) : c \in ]c_*, c_* + \varepsilon_*]\}\).

- \(m^*\) is a manifold of stable solutions, which is a graph \(\{(u^*(c), c) : c \in ]c_* - \varepsilon_*, c_*]\}\) \(\{(u^*(c), c) : c \in ]c_*, c_* + \varepsilon_*]\}\).

The value \(\varepsilon_*\) is positive. The manifolds \(m^2\) and \(m^*\) are connected by \(\{p_*\}\).

**Sketch of the proof.** The function \((a, u_*, c_*)\) satisfies (1) and there exists \(w_* > 0\) in \(\Omega\) such that

\[
\Delta w_* + aw_* - f'(u_*)w_* = 0. \quad (16)
\]

Combining these equalities, we obtain

\[
\int (f'(u_*)u_* - f(u_*))w_* = c_* \int h w_*.
\]

Observe that

\[
\int h w_* \neq 0. \quad (17)
\]

Otherwise, \(f'(u_*)u_* - f(u_*) = 0\) which implies \(u_* \leq M\), because of (i)–(iii). The term \(f'(u_*)w_*\) in equation (16) would vanish and so \(a = \lambda_1\), contrary to our assumption. Therefore, if \(c_* > 0\), then \(\int h w_* > 0\), and if \(c_* < 0\), then \(\int h w_* < 0\). The proof is completed following the argument of Lemma 4.3 of [6], parametrizing solutions by \(t\), where \(u = tw_* + y\), and taking into account

\[
c''(t_*)(1) = -\frac{\int f''(u_*)w_*^3}{\int h w_*}.
\]

which is negative if \(c_* > 0\) and positive if \(c_* < 0\).

The next proposition guarantees the degenerate solutions vary smoothly with \(a\).

**Proposition 4.7.** The set of degenerate solutions \((a, u, c)\) of (1) with Morse index equal to zero in \([\lambda_1, +\infty[ \times \mathcal{H} \times \mathbb{R}\) is the disjoint union of two connected one dimensional manifolds \(D_*^-\) and \(D_*^+\) of class \(C^1\). Each manifold is a graph \(\{(a, u_-(a), c_-(a)) : a \in ]\lambda_1, +\infty]\}\) and \(\{(a, u_+(a), c_+(a)) : a \in ]\lambda_1, +\infty]\}\).
Sketch of the proof. To prove the degenerate solutions can be followed using the parameter $a$, we apply (17) and the argument in the proof of Theorem 3.1 in [6]. On the other hand, suppose there were more than the two degenerate solutions with Morse index equal to zero, $(a, u^*_-(a), c^*_-(a))$ and $(a, u^*_+(a), c^*_+(a))$, for each value of $a$. Then, because of Lemma 4.6, each additional degenerate solution would give rise to a branch of stable solutions, which could be followed using the parameter $c$, to $c = 0$. However, from Lemma 4.1, at $c = 0$ there exists only one stable solution $(a, u_1(a), 0)$. This would yield a contradiction. □

We now restrict our attention to $\lambda_1 < a < \lambda_2$. Considering there are no degenerate solutions with Morse index greater than zero, the above results lead to the following

**Theorem 4.8.** Let $\lambda_1 < a < \lambda_2$. The set of solutions of (1) is a compact connected one dimensional manifold in $\{a\} \times \mathcal{H} \times \mathbb{R}$. There exist precisely two solutions for each $c \in ]c^-_*, c^+_[*$, one stable, $(a, u^*_a(c), c)$, and the other nondegenerate with Morse index equal to one, $(a, u^*_a(c), c)$. In addition, there exists exactly one solution when $c = c^-_*$ and $c = c^+_*$.

In the next results, we consider the case $M > 0$. Recall the definitions of $\Lambda$ in (2) and of $T$ in (3). In parallel to Lemma 3.1, we can prove

**Lemma 4.9.** Suppose $M > 0$. For $0 < t < T$, define

$$
\Lambda_t = \Lambda \cap \{(t, c) \in \mathbb{R}^2 : t \leq t\} \quad \text{and} \quad \Lambda_t^C = \{(t, c) \in \mathbb{R}^2 : t \geq 0\} \setminus \Lambda_t.
$$

There exists $\delta > 0$ such that for all $\lambda_1 < a < \lambda_1 + \delta$ and $(a, u, c)$ solution of (1), we have $(t, c) \in \Lambda_t^C$.

The proof is analogous to the one of Lemma 3.1, but ones uses the fact the last term in (11) is bounded, since $(u_n)$ is uniformly bounded above, according to Remark 2.2.

We examine the behavior of $D^-_s$ and $D^+_s$ as $a$ decreases to $\lambda_1$.

**Proposition 4.10.** Suppose $M > 0$. As $a$ decreases to $\lambda_1$, we have

$$
\lim_{a \searrow \lambda_1} (a, u^*_-(a), c^*_-(a)) = (\lambda_1, 0, \phi + c^-_{s, \lambda_1}(\Delta + \lambda_1)^{-1} h, c^-_{s, \lambda_1}), \tag{18}
$$

$$
\lim_{a \searrow \lambda_1} (a, u^*_+(a), c^*_+(a)) = (\lambda_1, 0, \phi + c^+_{s, \lambda_1}(\Delta + \lambda_1)^{-1} h, c^+_{s, \lambda_1}), \tag{19}
$$

i.e.

$$
\lim_{a \searrow \lambda_1} (t^*_-(a), c^*_-(a)) = (0, c^-_{s, \lambda_1}), \quad \lim_{a \searrow \lambda_1} (t^*_+(a), c^*_+(a)) = (0, c^+_{s, \lambda_1}),
$$

where $t^\pm_s(a) := \int \frac{u^\pm_s(a)}{\phi^s} \, ds$ and $c^-_{s, \lambda_1}$ and $c^+_{s, \lambda_1}$ are given in (6).
Proof. By Proposition 4.4, $c^+_s(a)$ is bounded. By Lemma 4.5, we may assume, as $a \searrow \lambda_1$, $(a, u^*_a(a), c^+_s(a))$ converges, say to $(\lambda_1, u^*_a(\lambda_1), c^+_s(\lambda_1))$. From equality (7), $t^+_s(\lambda_1) \equiv \frac{\int u^*_a(\lambda_1) f}{\int f^2}$ is nonnegative. It is enough to prove $c^+_s(\lambda_1) = c^+_{\lambda_1}(0)$. Since $t^+_s(\lambda_1) \geq 0$ and $c^+_s$ is strictly decreasing, $c^+_s(\lambda_1) \leq c^+_{\lambda_1}(0)$. Suppose, by contradiction, that $\delta = c^+_{\lambda_1}(0) - c^+_s(\lambda_1) > 0$. According to the definition of $c^+_s(\lambda_1)$, there exists $\varepsilon > 0$ such that for all $\lambda_1 < a < \lambda_1 + \varepsilon$, we have $c^+_s(a) < c^+_s(\lambda_1) + \delta/2$. Lemma 4.6 implies that for all $\lambda_1 < a < \lambda_1 + \varepsilon$ and $(a, u, c)$ solution of (1), $c \leq c^+_s(a) < c^+_s(\lambda_1) + \delta/2$. Now choose $(t_0, c_0) \in \partial \Lambda$, with $t_0 > 0$ and $c_0 > c^+_{\lambda_1}(0) - \delta/2$. Applying Lemma 2.1 at $(t_0, c_0)$, we reach a contradiction to $c^+_s(a) < c^+_s(\lambda_1) + \delta/2$ for $a$ in a right neighborhood of $\lambda_1$. This proves $c^+_s(\lambda_1) = c^+_{\lambda_1}(0)$.

To conclude the analysis of the case $M > 0$, we state a result on uniform convergence of the curves of solutions, for fixed $\lambda_1 < a < \lambda_2$, as $a$ decreases to $\lambda_1$, taking into account that the set of values $c$ for which there exists a solution, $[c^+_s(a), c^+_s(a)]$, in general depends on $a$. Of course, due to (18) and (19), for each $c^+_{\lambda_1} < c < c^+_{\lambda_2}$, there exist solutions for $a$ in a right neighborhood of $\lambda_1$, and for each $c < c^+_{\lambda_1}$ and $c > c^+_{\lambda_2}$, there do not exist solutions for $a$ in a right neighborhood of $\lambda_1$.

Theorem 4.11. Suppose $M > 0$ and let $T$ be as in (3). When $c^+_{\lambda_1} < c \leq c^+_{\lambda_2}$, define $t_c$ by

\[
t_c = \begin{cases} 
(c^+_{\lambda_1})^{-1}(c) & \text{if } c < c^+_{\lambda_1}(T), \\
T & \text{if } c^+_{\lambda_1}(T) \leq c \leq c^+_{\lambda_2}(T), \\
(c^+_{\lambda_2})^{-1}(c) & \text{if } c > c^+_{\lambda_2}(T).
\end{cases}
\]

For all $\delta > 0$, there exists $\varepsilon > 0$ satisfying for all $\lambda_1 < a < \lambda_1 + \varepsilon$ if the value $c$ is such that there exists a solution $(a, u_a(c), c)$ then, in the case $c < c^+_{\lambda_1}$ we have

\[
\|u_a(c) - (0 + c^+_{\lambda_1}(\Delta + \lambda_1)^{-1}h)\|_{\mathcal{H}} < \delta,
\]

and in the case $c > c^+_{\lambda_1}$ we have

\[
\|u_a(c) - (0 + c^+_{\lambda_1}(\Delta + \lambda_1)^{-1}h)\|_{\mathcal{H}} < \delta;
\]

if the value $c$ is such that there exist solutions $(a, u^*_a(c), c)$ and $(a, u^*_a(c), c)$ then, in the case $c^+_{\lambda_1} < c \leq c^+_{\lambda_2}$, we have

\[
\|u^*_a(c) - (t_c \phi + c(\Delta + \lambda_1)^{-1}h)\|_{\mathcal{H}} < \delta,
\]

\[
\|u^*_a(c) - (0 + c(\Delta + \lambda_1)^{-1}h)\|_{\mathcal{H}} < \delta.
\]
The proof is similar to the one of Theorem 3.2.

To conclude this section we consider the case $M = 0$.

**Theorem 4.12.** Suppose $M = 0$. For all $\delta > 0$, there exists $\varepsilon > 0$ satisfying for all $\lambda_1 < a < \lambda_1 + \varepsilon$ if the value $c$ is such that there exists a solution $(a, u_a(c), c)$, then

$$\|u_a(c)\|_H < \delta.$$ 

5 Linear growth a greater than or equal to $\lambda_2$

We recall the assumption that the second eigenvalue of the Dirichlet Laplacian on $\Omega$ is simple and we call $\psi$ an associated eigenfunction, normalized so $\max_{\Omega} \psi = 1$. We define

$$\beta = -\min_{\Omega} \psi,$$

so that $\beta > 0$. To fix ideas, without loss of generality, we suppose

$$\int h\psi < 0. \quad (20)$$

For $a = \lambda_2$, the set of solutions of (1) is completely described by

**Theorem 5.1.** Suppose $f$ satisfies (i)-(iv) and $h$ satisfies (a)-(c). Fix $a = \lambda_2$. The set of solutions $(\lambda_2, u, c)$ of (1) is a compact connected one dimensional manifold $\mathcal{M}$ of class $C^1$ in $\{\lambda_2\} \times \mathcal{H} \times \mathbb{R}$. We have

$$\mathcal{M} = \mathcal{M}^\circ \cup \mathcal{L} \cup \mathcal{M}^\sharp \cup \{p_+^*\} \cup \mathcal{M}^* \cup \{p_-^*\},$$

where $\mathcal{L}$ connects $\mathcal{M}^\circ$ and $\mathcal{M}^\sharp$, $\{p_+^*\}$ connects $\mathcal{M}^\sharp$ and $\mathcal{M}^*$, and, finally, $\{p_-^*\}$ connects $\mathcal{M}^*$ and $\mathcal{M}^\circ$. Here

- $\mathcal{M}^\circ$ is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(\lambda_2, u_{\lambda_2}^*(c), c) : c \in [c^-(\lambda_2), 0]\}$.

- $\mathcal{L}$ is a segment (a point in the case $M = 0$) of degenerate solutions with Morse index equal to one, $\{(\lambda_2, t\psi, 0) : t \in [-\frac{M}{\beta}, M]\}$.

- $\mathcal{M}^\sharp$ is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(\lambda_2, u_{\lambda_2}^*(c), c) : c \in [0, c^+_*(\lambda_2)]\}$.

- $p_+^* = (\lambda_2, u_+^*(\lambda_2), c^+_*(\lambda_2))$ is a degenerate solution with Morse index equal to zero.

- $\mathcal{M}^*$ is a manifold of stable solutions, which is a graph $\{(\lambda_2, u_+^*(\lambda_2), c) : c \in [c^-_*(\lambda_2), c^+_*(\lambda_2)]\}$.

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Figure 1: A bifurcation curve for $a = \lambda_2$.

- $p^*_s = (\lambda_2, u^*_s(\lambda_2), c^*_s(\lambda_2))$ is a degenerate solution with Morse index equal to zero.

Theorem 5.1 is illustrated in Figure 1.

Sketch of the proof. We start at $(\lambda_2, u_+(\lambda_2), 0)$, the stable solution at $c = 0$. We may use the Implicit Function Theorem to follow the solutions for $c \in [c^*_-(\lambda_2), c^*_+(\lambda_2)]$, arriving at the left at $p^*_-$ and at the right at $p^*_+$. Arguing as in Theorem 1.2 of [6], $p^*_+$ is connected successively to $M^\flat$, $L$ and $M^*$. By Lemma 6.1 in [6], we may use the parameter $c$ to follow the branch $M^\flat$ until we reach a degenerate solution with Morse index equal to zero, as all solutions with Morse index equal to one are nondegenerate, except for $(\lambda_2, t\psi, 0)$ for $t \in [-\frac{M}{\beta}, M]$. The degenerate solution with Morse index equal to zero must be $p^*_-$, since it is the only one corresponding to a negative value of $c$. The branches $M^\flat$ and $M^*$ connect at $p^*_+$. This is in accordance to Lemma 4.6.

When $a > \lambda_2$, the set of solutions of (1) is characterized in

**Theorem 5.2.** Suppose $f$ satisfies (i)-(iv) and $h$ satisfies (a)-(c). Without loss of generality, suppose (20) is true. There exists $\delta > 0$ such that the following holds. Fix $\lambda_2 < a < \lambda_2 + \delta$. The set of solutions $(a, u, c)$ of (1) is a compact connected one dimensional manifold $\mathcal{M}$ of class $C^1$ in $\{a\} \times \mathcal{H} \times \mathbb{R}$. We have $\mathcal{M}$ is the disjoint union

$$\mathcal{M} = \mathcal{M}^\flat \cup \{p_s\} \cup \mathcal{M}^\flat \cup \{p_1\} \cup \mathcal{M}^\sharp \cup \{p^*_s\} \cup \mathcal{M}^* \cup \{p^*_+\},$$

where $\{p_s\}$ connects $\mathcal{M}^\flat$ and $\mathcal{M}^\flat$, $\{p_1\}$ connects $\mathcal{M}^\flat$ and $\mathcal{M}^\sharp$, $\{p^*_s\}$ connects $\mathcal{M}^\sharp$ and $\mathcal{M}^*$, and $\{p^*_+\}$ connects $\mathcal{M}^*$ and $\mathcal{M}^\flat$. Here
• $M^\flat$ is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(a, u^\flat_a(c), c) : c \in ]c^-_a(a), c^+_a(a)[\}$.

• $p_\flat = (a, u_\flat(a), c_\flat(a))$ is a degenerate solution with Morse index equal to one.

• $M^\sharp$ is a manifold of solutions with Morse index equal to one or to two, 
  \[ \{(a, u^\sharp_a(t), c^\sharp_a(t)) : u^\sharp_a(t) = t\psi + y^\sharp_a(t), \quad t \in J\}, \]
  with $c^\sharp_a : J \to \mathbb{R}$, $y^\sharp_a : J \to \{y \in \mathcal{H} : \int y\psi = 0\}$ and $J = ] - \frac{M}{\beta} - \varepsilon, M + \varepsilon[\] for some $\varepsilon, \varepsilon_2 > 0$.

• $p_\sharp = (a, u_\sharp(a), c_\sharp(a))$ is a degenerate solution with Morse index equal to one.

• $M^\ast$ is a manifold of nondegenerate solutions with Morse index equal to one, which is a graph $\{(a, u^\ast_a(c), c) : c \in ]c^-\ast_a(a), c^+\ast(a)[\}$.

• $p_\ast = (a, u_\ast(a), c_\ast(a))$ is a degenerate solution with Morse index equal to zero.

• $M^*\ast\ast$ is the manifold of stable solutions, which is a graph $\{(a, u^\ast\ast_a(c), c) : c \in ]c^-\ast\ast_a(a), c^+\ast\ast(a)[\}$.

• $p^\ast_\ast = (a, u^\ast_\ast(a), c^\ast_\ast(a))$ is a degenerate solution with Morse index equal to zero.

We have $(c^\flat_a)'(0) < 0$ and 
  \[ \lim_{t \to - \frac{M}{\beta} - \varepsilon_\flat} (a, u^\flat_a(t), c^\flat_a(t)) = (a, u_\flat(a), c_\flat(a)), \]
  \[ \lim_{t \to M + \varepsilon_2} (a, u^\flat_a(t), c^\flat_a(t)) = (a, u_\flat(a), c_\flat(a)). \]

In particular, if $|c|$ is sufficiently small, then (1) has at least four solutions.

Theorem 5.2 is illustrated in Figure 2.

Sketch of the proof. Using Lemma 7.3 of [6], we introduce a chart $]\lambda_2 - \varepsilon, \lambda_2 + \varepsilon[\times] - \frac{M}{\beta} - \hat{\varepsilon}, M + \hat{\varepsilon}[\}$ in the $(a, t)$ plane, around $(\lambda_2, 0)$, to describe the solutions of (1) in a neighborhood of $(\lambda_2, 0, 0)$. Suppose $\lambda_2 < a < \lambda_2 + \delta$. We start at $(a, 0, 0)$ and vary the parameter $t$ in $] - \frac{M}{\beta} - \hat{\varepsilon}, M + \hat{\varepsilon}[\}$ to follow the solutions of (1). By choosing $\delta$ small enough, we can guarantee that when $t$ reaches $- \frac{M}{\beta} - \hat{\varepsilon}$ and $M + \hat{\varepsilon}$ we can switch to the parameter $c$ to follow
the solutions of (1). Indeed, consider the curve $D_\varsigma$ of degenerate solutions with Morse index equal to one, constructed in Lemma 7.2 of [6]. A small enough choice of $\delta$ will make the projection of $D_\varsigma$ in the chart not intersect either $(a, -\frac{M}{\beta} - \tilde{\epsilon})$ or $(a, M + \tilde{\epsilon})$. Arguing as in the proof of Proposition 7.5 of [6], we know the solutions with coordinates $(a, -\frac{M}{\beta} - \tilde{\epsilon})$ and $(a, M + \tilde{\epsilon})$ are nondegenerate and have Morse index equal to one. We also know when we arrive at the solution with coordinates $(a, M + \tilde{\epsilon})$ we have to increase $c$, and when we arrive at the solution with coordinates $(a, -\frac{M}{\beta} - \tilde{\epsilon})$ we have to decrease $c$, to follow the solutions out of the chart. By further reducing $\delta$, if necessary, Lemma 7.4 of [6] together with Proposition 4.4, assure there are no degenerate solutions with Morse index equal to one when we use the parameter $c$ to follow the solutions outside the chart. If we find a degenerate solution it will have to have Morse index equal to zero. But we know these lie on $D_-^\ast$ and $D_+^\ast$. So one can finish by arguing as in the proof of Theorem 5.1. 

References


