

# Asymptotic behaviour of solutions to evolution problems with nonlocal diffusion

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**Abstract** In these notes we review recent results concerning solutions to nonlocal evolution equations with different boundary conditions, Dirichlet or Neumann and even for the Cauchy problem. We deal with existence/uniqueness of solutions and their asymptotic behavior. We also review some recent results concerning limits of solutions to nonlocal equations when a rescaling parameter goes to zero. We recover in these limits some of the most frequently used diffusion models: the heat equation with Neumann or Dirichlet boundary conditions.

*To the memory of Fuensanta Andreu and her beautiful smile.*

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## CHAPTER 1

### Introduction

First, let us briefly introduce the prototype of nonlocal problem that will be considered along this work.

Let  $J : \mathbb{R}^N \rightarrow \mathbb{R}$  be a nonnegative, radial, continuous function with  $\int_{\mathbb{R}^N} J(z) dz = 1$ . Nonlocal evolution equations of the form

$$(1.1) \quad u_t(x, t) = (J * u - u)(x, t) = \int_{\mathbb{R}^N} J(x - y)u(y, t) dy - u(x, t),$$

and variations of it, have been recently widely used to model diffusion processes. More precisely, as stated in [27], if  $u(x, t)$  is thought of as a density at the point  $x$  at time  $t$  and  $J(x - y)$  is thought of as the probability distribution of jumping from location  $y$  to location  $x$ , then  $\int_{\mathbb{R}^N} J(y - x)u(y, t) dy = (J * u)(x, t)$  is the rate at which individuals are arriving at position  $x$  from all other places and  $-u(x, t) = -\int_{\mathbb{R}^N} J(y - x)u(x, t) dy$  is the rate at which they are leaving location  $x$  to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density  $u$  satisfies equation (1.1). For recent references on nonlocal diffusion see, [6], [7], [8], [11], [13], [27], and references therein.

These type of problems have been used to model very different applied situations, for example in biology ([11], [38]), image processing ([37], [30]), particle systems ([9]), coagulation models ([29]), etc.

Concerning boundary conditions for nonlocal problems we consider a bounded smooth domain  $\Omega \subset \mathbb{R}^N$  and look at the nonlocal problem

$$(1.2) \quad \begin{aligned} u_t(x, t) &= \int_{\mathbb{R}^N} J(x - y)u(y, t) dy - u(x, t), & x \in \Omega, \quad t > 0, \\ u(x, t) &= 0, & x \notin \Omega, \quad t > 0, \\ u(x, 0) &= u_0(x), & x \in \Omega. \end{aligned}$$

In this model we have that diffusion takes place in the whole  $\mathbb{R}^N$  but we impose that  $u$  vanishes outside  $\Omega$ . This is the analogous of what is called Dirichlet boundary conditions for the heat equation. However, the boundary data is not understood in the usual sense, since we are not imposing that  $u|_{\partial\Omega} = 0$ .

Let us turn our attention to Neumann boundary conditions. We study

$$(1.3) \quad \begin{aligned} u_t(x, t) &= \int_{\Omega} J(x - y)(u(y, t) - u(x, t)) dy, & x \in \Omega, t > 0, \\ u(x, 0) &= u_0(x), & x \in \Omega. \end{aligned}$$

In this model we have that the integral terms take into account the diffusion inside  $\Omega$ . In fact, as we have explained the integral  $\int J(x - y)(u(y, t) - u(x, t)) dy$  takes into account the individuals arriving or leaving position  $x$  from other places. Since we are integrating in  $\Omega$ , we are imposing that diffusion takes place only in  $\Omega$ . The individuals may not enter nor leave  $\Omega$ . This is the analogous of what is called homogeneous Neumann boundary conditions in the literature.

Here we review some results concerning existence and uniqueness for these models and their asymptotic behavior as  $t \rightarrow \infty$ . These results says that these problems are well posed in appropriate functional spaces although they do not have a smoothing property. Moreover, the asymptotic behavior for the linear nonlocal models coincide with the one that holds for the heat equation. We will also review some recent results concerning limits of nonlocal problems when a scaling parameter (that measures the radius of influence of the nonlocal term) goes to zero. We recover in these limits some well known diffusion problems, namely, the heat equation with Neumann or Dirichlet boundary conditions.

The content of these notes summarizes the research of the author in the last years and is contained in [1], [2], [3], [12], [18], [19], [20], [21], [34], [35], [40]. We refer to these papers or to the book

F. Andreu-Vaillo – J. M. Mazón – J. D. Rossi – J. J. Toledo-Melero. **Nonlocal Diffusion Problems**. American Mathematical Society. Mathematical Surveys and Monographs 2010. Vol. 165.

for extra details and further references.

There is a huge amount of papers dealing with nonlocal problems. Among them we quote [5], [8], [16], [13], [22], [23], [25], [26] and [42], devoted to travelling front type solutions to the parabolic problem in  $\Omega = \mathbb{R}$ , and [14], [15], [24], [39], which dealt with source term of logistic type, bistable or power-like nonlinearity. The particular instance of the parabolic problem in  $\mathbb{R}^N$  is considered in [12], [35], while the “Neumann” boundary condition for the same problem is treated in [1], [20] and [21]. See also [34] for the appearance of convective terms and [17], [18] for interesting features in other related nonlocal problems. We finally mention the paper [33], where some logistic equations and systems of Lotka-Volterra type are studied. There is also an increasing interest in free boundary problems and regularity issues for nonlocal problems. We refer to [4], [10], [41], but we are not dealing with such issues in the present work.

The Bibliography of this work does not escape the usual rule of being incomplete. In general, we have listed those papers which are more close to the topics discussed here. But, even for those papers, the list is far from being exhaustive and we apologize for omissions.

## CHAPTER 2

### The Cauchy problem

The aim of this chapter is to study the asymptotic behavior of solutions of a nonlocal diffusion operator in the whole  $\mathbb{R}^N$ .

**0.1. The Cauchy problem.** We will consider the linear nonlocal diffusion problem presented in the Introduction

$$(2.1) \quad \begin{aligned} u_t(x, t) &= J * u - u(x, t) = \int_{\mathbb{R}^N} J(x - y)u(y, t) dy - u(x, t), \\ u(x, 0) &= u_0(x), \end{aligned}$$

We will understand a solution of (2.1) as a function

$$u \in C^0([0, +\infty); L^1(\mathbb{R}^N))$$

that verifies (2.1) in the integral sense, see Theorem 3. Our first result states that the decay rate as  $t$  goes to infinity of solutions of this nonlocal problem is determined by the behavior of the Fourier transform of  $J$  near the origin. The asymptotic decays are the same as the ones that hold for solutions of the evolution problem with right hand side given by a power of the laplacian.

In the sequel we denote by  $\widehat{f}$  the Fourier transform of  $f$ . Let us recall our hypotheses on  $J$  that we will assume throughout this chapter,

$$(H) \quad J \in C(\mathbb{R}^N, \mathbb{R}) \text{ is a nonnegative, radial function with } \int_{\mathbb{R}^N} J(x) dx = 1.$$

This means that  $J$  is a radial density probability which implies obviously that  $|\widehat{J}(\xi)| \leq 1$  with  $\widehat{J}(0) = 1$ , and we shall assume that  $\widehat{J}$  has an expansion of the form

$$\widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$$

for  $\xi \rightarrow 0$  ( $A > 0$ ). Remark that in this case, (H) implies also that  $0 < \alpha \leq 2$  and  $\alpha \neq 1$  if  $J$  has a first momentum.

The main result of this chapter reads as follows,

**THEOREM 1.** *Let  $u$  be a solution of (2.1) with  $u_0, \widehat{u}_0 \in L^1(\mathbb{R}^N)$ . If there exist  $A > 0$  and  $0 < \alpha \leq 2$  such that*

$$(2.2) \quad \widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha), \quad \xi \rightarrow 0,$$

then the asymptotic behavior of  $u(x, t)$  is given by

$$\lim_{t \rightarrow +\infty} t^{N/\alpha} \max_x |u(x, t) - v(x, t)| = 0,$$

where  $v$  is the solution of  $v_t(x, t) = -A(-\Delta)^{\alpha/2}v(x, t)$  with initial condition  $v(x, 0) = u_0(x)$ . Moreover, we have

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C t^{-N/\alpha},$$

and the asymptotic profile is given by

$$\lim_{t \rightarrow +\infty} \max_y |t^{N/\alpha} u(yt^{1/\alpha}, t) - \|u_0\|_{L^1} G_A(y)| = 0,$$

where  $G_A(y)$  satisfies  $\widehat{G}_A(\xi) = e^{-A|\xi|^\alpha}$ .

In the special case  $\alpha = 2$ , the decay rate is  $t^{-N/2}$  and the asymptotic profile is a gaussian  $G_A(y) = (4\pi A)^{N/2} \exp(-A|y|^2/4)$  with  $A \cdot \text{Id} = -(1/2)D^2 \widehat{J}(0)$ . Note that in this case (that occurs, for example, when  $J$  is compactly supported) the asymptotic behavior is the same as the one for solutions of the heat equation and, as happens for the heat equation, the asymptotic profile is a gaussian.

The decay in  $L^\infty$  of the solutions together with the conservation of mass give the decay of the  $L^p$ -norms by interpolation. As a consequence of the previous theorem, we find that this decay is analogous to the decay of the evolution given by the fractional laplacian, that is,

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq C t^{-\frac{N}{\alpha}(1-\frac{1}{p})},$$

see Corollary 11. We refer to [16] for the decay of the  $L^p$ -norms for the fractional laplacian.

We shall make an extensive use of the Fourier transform in order to obtain explicit solutions in frequency formulation. Let us recall that if  $f \in L^1(\mathbb{R}^N)$  then  $\widehat{f}$  and  $\check{f}$  are bounded and continuous, where  $\widehat{f}$  is the Fourier transform of  $f$  and  $\check{f}$  its inverse Fourier transform. Moreover,

$$\lim_{|\xi| \rightarrow \infty} \widehat{f}(\xi) = 0 \quad \text{and} \quad \lim_{|x| \rightarrow \infty} \check{f}(x) = 0.$$

We begin by collecting some properties of the function  $J$ .

LEMMA 2. *Let  $J$  satisfy hypotheses (H). Then,*

- i)  $|\widehat{J}(\xi)| \leq 1$ ,  $\widehat{J}(0) = 1$ .
- ii) *If  $\int_{\mathbb{R}^N} J(x)|x| dx < +\infty$  then*

$$\left( \nabla_\xi \widehat{J} \right)_i (0) = -i \int_{\mathbb{R}^N} x_i J(x) dx = 0$$

*and if  $\int_{\mathbb{R}^N} J(x)|x|^2 dx < +\infty$  then*

$$\left( D^2 \widehat{J} \right)_{ij} (0) = - \int_{\mathbb{R}^N} x_i x_j J(x) dx,$$

therefore  $(D^2\widehat{J})_{ij}(0) = 0$  when  $i \neq j$  and  $(D^2\widehat{J})_{ii}(0) \neq 0$ . Hence the Hessian matrix of  $\widehat{J}$  at the origin is given by

$$D^2\widehat{J}(0) = -\left(\frac{1}{N} \int_{\mathbb{R}^N} |x|^2 J(x) dx\right) \cdot \text{Id}.$$

iii) If  $\widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$  then necessarily  $\alpha \in (0, 2]$ , and if  $J$  has a first momentum, then  $\alpha \neq 1$ . Finally, if  $\alpha = 2$ , then

$$A \cdot \text{Id} = -(1/2)(D^2\widehat{J})_{ij}(0).$$

PROOF. Points i) and ii) are rather straightforward (recall that  $J$  is radially symmetric). Now we turn to iii). Let us recall a well-known probability lemma that says that if  $\widehat{J}$  has an expansion of the form,

$$\widehat{J}(\xi) = 1 + i\langle a, \xi \rangle - \frac{1}{2}\langle \xi, B\xi \rangle + o(|\xi|^2),$$

then  $J$  has a second momentum and we have

$$a_i = \int x_i J(x) dx, \quad B_{ij} = \int x_i x_j J(x) dx < \infty.$$

Thus if iii) holds for some  $\alpha > 2$ , it would turn out that the second moment of  $J$  is null, which would imply that  $J \equiv 0$ , a contradiction. Finally, when  $\alpha = 2$ , then clearly  $B_{ij} = -(D^2\widehat{J})_{ij}(0)$  hence the result since by symmetry, the Hessian is diagonal.  $\square$

Now, we prove existence and uniqueness of solutions using the Fourier transform.

**THEOREM 3.** *Let  $u_0 \in L^1(\mathbb{R}^N)$  such that  $\widehat{u}_0 \in L^1(\mathbb{R}^N)$ . There exists a unique solution  $u \in C^0([0, \infty); L^1(\mathbb{R}^N))$  of (2.1), and it is given by*

$$\widehat{u}(\xi, t) = e^{(\widehat{J}(\xi)-1)t} \widehat{u}_0(\xi).$$

PROOF. We have

$$u_t(x, t) = J * u - u(x, t) = \int_{\mathbb{R}^N} J(x-y)u(y, t) dy - u(x, t).$$

Applying the Fourier transform to this equation we obtain

$$\widehat{u}_t(\xi, t) = \widehat{u}(\xi, t)(\widehat{J}(\xi) - 1).$$

Hence,

$$\widehat{u}(\xi, t) = e^{(\widehat{J}(\xi)-1)t} \widehat{u}_0(\xi).$$

Since  $\widehat{u}_0 \in L^1(\mathbb{R}^N)$  and  $e^{(\widehat{J}(\xi)-1)t}$  is continuous and bounded, the result follows by taking the inverse of the Fourier transform.  $\square$

**REMARK 4.** One can also understand solutions of (2.1) directly in Fourier variables. This concept of solution is equivalent to the integral one in the original variables under our hypotheses on the initial condition.

Now we prove a lemma concerning the fundamental solution of (2.1).

LEMMA 5. *Let  $J \in \mathcal{S}(\mathbb{R}^N)$ , the space of rapidly decreasing functions. The fundamental solution of (2.1), that is the solution of (2.1) with initial condition  $u_0 = \delta_0$ , can be decomposed as*

$$(2.3) \quad w(x, t) = e^{-t}\delta_0(x) + v(x, t),$$

with  $v(x, t)$  smooth. Moreover, if  $u$  is a solution of (2.1) it can be written as

$$u(x, t) = (w * u_0)(x, t) = \int_{\mathbb{R}^N} w(x - z, t)u_0(z) dz.$$

PROOF. By the previous result we have

$$\widehat{w}_t(\xi, t) = \widehat{w}(\xi, t)(\widehat{J}(\xi) - 1).$$

Hence, as the initial datum verifies  $\widehat{u}_0 = \widehat{\delta}_0 = 1$ ,

$$\widehat{w}(\xi, t) = e^{(\widehat{J}(\xi)-1)t} = e^{-t} + e^{-t}(e^{\widehat{J}(\xi)t} - 1).$$

The first part of the lemma follows applying the inverse Fourier transform in  $\mathcal{S}(\mathbb{R}^N)$ .

To finish the proof we just observe that  $w * u_0$  is a solution of (2.1) (just use Fubini's theorem) with  $(w * u_0)(x, 0) = u_0(x)$ .  $\square$

REMARK 6. The above proof together with the fact that  $\widehat{J}(\xi) \rightarrow 0$  (since  $J \in L^1(\mathbb{R}^N)$ ) shows that if  $\widehat{J} \in L^1(\mathbb{R}^N)$  then the same decomposition (2.3) holds and the result also applies.

Next, we prove the first part of our main result.

THEOREM 7. *Let  $u$  be a solution of (2.1) with  $u_0, \widehat{u}_0 \in L^1(\mathbb{R}^N)$ . If*

$$\widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha), \quad \xi \rightarrow 0,$$

the asymptotic behavior of  $u(x, t)$  is given by

$$\lim_{t \rightarrow +\infty} t^{N/\alpha} \max_x |u(x, t) - v(x, t)| = 0,$$

where  $v$  is the solution of  $v_t(x, t) = -A(-\Delta)^{\alpha/2}v(x, t)$  with initial condition  $v(x, 0) = u_0(x)$ .

PROOF. As in the proof of the previous lemma we have

$$\widehat{u}_t(\xi, t) = \widehat{u}(\xi, t)(\widehat{J}(\xi) - 1).$$

Hence

$$\widehat{u}(\xi, t) = e^{(\widehat{J}(\xi)-1)t}\widehat{u}_0(\xi).$$

On the other hand, let  $v(x, t)$  be a solution of

$$v_t(x, t) = -A(-\Delta)^{\alpha/2}v(x, t),$$



with the same initial datum  $v(x, 0) = u_0(x)$ . Solutions of this equation are understood in the sense that

$$\widehat{v}(\xi, t) = e^{-A|\xi|^\alpha t} \widehat{u}_0(\xi).$$

Hence in Fourier variables,

$$\begin{aligned} \int_{\mathbb{R}^N} |\widehat{u} - \widehat{v}|(\xi, t) d\xi &= \int_{\mathbb{R}^N} \left| \left( e^{t(\widehat{J}(\xi)-1)} - e^{-A|\xi|^\alpha t} \right) \widehat{u}_0(\xi) \right| d\xi \\ &\leq \int_{|\xi| \geq r(t)} \left| \left( e^{t(\widehat{J}(\xi)-1)} - e^{-A|\xi|^\alpha t} \right) \widehat{u}_0(\xi) \right| d\xi \\ &\quad + \int_{|\xi| < r(t)} \left| \left( e^{t(\widehat{J}(\xi)-1)} - e^{-A|\xi|^\alpha t} \right) \widehat{u}_0(\xi) \right| d\xi = I + II. \end{aligned}$$

To get a bound for  $I$  we proceed as follows, we decompose it in two parts,

$$I \leq \int_{|\xi| \geq r(t)} |e^{-A|\xi|^\alpha t} \widehat{u}_0(\xi)| d\xi + \int_{|\xi| \geq r(t)} \left| e^{t(\widehat{J}(\xi)-1)} \widehat{u}_0(\xi) \right| d\xi = I_1 + I_2.$$

First, we deal with  $I_1$ . We have,

$$t^{N/\alpha} \int_{|\xi| > r(t)} e^{-A|\xi|^\alpha t} |\widehat{u}_0(\xi)| d\xi \leq \|\widehat{u}_0\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta| > r(t)t^{1/\alpha}} e^{-A|\eta|^\alpha} \rightarrow 0,$$

as  $t \rightarrow \infty$  if we impose that

$$(2.4) \quad r(t)t^{1/\alpha} \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

Now, remark that from our hypotheses on  $J$  we have that  $\widehat{J}$  verifies

$$\widehat{J}(\xi) \leq 1 - A|\xi|^\alpha + |\xi|^\alpha h(\xi),$$

where  $h$  is bounded and  $h(\xi) \rightarrow 0$  as  $\xi \rightarrow 0$ . Hence there exists  $D > 0$  such that

$$\widehat{J}(\xi) \leq 1 - D|\xi|^\alpha, \quad \text{for } |\xi| \leq a,$$

and  $\delta > 0$  such that

$$\widehat{J}(\xi) \leq 1 - \delta, \quad \text{for } |\xi| \geq a.$$

Therefore,  $I_2$  can be bounded by

$$\begin{aligned} \int_{|\xi| \geq r(t)} \left| e^{t(\widehat{J}(\xi)-1)} \widehat{u}_0(\xi) \right| d\xi &\leq \int_{a \geq |\xi| \geq r(t)} \left| e^{t(\widehat{J}(\xi)-1)} \widehat{u}_0(\xi) \right| d\xi \\ &+ \int_{|\xi| \geq a} \left| e^{t(\widehat{J}(\xi)-1)} \widehat{u}_0(\xi) \right| d\xi \leq \int_{a \geq |\xi| \geq r(t)} \left| e^{t(\widehat{J}(\xi)-1)} \widehat{u}_0(\xi) \right| d\xi + Ce^{-\delta t}. \end{aligned}$$

Using this bound and changing variables,  $\eta = \xi t^{1/\alpha}$ ,

$$\begin{aligned} t^{N/\alpha} I_2 &\leq C \int_{at^{1/\alpha} \geq |\eta| \geq t^{1/\alpha} r(t)} |e^{-D|\eta|^\alpha} \widehat{u}_0(\eta t^{-1/\alpha})| d\eta + t^{N/\alpha} Ce^{-\delta t} \\ &\leq C \int_{|\eta| \geq t^{1/\alpha} r(t)} e^{-D|\eta|^\alpha} d\eta + t^{N/\alpha} Ce^{-\delta t}, \end{aligned}$$

and then

$$t^{N/\alpha} I_2 \rightarrow 0, \quad \text{as } t \rightarrow \infty,$$

if (2.4) holds.

Now we estimate  $II$  as follows,

$$\begin{aligned} & t^{N/\alpha} \int_{|\xi| < r(t)} |e^{(\widehat{J}(\xi) - 1 + A|\xi|^\alpha)t} - 1| e^{-A|\xi|^\alpha t} |\widehat{u}_0(\xi)| d\xi \\ & \leq C t^{N/\alpha} \int_{|\xi| < r(t)} t |\xi|^\alpha h(\xi) e^{-A|\xi|^\alpha t} d\xi, \end{aligned}$$

provided we impose

$$(2.5) \quad t (r(t))^\alpha h(r(t)) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In this case, we have

$$t^{N/\alpha} II \leq C \int_{|\eta| < r(t)t^{1/\alpha}} |\eta|^\alpha h(\eta/t^{1/\alpha}) e^{-A|\eta|^\alpha} d\eta,$$

and we use dominated convergence,  $h(\eta/t^{1/\alpha}) \rightarrow 0$  as  $t \rightarrow \infty$  while the integrand is dominated by  $\|h\|_\infty |\eta|^\alpha \exp(-c|\eta|^\alpha)$ , which belongs to  $L^1(\mathbb{R}^N)$ .

This shows that

$$(2.6) \quad t^{N/\alpha} (I + II) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

provided we can find a  $r(t) \rightarrow 0$  as  $t \rightarrow \infty$  which fulfills both conditions (2.4) and (2.5). This is done in Lemma 8, which is postponed just after the end of the present proof. To conclude, we only have to observe that from (2.6) we obtain

$$t^{N/\alpha} \max_x |u(x, t) - v(x, t)| \leq t^{N/\alpha} \int_{\mathbb{R}^N} |\widehat{u} - \widehat{v}|(\xi, t) d\xi \rightarrow 0, \quad t \rightarrow \infty,$$

which ends the proof of the theorem.  $\square$

The following Lemma shows that there exists a function  $r(t)$  satisfying (2.4) and (2.5), as required in the proof of the previous theorem.

LEMMA 8. *Given a function  $h \in C(\mathbb{R}, \mathbb{R})$  such that  $h(\rho) \rightarrow 0$  as  $\rho \rightarrow 0$  with  $h(\rho) > 0$  for small  $\rho$ , there exists a function  $r$  with  $r(t) \rightarrow 0$  as  $t \rightarrow \infty$  which satisfies*

$$\lim_{t \rightarrow \infty} r(t) t^{1/\alpha} = \infty$$

and

$$\lim_{t \rightarrow \infty} t (r(t))^\alpha h(r(t)) = 0.$$

PROOF. For fixed  $t$  large enough, we choose  $r(t)$  as a small solution of

$$(2.7) \quad r(h(r))^{1/(2\alpha)} = t^{-1/\alpha}.$$

This equation defines a function  $r = r(t)$  which, by continuity arguments, goes to zero as  $t$  goes to infinity. Indeed, if there exists  $t_n \rightarrow \infty$  with no solution of (2.7) for  $r \in (0, \delta)$  then  $h(r) \equiv 0$  in  $(0, \delta)$  a contradiction.  $\square$

REMARK 9. In the case when  $h(t) = t^s$  with  $s > 0$ , we can look for a function  $h$  of power-type,  $r(t) = t^\beta$  with  $\beta < 0$  and the two conditions read as follows:

$$(2.8) \quad \beta + 1/\alpha > 0, \quad 1 + \beta\alpha + s\beta < 0.$$

This implies that  $\beta \in (-1/\alpha, -1/(\alpha + s))$  which is of course always possible.

As a consequence of Theorem 7, we obtain the following corollary which completes the results gathered in the main theorem.

COROLLARY 10. *If  $\widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$ ,  $\xi \rightarrow 0$ ,  $0 < \alpha \leq 2$ , the asymptotic behavior of solutions of (2.1) is given by*

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{t^{N/\alpha}}.$$

Moreover, the asymptotic profile is given by

$$\lim_{t \rightarrow +\infty} \max_y |t^{N/\alpha} u(yt^{1/\alpha}, t) - \|u_0\|_{L^1} G_A(y)| = 0,$$

where  $G_A(y)$  satisfies  $\widehat{G}_A(\xi) = e^{-A|\xi|^\alpha}$ .

PROOF. From Theorem 7 we obtain that the asymptotic behavior is the same as the one for solutions of the evolution given by the fractional laplacian.

It is easy to check that this asymptotic behavior is exactly the one described in the statement of the corollary. Indeed, in Fourier variables we have for  $t \rightarrow \infty$

$$\widehat{v}(t^{-1/\alpha}\eta, t) = e^{-A|\eta|^\alpha} \widehat{u}_0(\eta t^{-1/\alpha}) \longrightarrow e^{-A|\eta|^\alpha} \widehat{u}_0(0) = e^{-A|\eta|^\alpha} \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Therefore

$$\lim_{t \rightarrow +\infty} \max_y |t^{N/\alpha} v(yt^{1/\alpha}, t) - \|u_0\|_{L^1} G_A(y)| = 0,$$

where  $G_A(y)$  satisfies  $\widehat{G}_A(\xi) = e^{-A|\xi|^\alpha}$ .  $\square$

Now we find the decay rate in  $L^p$  of solutions of (2.1).

COROLLARY 11. *Let  $1 < p < \infty$ . If  $\widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$ ,  $\xi \rightarrow 0$ ,  $0 < \alpha \leq 2$ , then, the decay of the  $L^p$ -norm of the solution of (2.1) is given by*

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq C t^{-\frac{N}{\alpha} \left(1 - \frac{1}{p}\right)}.$$

PROOF. By interpolation, we have

$$\|u\|_{L^p(\mathbb{R}^N)} \leq \|u\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p}} \|u\|_{L^\infty(\mathbb{R}^N)}^{1 - \frac{1}{p}}.$$

As (2.1) preserves the  $L^1$  norm, the result follows from the previous results that give the decay in  $L^\infty$  of the solutions.  $\square$



## CHAPTER 3

### The Dirichlet problem.

Next we consider a bounded smooth domain  $\Omega \subset \mathbb{R}^N$  and impose boundary conditions to our model. From now on we assume that  $J$  is continuous.

Consider the nonlocal problem

$$(3.1) \quad \begin{aligned} u_t(x, t) &= \int_{\mathbb{R}^N} J(x-y)u(y, t) dy - u(x, t), & x \in \Omega, \quad t > 0, \\ u(x, t) &= 0, & x \notin \Omega, \quad t > 0, \\ u(x, 0) &= u_0(x), & x \in \Omega. \end{aligned}$$

In this model we have that diffusion takes place in the whole  $\mathbb{R}^N$  but we impose that  $u$  vanishes outside  $\Omega$ . This is the analogous of what is called Dirichlet boundary conditions for the heat equation. However, the boundary data is not understood in the usual sense, see Remark 17. As for the Cauchy problem we understand solutions in an integral sense, see Theorem 14.

In this case we find an exponential decay given by the first eigenvalue of an associated problem and the asymptotic behavior of solutions is described by the unique (up to a constant) associated eigenfunction. Let  $\lambda_1 = \lambda_1(\Omega)$  be given by

$$(3.2) \quad \lambda_1 = \inf_{u \in L^2(\Omega)} \frac{\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)(u(x) - u(y))^2 dx dy}{\int_{\Omega} (u(x))^2 dx}$$

and  $\phi_1$  an associated eigenfunction (a function where the infimum is attained).

**THEOREM 12.** *For every  $u_0 \in L^1(\Omega)$  there exists a unique solution  $u$  of (3.1) such that  $u \in C([0, \infty); L^1(\Omega))$ . Moreover, if  $u_0 \in L^2(\Omega)$ , solutions decay to zero as  $t \rightarrow \infty$  with an exponential rate*

$$(3.3) \quad \|u(\cdot, t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} e^{-\lambda_1 t}.$$

*If  $u_0$  is continuous, positive and bounded then there exist positive constants  $C$  and  $C^*$  such that*

$$(3.4) \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C e^{-\lambda_1 t}$$

and

$$(3.5) \quad \lim_{t \rightarrow \infty} \max_x |e^{\lambda_1 t} u(x, t) - C^* \phi_1(x)| = 0.$$

A solution of the Dirichlet problem is defined as follows:  $u \in C([0, \infty); L^1(\Omega))$  satisfying

$$\begin{aligned} u_t(x, t) &= \int_{\mathbb{R}^N} J(x-y)u(y, t) dy - u(x, t), & x \in \Omega, \quad t > 0, \\ u(x, t) &= 0, & x \notin \Omega, \quad t > 0, \\ u(x, 0) &= u_0(x) & x \in \Omega. \end{aligned}$$

Before studying the asymptotic behavior, we shall first derive existence and uniqueness of solutions, which is a consequence of Banach's fixed point theorem.

Fix  $t_0 > 0$  and consider the Banach space

$$X_{t_0} = \{w \in C([0, t_0]; L^1(\Omega))\}$$

with the norm

$$|||w||| = \max_{0 \leq t \leq t_0} \|w(\cdot, t)\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator  $\mathcal{T} : X_{t_0} \rightarrow X_{t_0}$  defined by

$$\begin{aligned} \mathcal{T}_{w_0}(w)(x, t) &= w_0(x) + \int_0^t \int_{\mathbb{R}^N} J(x-y)(w(y, s) - w(x, s)) dy ds, \\ \mathcal{T}_{w_0}(w)(x, t) &= 0, \quad x \notin \Omega. \end{aligned}$$

LEMMA 13. *Let  $w_0, z_0 \in L^1(\Omega)$  and  $w, z \in X_{t_0}$ , then there exists a constant  $C$  depending on  $J$  and  $\Omega$  such that*

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \leq Ct_0 |||w - z||| + \|w_0 - z_0\|_{L^1(\Omega)}.$$

PROOF. We have

$$\begin{aligned} \int_{\Omega} |\mathcal{T}_{w_0}(w)(x, t) - \mathcal{T}_{z_0}(z)(x, t)| dx &\leq \int_{\Omega} |w_0 - z_0|(x) dx \\ &+ \int_{\Omega} \left| \int_0^t \int_{\mathbb{R}^N} J(x-y) \left[ (w(y, s) - z(y, s)) \right. \right. \\ &\quad \left. \left. - (w(x, s) - z(x, s)) \right] dy ds \right| dx. \end{aligned}$$

Hence, taking into account that  $w$  and  $z$  vanish outside  $\Omega$ ,

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \leq \|w_0 - z_0\|_{L^1(\Omega)} + Ct_0 |||w - z|||,$$

as we wanted to prove.  $\square$

THEOREM 14. *For every  $u_0 \in L^1(\Omega)$  there exists a unique solution  $u$ , such that  $u \in C([0, \infty); L^1(\Omega))$ .*

PROOF. We check first that  $\mathcal{T}_{u_0}$  maps  $X_{t_0}$  into  $X_{t_0}$ . Taking  $z_0, z \equiv 0$  in Lemma 13 we get that  $\mathcal{T}(w) \in C([0, t_0]; L^1(\Omega))$ .

Choose  $t_0$  such that  $Ct_0 < 1$ . Now taking  $z_0 \equiv w_0 \equiv u_0$  in Lemma 13 we get that  $\mathcal{T}_{u_0}$  is a strict contraction in  $X_{t_0}$  and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem in the interval  $[0, t_0]$ . To extend the solution to  $[0, \infty)$  we may take as initial data  $u(x, t_0) \in L^1(\Omega)$  and obtain a solution up to  $[0, 2t_0]$ . Iterating this procedure we get a solution defined in  $[0, \infty)$ .  $\square$

Next we look for steady states of (3.1).

PROPOSITION 15.  $u \equiv 0$  is the unique stationary solution of (3.1).

PROOF. Let  $u$  be a stationary solution of (3.1). Then

$$0 = \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x)) dy, \quad x \in \Omega,$$

and  $u(x) = 0$  for  $x \notin \Omega$ . Hence, using that  $\int J = 1$  we obtain that for every  $x \in \mathbb{R}^N$  it holds,

$$u(x) = \int_{\mathbb{R}^N} J(x-y)u(y) dy.$$

This equation, together with  $u(x) = 0$  for  $x \notin \Omega$ , implies that  $u \equiv 0$ .  $\square$

Now, let us analyze the asymptotic behavior of the solutions. As there exists a unique stationary solution, it is expected that solutions converge to zero as  $t \rightarrow \infty$ . Our main concern will be the rate of convergence.

First, let us look the eigenvalue given by (3.2), that is we look for the first eigenvalue of

$$(3.6) \quad u(x) - \int_{\mathbb{R}^N} J(x-y)u(y) dy = \lambda_1 u(x).$$

This is equivalent to,

$$(3.7) \quad (1 - \lambda_1)u(x) = \int_{\mathbb{R}^N} J(x-y)u(y) dy.$$

Let  $T : L^2(\Omega) \rightarrow L^2(\Omega)$  be the operator given by

$$T(u)(x) := \int_{\mathbb{R}^N} J(x-y)u(y) dy.$$

In this definition we have extended by zero a function in  $L^2(\Omega)$  to the whole  $\mathbb{R}^N$ . Hence we are looking for the largest eigenvalue of  $T$ . Since  $T$  is compact this eigenvalue is attained at some function  $\phi_1(x)$  that turns out to be an eigenfunction for our original problem (3.6).

By taking  $|\phi_1|$  instead of  $\phi_1$  in (3.2) we may assume that  $\phi_1 \geq 0$  in  $\Omega$ . Indeed, one simply has to use the fact that  $(a - b)^2 \geq (|a| - |b|)^2$ .

Next, we analyze some properties of the eigenvalue problem (3.6).

PROPOSITION 16. *Let  $\lambda_1$  the first eigenvalue of (3.6) and denote by  $\phi_1(x)$  a corresponding non-negative eigenfunction. Then  $\phi_1(x)$  is strictly positive in  $\Omega$  and  $\lambda_1$  is a positive simple eigenvalue with  $\lambda_1 < 1$ .*

PROOF. In what follows, we denote by  $\bar{\phi}_1$  the natural continuous extension of  $\phi_1$  to  $\bar{\Omega}$ . We begin with the positivity of the eigenfunction  $\phi_1$ . Assume for contradiction that the set  $\mathbf{B} = \{x \in \Omega : \phi_1(x) = 0\}$  is non-void. Then, from the continuity of  $\phi_1$  in  $\Omega$ , we have that  $\mathbf{B}$  is closed. We next prove that  $\mathbf{B}$  is also open, and hence, since  $\Omega$  is connected, standard topological arguments allows to conclude that  $\Omega \equiv \mathbf{B}$  yielding to a contradiction. Consider  $x_0 \in \mathbf{B}$ . Since  $\phi_1 \geq 0$ , we obtain from (3.7) that  $\Omega \cap B_1(x_0) \in \mathbf{B}$ . Hence  $\mathbf{B}$  is open and the result follows. Analogous arguments apply to prove that  $\bar{\phi}_1$  is positive in  $\bar{\Omega}$ .

Assume now for contradiction that  $\lambda_1 \leq 0$  and denote by  $M^*$  the maximum of  $\bar{\phi}_1$  in  $\bar{\Omega}$  and by  $x^*$  a point where such maximum is attained. Assume for the moment that  $x^* \in \Omega$ . From Proposition 15, one can choose  $x^*$  in such a way that  $\phi_1(x) \neq M^*$  in  $\Omega \cap B_1(x^*)$ . By using (3.7) we obtain that,

$$M^* \leq (1 - \lambda_1)\phi_1(x^*) = \int_{\mathbb{R}^N} J(x^* - y)\phi_1(y) < M^*$$

and a contradiction follows. If  $x^* \in \partial\Omega$ , we obtain a similar contradiction after substituting and passing to the limit in (3.7) on a sequence  $\{x_n\} \in \Omega$ ,  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$ . To obtain the upper bound, assume that  $\lambda_1 \geq 1$ . Then, from (3.7) we obtain for every  $x \in \Omega$  that

$$0 \geq (1 - \lambda_1)\phi_1(x^*) = \int_{\mathbb{R}^N} J(x^* - y)\phi_1(y)$$

a contradiction with the positivity of  $\phi_1$ .

Finally, to prove that  $\lambda_1$  is a simple eigenvalue, let  $\phi_1 \neq \phi_2$  be two different eigenfunctions associated to  $\lambda_1$  and define

$$C^* = \inf\{C > 0 : \bar{\phi}_2(x) \leq C\bar{\phi}_1(x), x \in \bar{\Omega}\}.$$

The regularity of the eigenfunctions and the previous analysis shows that  $C^*$  is nontrivial and bounded. Moreover from its definition, there must exists  $x^* \in \bar{\Omega}$  such that  $\bar{\phi}_2(x^*) = C^*\bar{\phi}_1(x^*)$ . Define  $\phi(x) = C^*\phi_1(x) - \phi_2(x)$ . From the linearity of (3.6), we have that  $\phi$  is a non-negative eigenfunction associated to  $\lambda_1$  with  $\bar{\phi}(x^*) = 0$ . From the positivity of the eigenfunctions stated above, it must be  $\phi \equiv 0$ . Therefore,  $\phi_2(x) = C^*\phi_1(x)$  and the result follows. This completes the proof.  $\square$

REMARK 17. Note that the first eigenfunction  $\phi_1$  is strictly positive in  $\Omega$  (with positive continuous extension to  $\bar{\Omega}$ ) and vanishes outside  $\Omega$ . Therefore a discontinuity occurs on  $\partial\Omega$  and the boundary value is not taken in the usual "classical" sense.



PROOF OF THEOREM 12. Using the symmetry of  $J$ , we have

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{1}{2} \int_{\Omega} u^2(x, t) dx \right) &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)[u(y, t) - u(x, t)]u(x, t) dy dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)[u(y, t) - u(x, t)]^2 dy dx. \end{aligned}$$

From the definition of  $\lambda_1$ , (3.2), we get

$$\frac{\partial}{\partial t} \int_{\Omega} u^2(x, t) dx \leq -2\lambda_1 \int_{\Omega} u^2(x, t) dx.$$

Therefore

$$\int_{\Omega} u^2(x, t) dx \leq e^{-2\lambda_1 t} \int_{\Omega} u_0^2(x) dx$$

and we have obtained (3.3).

We now establish the decay rate and the convergence stated in (3.4) and (3.5) respectively. Consider a nontrivial and non-negative continuous initial data  $u_0(x)$  and let  $u(x, t)$  be the corresponding solution to (1.1). We first note that  $u(x, t)$  is a continuous function satisfying  $u(x, t) > 0$  for every  $x \in \Omega$  and  $t > 0$ , and the same holds for  $\bar{u}(x, t)$ , the unique natural continuous extension of  $u(x, t)$  to  $\bar{\Omega}$ . This instantaneous positivity can be obtained by using analogous topological arguments to those in Proposition 16.

In order to deal with the asymptotic analysis, it is more convenient to introduce the rescaled function  $v(x, t) = e^{\lambda_1 t} u(x, t)$ . By substituting in (1.1), we find that the function  $v(x, t)$  satisfies

$$(3.8) \quad v_t(x, t) = \int_{\mathbb{R}^N} J(x-y)v(y, t) dy - (1 - \lambda_1)v(x, t).$$

On the other hand, we have that  $C\phi_1(x)$  is a solution of (3.8) for every  $C \in \mathbb{R}$  and moreover, it follows from the eigenfunction analysis above, that the set of stationary solutions of (3.8) is given by  $\mathbf{S}^* = \{C\phi_1, C \in \mathbb{R}\}$ .

Define now for every  $t > 0$ , the function

$$C^*(t) = \inf\{C > 0 : v(x, t) \leq C\phi_1(x), x \in \Omega\}.$$

By definition and by using the linearity of equation (3.8), we have that  $C^*(t)$  is a non-increasing function. In fact, this is a consequence of the comparison principle applied to the solutions  $C^*(t_1)\phi_1(x)$  and  $v(x, t)$  for  $t$  larger than any fixed  $t_1 > 0$ . It implies that  $C^*(t_1)\phi_1(x) \geq v(x, t)$  for every  $t \geq t_1$ , and therefore,  $C^*(t_1) \geq C^*(t)$  for every  $t \geq t_1$ . In an analogous way, one can see that the function

$$C_*(t) = \sup\{C > 0 : v(x, t) \geq C\phi_1(x), x \in \Omega\},$$

is non-decreasing. These properties imply that both limits exist,

$$\lim_{t \rightarrow \infty} C^*(t) = K^* \quad \text{and} \quad \lim_{t \rightarrow \infty} C_*(t) = K_*,$$

and also provides the compactness of the orbits necessary in order passing to the limit (after subsequences if needed) to obtain that  $v(\cdot, t + t_n) \rightarrow w(\cdot, t)$  as  $t_n \rightarrow \infty$  uniformly on compact subsets in  $\Omega \times \mathbb{R}_+$  and that  $w(x, t)$  is a continuous function which satisfies (3.8). We also have for every  $g \in \omega(u_0)$  there holds,

$$K_*\phi_1(x) \leq g(x) \leq K^*\phi_1(x).$$

Moreover,  $C^*(t)$  plays a role of a Lyapunov function and this fact allows to conclude that  $\omega(u_0) \subset \mathbf{S}^*$  and the uniqueness of the convergence profile. In more detail, assume that  $g \in \omega(u_0)$  does not belong to  $\mathbf{S}^*$  and consider  $w(x, t)$  the solution of (3.8) with initial data  $g(x)$  and define

$$C^*(w)(t) = \inf\{C > 0 : w(x, t) \leq C\phi_1(x), x \in \Omega\}.$$

It is clear that  $W(x, t) = K^*\phi_1(x) - w(x, t)$  is a non-negative continuous solution of (3.8) and it becomes strictly positive for every  $t > 0$ . This implies that there exists  $t^* > 0$  such that  $C^*(w)(t^*) < K^*$  and by the convergence, the same holds before passing to the limit. Hence,  $C^*(t^* + t_j) < K^*$  if  $j$  is large enough and a contradiction with the properties of  $C^*(t)$  follows. The same arguments allow to establish the uniqueness of the convergence profile.  $\square$

**0.2. A linear Dirichlet problem with a rescale of the kernel.** Now, we study the following nonlocal nonhomogeneous ‘‘Dirichlet’’ boundary value problem: Given  $g(x, t)$  defined for  $x \in \mathbb{R}^N \setminus \Omega$  and  $u_0(x)$  defined for  $x \in \Omega$ , find  $u(x, t)$  such that

$$(3.9) \quad \begin{cases} u_t(x, t) = \int_{\mathbb{R}^N} J(x-y)(u(y, t) - u(x, t))dy, & x \in \Omega, t > 0, \\ u(x, t) = g(x, t), & x \notin \Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases}$$

In this model we prescribe the values of  $u$  outside  $\Omega$  which is the analogous of prescribing the so called Dirichlet boundary conditions for the classical heat equation. However, the boundary data is not understood in the usual sense as we will see in Remark 17 below. As explained before in this model the right hand side models the diffusion, the integral  $\int J(x-y)(u(y, t) - u(x, t)) dy$  takes into account the individuals arriving or leaving position  $x \in \Omega$  from or to other places while we are prescribing the values of  $u$  outside the domain  $\Omega$  by imposing  $u = g$  for  $x \notin \Omega$ . When  $g = 0$  we get that any individuals that leave  $\Omega$  die, this is the case when  $\Omega$  is surrounded by a hostile environment.

Existence and uniqueness of solutions of (3.9) is proved by a fixed point argument. Also a comparison principle can be obtained. The proofs of these facts is analogous to the previous ones and hence we omit them.

Let us consider the classical Dirichlet problem for the heat equation,

$$(3.10) \quad \begin{cases} v_t(x, t) - \Delta v(x, t) = 0, & x \in \Omega, t > 0, \\ v(x, t) = g(x, t), & x \in \partial\Omega, t > 0, \\ v(x, 0) = u_0(x), & x \in \Omega. \end{cases}$$

The nonlocal Dirichlet model (3.9) and the classical Dirichlet problem (3.10) share many properties, among them the asymptotic behavior of their solutions as  $t \rightarrow \infty$  is similar as was proved in [12].

The main goal now is to show that the Dirichlet problem for the heat equation (3.10) can be approximated by suitable nonlocal problems of the form of (3.9).

More precisely, for a given  $J$  and a given  $\varepsilon > 0$  we consider the rescaled kernel

$$(3.11) \quad J_\varepsilon(\xi) = C_1 \frac{1}{\varepsilon^N} J\left(\frac{\xi}{\varepsilon}\right), \quad \text{with} \quad C_1^{-1} = \frac{1}{2} \int_{B(0,d)} J(z) z_N^2 dz.$$

Here  $C_1$  is a normalizing constant in order to obtain the Laplacian in the limit instead of a multiple of it. Let  $u^\varepsilon(x, t)$  be the solution of

$$(3.12) \quad \begin{cases} u_t^\varepsilon(x, t) = \int_{\Omega} \frac{J_\varepsilon(x-y)}{\varepsilon^2} (u^\varepsilon(y, t) - u^\varepsilon(x, t)) dy, & x \in \Omega, t > 0, \\ u(x, t) = g(x, t), & x \notin \Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases}$$

Our main result now reads as follows.

**THEOREM 18.** *Let  $\Omega$  be a bounded  $C^{2+\alpha}$  domain for some  $0 < \alpha < 1$ .*

*Let  $v \in C^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times [0, T])$  be the solution to (3.10) and let  $u^\varepsilon$  be the solution to (3.12) with  $J_\varepsilon$  as above. Then, there exists  $C = C(T)$  such that*

$$(3.13) \quad \sup_{t \in [0, T]} \|v - u^\varepsilon\|_{L^\infty(\Omega)} \leq C\varepsilon^\alpha \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Related results for the Neumann are presented in a different chapter (see also [21]).

Note that the assumed regularity of  $v$  is a consequence of regularity assumptions on the boundary data  $g$ , the domain  $\Omega$  and the initial condition  $u_0$ .

In order to prove Theorem 18 let  $\tilde{v}$  be a  $C^{2+\alpha, 1+\alpha/2}$  extension of  $v$  to  $\mathbb{R}^N \times [0, T]$ .

Let us define the operator

$$\tilde{L}_\varepsilon(z) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N} J_\varepsilon(x-y) (z(y, t) - z(x, t)) dy.$$

Then  $\tilde{v}$  verifies

$$(3.14) \quad \begin{cases} \tilde{v}_t(x, t) = \tilde{L}_\varepsilon(\tilde{v})(x, t) + F_\varepsilon(x, t) & x \in \Omega, (0, T], \\ \tilde{v}(x, t) = g(x, t) + G(x, t), & x \notin \Omega, (0, T], \\ \tilde{v}(x, 0) = u_0(x), & x \in \Omega. \end{cases}$$

where, since  $\Delta v = \Delta \tilde{v}$  in  $\Omega$ ,

$$F_\varepsilon(x, t) = -\tilde{L}_\varepsilon(\tilde{v})(x, t) + \Delta \tilde{v}(x, t).$$

Moreover as  $G$  is smooth and  $G(x, t) = 0$  if  $x \in \partial\Omega$  we have

$$G(x, t) = O(\varepsilon), \quad \text{for } x \text{ such that } \text{dist}(x, \partial\Omega) \leq \varepsilon d.$$

We set  $w^\varepsilon = \tilde{v} - u^\varepsilon$  and we note that

$$(3.15) \quad \begin{cases} w_t^\varepsilon(x, t) = \tilde{L}_\varepsilon(w^\varepsilon)(x, t) + F_\varepsilon(x, t) & x \in \Omega, (0, T], \\ w^\varepsilon(x, t) = G(x, t), & x \notin \Omega, (0, T], \\ w^\varepsilon(x, 0) = 0, & x \in \Omega. \end{cases}$$

First, we claim that, by the choice of  $C_1$ , the fact that  $J$  is radially symmetric and  $\tilde{u} \in C^{2+\alpha, 1+\alpha/2}(\mathbb{R}^N \times [0, T])$ , we have that

$$(3.16) \quad \sup_{t \in [0, T]} \|F_\varepsilon\|_{L^\infty(\Omega)} = \sup_{t \in [0, T]} \|\Delta \tilde{v} - \tilde{L}_\varepsilon(\tilde{v})\|_{L^\infty(\Omega)} = O(\varepsilon^\alpha).$$

In fact,

$$\Delta \tilde{v}(x, t) - \frac{C_1}{\varepsilon^{N+2}} \int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right) (\tilde{v}(y, t) - \tilde{v}(x, t)) dy$$

becomes, under the change variables  $z = (x-y)/\varepsilon$ ,

$$\Delta \tilde{v}(x, t) - \frac{C_1}{\varepsilon^2} \int_{\mathbb{R}^N} J(z) (\tilde{v}(x - \varepsilon z, t) - \tilde{v}(x, t)) dz$$

and hence (3.16) follows by a simple Taylor expansion. This proves the claim.

We proceed now to prove Theorem 18.

**PROOF OF THEOREM 18.** In order to prove the theorem by a comparison we first look for a supersolution. Let  $\bar{w}$  be given by

$$(3.17) \quad \bar{w}(x, t) = K_1 \varepsilon^\alpha t + K_2 \varepsilon.$$

For  $x \in \Omega$  we have, if  $K_1$  is large,

$$(3.18) \quad \bar{w}_t(x, t) - \tilde{L}(\bar{w})(x, t) = K_1 \varepsilon^\alpha \geq F_\varepsilon(x, t) = w_t^\varepsilon(x, t) - \tilde{L}_\varepsilon(w^\varepsilon)(x, t).$$

Since

$$G_\varepsilon(x, t) = O(\varepsilon) \quad \text{for } x \text{ such that } \text{dist}(x, \partial\Omega) \leq \varepsilon$$

choosing  $K_2$  large, we obtain

$$(3.19) \quad \bar{w}(x, t) \geq w^\varepsilon(x, t)$$

for  $x \notin \Omega$  such that  $\text{dist}(x, \partial\Omega) \leq \varepsilon d$  and  $t \in [0, T]$ . Moreover it is clear that

$$(3.20) \quad \bar{w}(x, 0) = K_2\varepsilon > w^\varepsilon(x, 0) = 0.$$

Thanks to (3.18), (3.19) and (3.20) we can apply the comparison result and conclude that

$$(3.21) \quad w^\varepsilon(x, t) \leq \bar{w}(x, t) = K_1\varepsilon^\alpha t + K_2\varepsilon.$$

In a similar fashion we prove that  $\underline{w}(x, t) = -K_1\varepsilon^\alpha t - K_2\varepsilon$  is a subsolution and hence

$$(3.22) \quad w^\varepsilon(x, t) \geq \underline{w}(x, t) = -K_1\varepsilon^\alpha t - K_2\varepsilon.$$

Therefore

$$(3.23) \quad \sup_{t \in [0, T]} \|u - u^\varepsilon\|_{L^\infty(\Omega)} \leq C(T)\varepsilon^\alpha,$$

as we wanted to prove. □



## CHAPTER 4

### The Neumann problem.

Let us turn our attention to Neumann boundary conditions. We study

$$(4.1) \quad \begin{aligned} u_t(x, t) &= \int_{\Omega} J(x-y)(u(y, t) - u(x, t)) dy, & x \in \Omega, t > 0, \\ u(x, 0) &= u_0(x), & x \in \Omega. \end{aligned}$$

Again solutions are to be understood in an integral sense, see Theorem 21. In this model we have that the integral terms take into account the diffusion inside  $\Omega$ . In fact, as we have explained the integral  $\int J(x-y)(u(y, t) - u(x, t)) dy$  takes into account the individuals arriving or leaving position  $x$  from other places. Since we are integrating in  $\Omega$ , we are imposing that diffusion takes place only in  $\Omega$ . The individuals may not enter nor leave  $\Omega$ . This is the analogous of what is called homogeneous Neumann boundary conditions in the literature.

Again in this case we find that the asymptotic behavior is given by an exponential decay determined by an eigenvalue problem. Let  $\beta_1$  be given by

$$(4.2) \quad \beta_1 = \inf_{u \in L^2(\Omega), \int_{\Omega} u = 0} \frac{\frac{1}{2} \int_{\Omega} \int_{\Omega} J(x-y)(u(y) - u(x))^2 dy dx}{\int_{\Omega} (u(x))^2 dx}.$$

Concerning the asymptotic behavior of solutions of (4.1) our last result reads as follows:

**THEOREM 19.** *For every  $u_0 \in L^1(\Omega)$  there exists a unique solution  $u$  of (4.1) such that  $u \in C([0, \infty); L^1(\Omega))$ . This solution preserves the total mass in  $\Omega$*

$$\int_{\Omega} u(y, t) dy = \int_{\Omega} u_0(y) dy.$$

Moreover, let  $\varphi = \frac{1}{|\Omega|} \int_{\Omega} u_0$ , then the asymptotic behavior of solutions of (4.1) is described as follows: if  $u_0 \in L^2(\Omega)$ ,

$$(4.3) \quad \|u(\cdot, t) - \varphi\|_{L^2(\Omega)} \leq e^{-\beta_1 t} \|u_0 - \varphi\|_{L^2(\Omega)},$$

and if  $u_0$  is continuous and bounded there exist a positive constant  $C$  such that

$$(4.4) \quad \|u(\cdot, t) - \varphi\|_{L^\infty(\Omega)} \leq C e^{-\beta_1 t}.$$

Solutions of the Neumann problem are functions  $u \in C([0, \infty); L^1(\Omega))$  which satisfy

$$\begin{aligned} u_t(x, t) &= \int_{\Omega} J(x-y)(u(y, t) - u(x, t)) dy, & x \in \Omega, t > 0, \\ u(x, 0) &= u_0(x) & x \in \Omega. \end{aligned}$$

As in the previous chapter, see also [20], existence and uniqueness will be a consequence of Banach's fixed point theorem. The main arguments are basically the same but we repeat them here to make this chapter self-contained.

Fix  $t_0 > 0$  and consider the Banach space

$$X_{t_0} = C([0, t_0]; L^1(\Omega))$$

with the norm

$$|||w||| = \max_{0 \leq t \leq t_0} \|w(\cdot, t)\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator  $\mathcal{T} : X_{t_0} \rightarrow X_{t_0}$  defined by

$$(4.5) \quad \mathcal{T}_{w_0}(w)(x, t) = w_0(x) + \int_0^t \int_{\Omega} J(x-y)(w(y, s) - w(x, s)) dy ds.$$

The following lemma is the main ingredient in the proof of existence.

LEMMA 20. *Let  $w_0, z_0 \in L^1(\Omega)$  and  $w, z \in X_{t_0}$ , then there exists a constant  $C$  depending only on  $\Omega$  and  $J$  such that*

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \leq Ct_0 |||w - z||| + \|w_0 - z_0\|_{L^1(\Omega)}.$$

PROOF. We have

$$\begin{aligned} & \int_{\Omega} |\mathcal{T}_{w_0}(w)(x, t) - \mathcal{T}_{z_0}(z)(x, t)| dx \leq \int_{\Omega} |w_0 - z_0|(x) dx \\ & + \int_{\Omega} \left| \int_0^t \int_{\Omega} J(x-y) \left[ (w(y, s) - z(y, s)) - (w(x, s) - z(x, s)) \right] dy ds \right| dx. \end{aligned}$$

Hence

$$\begin{aligned} & \int_{\Omega} |\mathcal{T}_{w_0}(w)(x, t) - \mathcal{T}_{z_0}(z)(x, t)| dx \leq \|w_0 - z_0\|_{L^1(\Omega)} \\ & + \int_0^t \int_{\Omega} |(w(y, s) - z(y, s))| dy + \int_0^t \int_{\Omega} |(w(x, s) - z(x, s))| dx. \end{aligned}$$

Therefore, we obtain,

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \leq Ct_0 |||w - z||| + \|w_0 - z_0\|_{L^1(\Omega)},$$

as we wanted to prove.  $\square$



**THEOREM 21.** *For every  $u_0 \in L^1(\Omega)$  there exists a unique solution  $u$  of (4.1) such that  $u \in C([0, \infty); L^1(\Omega))$ . Moreover, the total mass in  $\Omega$  verifies,*

$$(4.6) \quad \int_{\Omega} u(y, t) dy = \int_{\Omega} u_0(y) dy.$$

**PROOF.** We check first that  $\mathcal{T}_{u_0}$  maps  $X_{t_0}$  into  $X_{t_0}$ . From (4.5) we see that for  $0 < t_1 < t_2 \leq t_0$ ,

$$\|\mathcal{T}_{u_0}(w)(t_2) - \mathcal{T}_{u_0}(w)(t_1)\|_{L^1(\Omega)} \leq 2 \int_{t_1}^{t_2} \int_{\Omega} |w(y, s)| dx dy ds.$$

On the other hand, again from (4.5)

$$\|\mathcal{T}_{u_0}(w)(t) - w_0\|_{L^1(\Omega)} \leq Ct \|w\|.$$

These two estimates give that  $\mathcal{T}_{u_0}(w) \in C([0, t_0]; L^1(\Omega))$ . Hence  $\mathcal{T}_{u_0}$  maps  $X_{t_0}$  into  $X_{t_0}$ .

Choose  $t_0$  such that  $Ct_0 < 1$ . Now taking  $z_0 \equiv w_0 \equiv u_0$ , in Lemma 20 we get that  $\mathcal{T}_{u_0}$  is a strict contraction in  $X_{t_0}$  and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem in the interval  $[0, t_0]$ . To extend the solution to  $[0, \infty)$  we may take as initial data  $u(x, t_0) \in L^1(\Omega)$  and obtain a solution up to  $[0, 2t_0]$ . Iterating this procedure we get a solution defined in  $[0, \infty)$ .

We finally prove that if  $u$  is the solution, then the integral in  $\Omega$  of  $u$  satisfies (4.6). Since

$$u(x, t) - u_0(x) = \int_0^t \int_{\Omega} J(x - y) (u(y, s) - u(x, s)) dy ds.$$

We can integrate in  $x$  and apply Fubini's theorem to obtain

$$\int_{\Omega} u(x, t) dx - \int_{\Omega} u_0(x) dx = 0$$

and the theorem is proved.  $\square$

Now we study the asymptotic behavior as  $t \rightarrow \infty$ . We start by analyzing the corresponding stationary problem so we consider the equation

$$(4.7) \quad 0 = \int_{\Omega} J(x - y) (\varphi(y) - \varphi(x)) dy.$$

The only solutions are constants. In fact, in particular, (4.7) implies that  $\varphi$  is a continuous function. Set

$$K = \max_{x \in \bar{\Omega}} \varphi(x)$$

and consider the set

$$\mathcal{A} = \{x \in \bar{\Omega} \mid \varphi(x) = K\}.$$

The set  $\mathcal{A}$  is clearly closed and non empty. We claim that it is also open in  $\bar{\Omega}$ . Let  $x_0 \in \mathcal{A}$ . We have then

$$\varphi(x_0) = \left( \int_{\Omega} J(x_0 - y) dy \right)^{-1} \int_{\Omega} J(x_0 - y) \varphi(y) dy,$$

and  $\varphi(y) \leq \varphi(x_0)$  this implies  $\varphi(y) = \varphi(x_0)$  for all  $y \in \Omega \cap B(x_0, d)$ , and hence  $\mathcal{A}$  is open as claimed. Consequently, as  $\Omega$  is connected,  $\mathcal{A} = \overline{\Omega}$  and  $\varphi$  is constant.

We have proved the following proposition:

**PROPOSITION 22.** *Every stationary solution of (4.1) is constant in  $\Omega$ .*

Now we prove the exponential rate of convergence to steady states of solutions in  $L^2$ . Let us take  $\beta_1$  as

$$(4.8) \quad \beta_1 = \inf_{u \in L^2(\Omega), \int_{\Omega} u = 0} \frac{\frac{1}{2} \int_{\Omega} \int_{\Omega} J(x-y)(u(y) - u(x))^2 dy dx}{\int_{\Omega} (u(x))^2 dx}.$$

It is clear that  $\beta_1 \geq 0$ . Let us prove that  $\beta_1$  is in fact strictly positive. To this end we consider the subspace of  $L^2(\Omega)$  given by the orthogonal to the constants,  $H = \langle cts \rangle^{\perp}$  and the symmetric (self-adjoint) operator  $T : H \mapsto H$  given by

$$T(u) = \int_{\Omega} J(x-y)(u(x) - u(y)) dy = - \int_{\Omega} J(x-y)u(y) dy + A(x)u(x).$$

Note that  $T$  is the sum of an invertible operator and a compact operator. Since  $T$  is symmetric we have that its spectrum verifies  $\sigma(T) \subset [m, M]$ , where

$$m = \inf_{u \in H, \|u\|_{L^2(\Omega)}=1} \langle Tu, u \rangle$$

and

$$M = \sup_{u \in H, \|u\|_{L^2(\Omega)}=1} \langle Tu, u \rangle.$$

Remark that

$$\begin{aligned} m &= \inf_{u \in H, \|u\|_{L^2(\Omega)}=1} \langle Tu, u \rangle \\ &= \inf_{u \in H, \|u\|_{L^2(\Omega)}=1} \int_{\Omega} \int_{\Omega} J(x-y)(u(x) - u(y)) dy u(x) dx \\ &= \beta_1. \end{aligned}$$

Then  $m \geq 0$ . Now we just observe that

$$m > 0.$$

In fact, if not, as  $m \in \sigma(T)$ , we have that  $T : H \mapsto H$  is not invertible. Using Fredholm's alternative this implies that there exists a nontrivial  $u \in H$  such that  $T(u) = 0$ , but then  $u$  must be constant in  $\Omega$ . This is a contradiction with the fact that  $H$  is orthogonal to the constants.

To study the asymptotic behavior of the solutions we need an upper estimate on  $\beta_1$ .

LEMMA 23. Let  $\beta_1$  be given by (4.8) then

$$(4.9) \quad \beta_1 \leq \min_{x \in \bar{\Omega}} \int_{\Omega} J(x-y) dy.$$

PROOF. Let

$$A(x) = \int_{\Omega} J(x-y) dy.$$

Since  $\bar{\Omega}$  is compact and  $A$  is continuous there exists a point  $x_0 \in \bar{\Omega}$  such that

$$A(x_0) = \min_{x \in \bar{\Omega}} A(x).$$

For every  $\varepsilon$  small let us choose two disjoint balls of radius  $\varepsilon$  contained in  $\Omega$ ,  $B(x_1, \varepsilon)$  and  $B(x_2, \varepsilon)$  in such a way that  $x_i \rightarrow x_0$  as  $\varepsilon \rightarrow 0$ . We use

$$u_{\varepsilon}(x) = \chi_{B(x_1, \varepsilon)}(x) - \chi_{B(x_2, \varepsilon)}(x)$$

as a test function in the definition of  $\beta_1$ , (4.8). Then we get that for every  $\varepsilon$  small it holds

$$\begin{aligned} \beta_1 &\leq \frac{\frac{1}{2} \int_{\Omega} \int_{\Omega} J(x-y)(u_{\varepsilon}(y) - u_{\varepsilon}(x))^2 dy dx}{\int_{\Omega} (u_{\varepsilon}(x))^2 dx} \\ &= \frac{\int_{\Omega} A(x)u_{\varepsilon}^2(x) dx - \int_{\Omega} \int_{\Omega} J(x-y)u_{\varepsilon}(y) u_{\varepsilon}(x) dy dx}{\int_{\Omega} (u_{\varepsilon}(x))^2 dx} \\ &= \frac{\int_{\Omega} A(x)u_{\varepsilon}^2(x) dx - \int_{\Omega} \int_{\Omega} J(x-y)u_{\varepsilon}(y) u_{\varepsilon}(x) dy dx}{2|B(0, \varepsilon)|}. \end{aligned}$$

Using the continuity of  $A$  and the explicit form of  $u_{\varepsilon}$  we obtain

$$\lim_{\varepsilon \rightarrow 0} \frac{\int_{\Omega} A(x)u_{\varepsilon}^2(x) dx}{2|B(0, \varepsilon)|} = A(x_0)$$

and

$$\lim_{\varepsilon \rightarrow 0} \frac{\int_{\Omega} \int_{\Omega} J(x-y)u_{\varepsilon}(y) u_{\varepsilon}(x) dy dx}{2|B(0, \varepsilon)|} = 0.$$

Therefore, (4.9) follows.  $\square$

Now let us prove the exponential convergence of  $u(x, t)$  to the mean value of the initial datum.

THEOREM 24. For every  $u_0 \in L^2(\Omega)$  the solution  $u(x, t)$  of (4.1) satisfies

$$(4.10) \quad \|u(\cdot, t) - \varphi\|_{L^2(\Omega)} \leq e^{-\beta_1 t} \|u_0 - \varphi\|_{L^2(\Omega)}.$$

Moreover, if  $u_0$  is continuous and bounded, there exists a positive constant  $C > 0$  such that,

$$(4.11) \quad \|u(\cdot, t) - \varphi\|_{L^\infty(\Omega)} \leq C e^{-\beta_1 t}.$$

Here  $\beta_1$  is given by (4.8).

PROOF. Let

$$H(t) = \frac{1}{2} \int_{\Omega} (u(x, t) - \varphi)^2 dx.$$

Differentiating with respect to  $t$  and using (4.8) and the conservation of the total mass, we obtain

$$H'(t) = -\frac{1}{2} \int_{\Omega} \int_{\Omega} J(x-y) (u(y, t) - u(x, t))^2 dy dx \leq -\beta_1 \int_{\Omega} (u(x, t) - \varphi)^2 dx.$$

Hence

$$H'(t) \leq -2\beta_1 H(t).$$

Therefore, integrating we obtain,

$$(4.12) \quad H(t) \leq e^{-2\beta_1 t} H(0),$$

and (4.10) follows.

In order to prove (4.11) let  $w(x, t)$  denote the difference

$$w(x, t) = u(x, t) - \varphi.$$

We seek for an exponential estimate in  $L^\infty$  of the decay of  $w(x, t)$ . The linearity of the equation implies that  $w(x, t)$  is a solution of (4.1) and satisfies

$$w(x, t) = e^{-A(x)t} w_0(x) + e^{-A(x)t} \int_0^t e^{A(x)s} \int_{\Omega} J(x-y) w(y, s) dy ds.$$

Recall that  $A(x) = \int_{\Omega} J(x-y) dx$ . By using (4.10) and the Holder inequality it follows that

$$|w(x, t)| \leq e^{-A(x)t} w_0(x) + C e^{-A(x)t} \int_0^t e^{A(x)s - \beta_1 s} ds.$$

Integrating this inequality, we obtain that the solution  $w(x, t)$  decays to zero exponentially fast and moreover, it implies (4.11) thanks to Lemma 23.  $\square$

## CHAPTER 5

### Energy methods

In this chapter our main aim is to apply energy methods to obtain decay estimates for solutions to nonlocal evolution equations. Our motivation to introduce energy methods to deal with nonlocal problems is twofold, first we want to see how energy methods can be applied to equations possibly without any regularization effect and moreover we want to deal with nonlinear problems for which there are no explicit representation formula for the solution (in general, Fourier methods are not applicable to nonlinear problems).

To begin our analysis, we first deal with a linear nonlocal diffusion operator with a nonlinear source, that is, we consider the following evolution problem

$$(5.1) \quad u_t(x, t) = \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t)) dy + f(u)(x, t)$$

with  $f$  a locally Lipschitz function satisfying the sign condition  $f(s)s \leq 0$  and  $J(x, y)$  a symmetric nonnegative kernel.

We generalize the previous results in two ways, we allow a nonlinear term  $f(u)$  imposing only a dissipativity condition,  $f(s)s \leq 0$ , and, what is even more relevant, we can consider equations in which the nonlocal part is not given by a convolution but for a general operator of the form  $\int_{\mathbb{R}^d} J(x, y)(u(y) - u(x)) dy$ .

Our first result reads as follows: *under adequate hypothesis on  $J$  (see Theorem 26) and  $f$  a locally Lipschitz function satisfying the sign condition  $f(s)s \leq 0$ , consider an initial condition  $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  with  $d \geq 3$ . Then, for any  $1 \leq q < \infty$  the solution to (5.1) verifies the following decay bound,*

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^d)} \leq C t^{-\frac{d}{2}\left(1-\frac{1}{q}\right)}.$$

Our main hypotheses on  $J$  can be summarized as follows:  $J(x, y)$  is strictly positive ( $\geq c_1 > 0$ ) for  $|y - a(x)| \leq c_2$ , where  $a$  is a function with bounded derivatives.

We remark that this decay bound need not be optimal, in the final section we present examples of functions  $J$  that give exponential decay in  $L^2(\mathbb{R})$ . To obtain a complete classification of all possible decay rates seems a very difficult but challenging problem.

**Preliminaries.** In this section we collect some preliminaries and state and prove a crucial decomposition theorem. In what follows we denote by

$$p^* = \frac{pd}{(d-p)}$$

the usual Sobolev exponent, while

$$p' = \frac{p}{p-1}$$

denotes the usual conjugate exponent.

First, let us describe briefly how the energy method can be applied to obtain decay estimates for local problems. Let us begin with the simpler case of the estimate for solutions to the heat equation in  $L^2(\mathbb{R}^d)$ -norm,

$$u_t = \Delta u.$$

If we multiply by  $u$  and integrate in  $\mathbb{R}^d$ , we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^2(x, t) dx = - \int_{\mathbb{R}^d} |\nabla u(x, t)|^2 dx.$$

Now we use Sobolev's inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2(x, t) dx \geq C \left( \int_{\mathbb{R}^d} |u|^{2^*}(x, t) dx \right)^{2/2^*}$$

to obtain

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^2(x, t) dx \leq -C \left( \int_{\mathbb{R}^d} |u|^{2^*}(x, t) dx \right)^{2/2^*}.$$

If we use interpolation and conservation of mass, that implies  $\|u(t)\|_{L^1(\mathbb{R}^d)} \leq C$  for any  $t > 0$ , we have

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq \|u(t)\|_{L^1(\mathbb{R}^d)}^\alpha \|u(t)\|_{L^{2^*}(\mathbb{R}^d)}^{1-\alpha} \leq C \|u(t)\|_{L^{2^*}(\mathbb{R}^d)}^{1-\alpha}$$

with  $\alpha$  determined by

$$\frac{1}{2} = \alpha + \frac{1-\alpha}{2^*}, \quad \text{that is,} \quad \alpha = \frac{2^* - 2}{2(2^* - 1)}.$$

Hence we get

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^2(x, t) dx \leq -C \left( \int_{\mathbb{R}^d} u^2(x, t) dx \right)^{\frac{1}{1-\alpha}}$$

from where the decay estimate

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq C t^{-\frac{d}{2}(1-\frac{1}{2})}, \quad t > 0,$$

follows.

We want to mimic the steps for the nonlocal evolution problem

$$u_t(x, t) = \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t)) dy.$$

Hence, we multiply by  $u$  and integrate in  $\mathbb{R}^d$  to obtain,

$$(5.2) \quad \frac{d}{dt} \int_{\mathbb{R}^d} u^2(x, t) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t)) dy u(x, t) dx.$$

Now, we need to “integrate by parts”. Therefore, let us begin by a simple algebraic identity (whose proof is immediate) that plays the role of an integration by parts formula for nonlocal operators.

LEMMA 25. *If  $J$  is symmetric,  $J(x, y) = J(y, x)$  then it holds*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)(\varphi(y) - \varphi(x))\psi(x)dydx = -\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)(\varphi(y) - \varphi(x))(\psi(y) - \psi(x))dydx.$$

If we apply this lemma to (5.2) we get

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^2(x, t)dx = -\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t))^2 dy dx,$$

but now we run into troubles since there is no analogous to Sobolev inequality. In fact, an inequality of the form

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t))^2 dy dx \geq C \left( \int_{\mathbb{R}^d} u^q(x, t) dx \right)^{2/q}$$

can not hold for any  $q > 2$ .

Now the idea is to split the function  $u$  as the sum of two functions  $u = v + w$ , where on the function  $v$  (the “smooth” part of the solution) the nonlocal operator acts as a gradient and on the function  $w$  (the “rough” part) it does not increase its norm significantly.

Therefore, we need to obtain estimates for the  $L^p(\mathbb{R}^d)$ -norm of the nonlocal operators. The main result of this section is the following.

THEOREM 26. *Let  $p \in [1, \infty)$  and  $J(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$  be a symmetric nonnegative function satisfying*

HJ1) *There exists a positive constant  $C < \infty$  such that*

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y) dx \leq C.$$

HJ2) *There exist positive constants  $c_1, c_2$  and a function  $a \in C^1(\mathbb{R}^d, \mathbb{R}^d)$  satisfying*

$$(5.3) \quad \sup_{x \in \mathbb{R}^d} |\nabla a(x)| < \infty$$

*such that the set*

$$(5.4) \quad B_x = \{y \in \mathbb{R}^d : |y - a(x)| \leq c_2\}$$

*verifies*

$$B_x \subset \{y \in \mathbb{R}^d : J(x, y) > c_1\}.$$

*Then, for any function  $u \in L^p(\mathbb{R}^d)$  there exist two functions  $v$  and  $w$  such that  $u = v + w$  and*

$$(5.5) \quad \|\nabla v\|_{L^p(\mathbb{R}^d)}^p + \|w\|_{L^p(\mathbb{R}^d)}^p \leq C(J, p) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)|u(x) - u(y)|^p dx dy.$$

Moreover, if  $u \in L^q(\mathbb{R}^d)$  with  $q \in [1, \infty]$  then the functions  $v$  and  $w$  satisfy

$$(5.6) \quad \|v\|_{L^q(\mathbb{R}^d)} \leq C(J, q) \|u\|_{L^q(\mathbb{R}^d)}$$

and

$$(5.7) \quad \|w\|_{L^q(\mathbb{R}^d)} \leq C(J, q) \|u\|_{L^q(\mathbb{R}^d)}.$$

Before the proof we collect some remarks and a prove a corollary.

REMARK 27. *The above result says that there exists a decomposition of  $u$  in a smooth part,  $v$ , and a rough part,  $w$ , such that the action of the nonlocal operator is like a gradient on the smooth part and as the identity on the rough part.*

REMARK 28. *The constant  $C(J; q)$  in the theorem depends only on the constants of HJ1) and HJ2) and not on any other characteristic of the kernel  $J$ .*

REMARK 29. *We note that in the case  $1 \leq p < d$  using the classical Sobolev's inequality  $\|v\|_{L^{p^*}(\mathbb{R}^d)} \leq \|\nabla v\|_{L^p(\mathbb{R}^d)}$  we get that (5.5) implies*

$$\|v\|_{L^{p^*}(\mathbb{R}^d)}^p + \|w\|_{L^p(\mathbb{R}^d)}^p \leq C(J, p) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y) |u(x) - u(y)|^p dx dy.$$

REMARK 30. *In particular, we can consider  $a(x) = x$ , that is, the case of a convolution kernel,  $J(x, y) = G(x - y)$ , with  $G(0) > 0$ . In fact, it is reasonable to assume that  $J(x, x) > 0$  since in biological models this means that the probability that some individuals that are in  $x$  at time  $t$  remain at the same position is positive.*

To simplify the notation let us note by  $\langle A_p u, u \rangle$  the following quantity,

$$\langle A_p u, u \rangle := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y) |u(x) - u(y)|^p dx dy.$$

Observe that, in order that the above quantity to be finite, we have to assume a priori that  $u$  belongs to  $L^p(\mathbb{R}^d)$ .

Note that our main result of this section, Theorem 26 gives estimates from below for  $\langle A_p u, u \rangle$ . A corollary of this result is the following.

COROLLARY 31. *Let  $J(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a symmetric nonnegative function satisfying hypotheses HJ1) and HJ2) in Theorem 26 and  $p \in [1, d)$ . There exist two positive constants  $C_1 = C_1(J, p)$  and  $C_2 = C_2(J, p)$  such that for any  $u \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$  the following holds:*

$$(5.8) \quad \|u\|_{L^p(\mathbb{R}^d)}^p \leq C_1 \|u\|_{L^1(\mathbb{R}^d)}^{p(1-\alpha(p))} \langle A_p u, u \rangle^{\alpha(p)} + C_2 \langle A_p u, u \rangle,$$

where  $\alpha(p)$  satisfies:

$$\frac{1}{p} = \frac{\alpha(p)}{p^*} + 1 - \alpha(p).$$



REMARK 32. *The explicit value of  $\alpha(p)$  is given by*

$$(5.9) \quad \alpha(p) = \frac{p^*}{p'(p^* - 1)} = \frac{d(p-1)}{d(p-1) + p}.$$

PROOF OF COROLLARY 31. We use the decomposition  $u = v + w$  given by Theorem 26 to obtain

$$\|u\|_{L^p(\mathbb{R}^d)}^p \leq \|v\|_{L^p(\mathbb{R}^d)}^p + \|w\|_{L^p(\mathbb{R}^d)}^p.$$

Also, by (5.5), we have

$$\|\nabla v\|_{L^p(\mathbb{R}^d)}^p \leq C(J, p) \langle A_p u, u \rangle$$

and

$$\|w\|_{L^p(\mathbb{R}^d)}^p \leq C(J, p) \langle A_p u, u \rangle.$$

Then, from the interpolation inequality

$$\|v\|_{L^p(\mathbb{R}^d)} \leq \|v\|_{L^{p^*}(\mathbb{R}^d)}^{\alpha(p)} \|v\|_{L^1(\mathbb{R}^d)}^{1-\alpha(p)},$$

we obtain that the  $L^p(\mathbb{R}^d)$ -norm of  $u$  satisfies

$$\begin{aligned} \|u\|_{L^p(\mathbb{R}^d)}^p &\leq \|v\|_{L^{p^*}(\mathbb{R}^d)}^{\alpha(p)p} \|v\|_{L^1(\mathbb{R}^d)}^{(1-\alpha(p))p} + \|w\|_{L^p(\mathbb{R}^d)}^p \\ &\leq \|\nabla v\|_{L^p(\mathbb{R}^d)}^{\alpha(p)p} \|u\|_{L^1(\mathbb{R}^d)}^{(1-\alpha(p))p} + C(J, p) \langle A_p u, u \rangle \\ &\leq C_1 \|u\|_{L^1(\mathbb{R}^d)}^{(1-\alpha(p))p} \langle A_p u, u \rangle^{\alpha(p)} + C_2 \langle A_p u, u \rangle, \end{aligned}$$

as we wanted to prove. □

Now we proceed with the proof of the decomposition theorem.

PROOF OF THEOREM 26. We divide the proof in two steps. First of all, we prove under the assumptions HJ1)-HJ2) the existence of a function  $\rho(\cdot, \cdot)$  satisfying

- H1)  $\rho(x, \cdot) \in C_c^\infty(\mathbb{R}^d)$  for a.e.  $x \in \mathbb{R}^d$ ,
- H2)  $\int_{\mathbb{R}^d} \rho(x, y) dy = 1$  for a.e.  $x \in \mathbb{R}^d$ ,
- H3)  $\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x, y) dx \leq M < \infty$ ,
- H4)  $\text{supp } \rho(x, \cdot) \subset B_x$  for a.e.  $x \in \mathbb{R}^d$ ,
- H5)  $\sup_{x \in \mathbb{R}^d} \|\rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)} \leq M < \infty$ ,
- H6)  $\sum_{k=1}^d \sup_{x \in \mathbb{R}^d} \|\partial_{x_k} \rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)} \leq M < \infty$ .

Next, we define

$$v(x) = \int_{\mathbb{R}^d} \rho(x, y) u(y) dy, \quad \text{and} \quad w = u - v,$$

and prove (5.5), (5.6) and (5.7).

**Step I. Construction of  $\rho$ .** With  $c_2$  given by HJ2) we consider a smooth function  $\psi \in C_c^\infty(\mathbb{R}^d)$  supported in the ball  $B_{c_2}(0)$ ,  $0 \leq \psi \leq C$  and having mass one:

$$\int_{B_{c_2}(0)} \psi(x) dx = 1.$$

For any  $x \in \mathbb{R}^d$  we consider the function  $a(x)$  and the set  $B_x$  as in (5.4), see HJ2). We then define  $\rho(x, y)$  by

$$(5.10) \quad \rho(x, y) = \psi(y - a(x)).$$

We will prove properties H3) and H6) since the others easily follow with a constant  $M(J)$ . We point out that the assumption on the existence of a ball  $B_x$  centered at  $a(x)$  with radius  $c_2$  is necessary in proving H5). Otherwise,  $\inf_{x \in \mathbb{R}^d} |B_x| = 0$  and by Hölder inequality, we get

$$\|\rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)} \geq \frac{\int_{\mathbb{R}^d} \rho(x, y) dy}{|B_x|^{1/p}} = \frac{1}{|B_x|^{1/p}}$$

and then

$$\sup_{x \in \mathbb{R}^d} \|\rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)} \geq \frac{1}{\inf_{x \in \mathbb{R}^d} |B_x|^{1/p}} = \infty.$$

Therefore, we cannot obtain property H5).

We now prove property H3). Observe that, by definition (5.10) of the function  $\rho(\cdot, \cdot)$  and the fact that  $\psi \leq C$  we have

$$\begin{aligned} \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x, y) dx &= \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \psi(y - a(x)) dx = \sup_{y \in \mathbb{R}^d} \int_{|y - a(x)| \leq c_2} \psi(y - a(x)) dx \\ &\leq C \sup_{y \in \mathbb{R}^d} |\{x : |y - a(x)| \leq c_2\}|. \end{aligned}$$

It remains to show that the last term in the right hand side is finite. Indeed, given  $y$ , we have

$$|\{x : |y - a(x)| \leq c_2\}| \leq \int_{\{x : |y - a(x)| \leq c_2\}} \frac{J(x, y)}{c_1} dx \leq \frac{1}{c_1} \int_{\mathbb{R}^d} J(x, y) dx \leq C.$$

We now prove HJ6). By definition (5.10) for any  $x \in \mathbb{R}^d$  we have

$$\|\partial_{x_k} \rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)} = \|\nabla \psi(\cdot - a(x)) \cdot \partial_{x_k} a(x)\|_{L^{p'}(\mathbb{R}^d)} \leq |\partial_{x_k} a(x)| \|\nabla \psi\|_{L^{p'}(\mathbb{R}^d)}.$$

Using (5.3) and the construction of  $\psi$  we obtain HJ6).

**Step II. Proof of the estimates on  $u$ ,  $v$  and  $w$ .** We have proved that there exists a function  $\rho$  satisfying hypotheses H1)-H6). Let us take

$$v(x) = \int_{\mathbb{R}^d} \rho(x, y) u(y) dy, \quad \text{and} \quad w = u - v.$$

First we prove (5.6) and (5.7). Hölder's inequality applied to the function  $v$  and H2) guarantee that

$$|v(x)|^q \leq \int_{\mathbb{R}^d} \rho(x, y) |u(y)|^q dy \left( \int_{\mathbb{R}^d} \rho(x, y) dy \right)^{\frac{q}{q'}} = \int_{\mathbb{R}^d} \rho(x, y) |u(y)|^q dy,$$

Then, property H3) gives us

$$\begin{aligned} \int_{\mathbb{R}^d} |v(x)|^q dx &\leq \int_{\mathbb{R}^d} |u(y)|^q \int_{\mathbb{R}^d} \rho(x, y) dx dy \leq \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x, y) dx \int_{\mathbb{R}^d} |u(y)|^q dy \\ &\leq M \int_{\mathbb{R}^d} |u(y)|^q dy \end{aligned}$$

which proves (5.6).

Also, we obviously have

$$\|w\|_{L^q(\mathbb{R}^d)} \leq \|u\|_{L^q(\mathbb{R}^d)} + \|v\|_{L^q(\mathbb{R}^d)} \leq (1 + M^{1/q}) \|u\|_{L^q(\mathbb{R}^d)}.$$

We now proceed to prove (5.5). To do that we prove the following inequalities:

$$(5.11) \quad \|w\|_{L^p(\mathbb{R}^d)}^p \leq c_1^{-1} \sup_{x \in \mathbb{R}^d} \|\rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)}^p \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y) |u(x) - u(y)|^p dx dy$$

and

$$(5.12) \quad \|\nabla v\|_{L^p(\mathbb{R}^d)}^p \leq \sum_{k=1}^d c_1^{-1} \sup_{x \in \mathbb{R}^d} \|\partial_{x_k} \rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)}^p \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y) |u(x) - u(y)|^p dx dy.$$

The fact that for any  $x \in \mathbb{R}^d$ ,  $\rho(x, \cdot)$  is supported in the set  $B_x$  and has mass one gives the following

$$\begin{aligned} w(x) &= u(x) - \int_{\mathbb{R}^d} \rho(x, y) u(y) dy = \int_{\mathbb{R}^d} \rho(x, y) (u(x) - u(y)) dy \\ &= \int_{B_x} \rho(x, y) (u(x) - u(y)) dy. \end{aligned}$$

Then by Hölder's inequality we get:

$$\begin{aligned} \|w\|_{L^p(\mathbb{R}^d)}^p &= \int_{\mathbb{R}^d} \left| \int_{B_x} \rho(x, y) (u(x) - u(y)) dy \right|^p dx \\ &\leq \int_{\mathbb{R}^d} \int_{B_x} |u(x) - u(y)|^p dy \left( \int_{B_x} \rho(x, y)^{p'} dy \right)^{\frac{p}{p'}} dx \\ &\leq \sup_{x \in \mathbb{R}^d} \left( \int_{B_x} \rho(x, y)^{p'} dy \right)^{\frac{p}{p'}} \int_{\mathbb{R}^d} \int_{B_x} |u(x) - u(y)|^p dy dx \\ &\leq \sup_{x \in \mathbb{R}^d} \|\rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)}^p \int_{\mathbb{R}^d} \int_{B_x} |u(x) - u(y)|^p dy dx. \end{aligned}$$

Using now that for any  $x \in \mathbb{R}^d$  and  $y \in B_x$  we have  $J(x, y) > c_1$  we obtain

$$\begin{aligned} \|w\|_{L^p(\mathbb{R}^d)}^p &\leq c_1^{-1} \sup_{x \in \mathbb{R}^d} \|\rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)}^p \int_{\mathbb{R}^d} \int_{B_x} J(x, y) |u(x) - u(y)|^p dy dx \\ &\leq c_1^{-1} \sup_{x \in \mathbb{R}^d} \|\rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)}^p \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y) |u(x) - u(y)|^p dy dx \end{aligned}$$

which proves (5.11).

In the case of  $v$  we proceed in a similar manner, by tacking into account that for any  $x \in \mathbb{R}$  the mass of  $\partial_{x_k} \rho(x, y)$ ,  $k = 1, \dots, d$  vanishes:

$$\int_{\mathbb{R}^d} \partial_{x_k} \rho(x, y) dy = \partial_{x_k} \left( \int_{\mathbb{R}^d} \rho(x, y) dy \right) = 0.$$

The definition of  $v$  and this mass property gives,

$$\partial_{x_k} v(x) = \int_{\mathbb{R}^d} \partial_{x_k} \rho(x, y) (u(y) - u(x)) dy = \int_{B_x} \partial_{x_k} \rho(x, y) (u(y) - u(x)) dy.$$

Thus, by Hölder inequality and the fact that  $J(x, y) > c_1$  for all  $y \in B_x$  we obtain,

$$\begin{aligned} \|\partial_{x_k} v\|_{L^p(\mathbb{R}^d)}^p &= \int_{\mathbb{R}^d} \left| \int_{B_x} \partial_{x_k} \rho(x, y) (u(y) - u(x)) dy \right|^p dx \\ &\leq \int_{\mathbb{R}^d} \int_{B_x} |u(y) - u(x)|^p dy \left( \int_{B_x} |\partial_{x_k} \rho(x, y)|^{p'} dy \right)^{\frac{p}{p'}} dx \\ &= \sup_{x \in \mathbb{R}^d} \|\partial_{x_k} \rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)}^p \int_{\mathbb{R}^d} \int_{B_x} |u(y) - u(x)|^p dx dy \\ &\leq c^{-1} \sup_{x \in \mathbb{R}^d} \|\partial_{x_k} \rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)}^p \int_{\mathbb{R}^d} \int_{B_x} J(x, y) |u(y) - u(x)|^p dx dy \\ &\leq c^{-1} \sup_{x \in \mathbb{R}^d} \|\partial_{x_k} \rho(x, \cdot)\|_{L^{p'}(\mathbb{R}^d)}^p \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y) |u(y) - u(x)|^p dx dy. \end{aligned}$$

Summing the above inequalities for all  $k = 1, \dots, d$  we get (5.12).

The proof is now finished since (5.11) and (5.12) imply (5.5).  $\square$

Now we present a similar result to Corollary 31 which can be used to obtain less accurate bounds (hence we prefer to use the more general result presented above) in the particular case of the nonlocal laplacian, i.e.  $p = 2$ , and  $J(x, y) = G(x - y)$ . The result is no so general as Corollary 31, but it is obtained using Fourier analysis tools and has the advantage that the previous decomposition  $u = v + w$  can be better understood. We include it here just for this purpose. In fact this decomposition can be viewed as a Fourier splitting of the function  $u$  in two parts, the first one,  $v$ , corresponding to the low frequencies (the smooth part) of  $u$ , and the second one,  $w$ , corresponds to the high frequencies component (the rough part) of  $u$ .

We will use that in the particular case  $p = 2$  and  $J(x, y) = G(x - y)$ ,  $G$  with mass one, the operator  $\langle A_2 u, u \rangle$  can be represented by means of the Fourier transform of  $G$  as follows

$$\langle A_2 u, u \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x - y) |u(x) - u(y)|^2 dx dy = \int_{\mathbb{R}^d} (1 - \widehat{G}(\xi)) |\widehat{u}(\xi)|^2 d\xi.$$

LEMMA 33. *Let  $d \geq 3$  and  $G$  be such that its Fourier transform  $\widehat{G}(\xi)$  satisfies*

$$(5.13) \quad \begin{cases} \widehat{G}(\xi) \leq 1 - \frac{|\xi|^2}{2}, & |\xi| \leq R, \\ \widehat{G}(\xi) \leq 1 - \delta, & |\xi| \geq R, \end{cases}$$

for some positive numbers  $R$  and  $\delta$ . Then, for any  $\varepsilon \in (0, 1)$  there exists a constant  $C = C(\varepsilon, \delta, R, d)$  such that the following

$$(5.14) \quad \|u\|_{L^2(\mathbb{R}^d)}^2 \leq C \|u\|_{L^{1+\varepsilon}(\mathbb{R}^d)}^{2(1-\beta(\varepsilon))} \langle A_2 u, u \rangle^{\beta(\varepsilon)} + \langle A_2 u, u \rangle$$

holds for all  $u \in L^{1+\varepsilon}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$  where

$$\beta(\varepsilon) = \frac{(1 - \varepsilon)d}{d + 2 - \varepsilon(d - 2)}.$$

REMARK 34. *The limit case  $\varepsilon = 0$  cannot be obtained since an estimate of the type*

$$\|(\mathbf{1}_{\{|\xi| \leq R\}} \widehat{u})^\vee\|_{L^1(\mathbb{R}^d)} \leq \|u\|_{L^1(\mathbb{R}^d)}$$

does not hold for all functions  $u \in L^1(\mathbb{R}^d)$ . In dimension one this can be seen by choosing a sequence  $u_\varepsilon$  with  $\|u_\varepsilon\|_{L^1(\mathbb{R}^d)} = 1$  such that  $u_\varepsilon \rightarrow \delta_0$ , the Dirac delta. Then

$$(\mathbf{1}_{\{|\xi| \leq R\}} \widehat{u}_\varepsilon)^\vee = u_\varepsilon * \frac{\sin(Rx)}{Rx} \rightarrow \frac{\sin Rx}{Rx}$$

and the last function does not belong to  $L^1(\mathbb{R}^d)$ . Thus  $\|(\mathbf{1}_{\{|\xi| \leq R\}} \widehat{u}_\varepsilon)^\vee\|_{L^1(\mathbb{R}^d)} \rightarrow \infty$  but  $\|u_\varepsilon\|_{L^1(\mathbb{R}^d)} = 1$ .

REMARK 35. *The same arguments can be used to obtain estimates for any function  $G$  which satisfies*

$$\begin{cases} \widehat{G}(\xi) \leq 1 - \frac{|\xi|^{2s}}{2}, & |\xi| \leq R, \\ \widehat{G}(\xi) \leq 1 - \delta, & |\xi| \geq R, \end{cases}$$

for some positive numbers  $R$ ,  $\delta$  and  $s$ .

PROOF OF LEMMA 33. For any function  $u \in L^2(\mathbb{R}^d)$  we define its projections on the low and high frequencies respectively,

$$(5.15) \quad v := (\mathbf{1}_{\{|\xi| \leq R\}} \widehat{u})^\vee, \quad w := (\mathbf{1}_{\{|\xi| \geq R\}} \widehat{u})^\vee.$$

Using that the function  $\widehat{G}$  satisfies (5.13) we obtain the following estimate for the operator  $A_2$ :

$$\begin{aligned}
(5.16) \quad \langle A_2 u, u \rangle &= \int_{\mathbb{R}^d} (1 - \widehat{G}(\xi)) |\widehat{u}(\xi)|^2 d\xi \geq \int_{|\xi| \leq R} \frac{|\xi|^2}{2} |\widehat{u}(\xi)|^2 d\xi + \delta \int_{|\xi| \geq R} |\widehat{u}(\xi)|^2 d\xi \\
&= \frac{1}{2} \int_{\mathbb{R}^d} |\xi|^2 |\widehat{v}(\xi)|^2 d\xi + \delta \int_{\mathbb{R}^d} |\widehat{w}(\xi)|^2 d\xi \\
&\geq c(\delta) \left( \|\nabla v\|_{L^2(\mathbb{R}^d)}^2 + \|w\|_{L^2(\mathbb{R}^d)}^2 \right) \\
&\geq c(\delta) \left( \|v\|_{L^{2^*}(\mathbb{R}^d)}^2 + \|w\|_{L^2(\mathbb{R}^d)}^2 \right).
\end{aligned}$$

In order to estimate from above the  $L^2(\mathbb{R}^d)$ -norm of  $u$  as in (5.14), using the orthogonality of  $v$  and  $w$  it is sufficient to estimate each projection  $v$  and  $w$  since

$$\|u\|_{L^2(\mathbb{R}^d)}^2 = \|v\|_{L^2(\mathbb{R}^d)}^2 + \|w\|_{L^2(\mathbb{R}^d)}^2.$$

In the case of  $w$ , using (5.15) and (5.16) we have the rough estimate:

$$(5.17) \quad \|w\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{1}{c(\delta)} \langle A_2 u, u \rangle.$$

Next we estimate the  $L^2(\mathbb{R}^d)$ -norm of  $v$ . We recall that classical results on Fourier multipliers give us that for any  $p \in (1, \infty)$  the  $L^p(\mathbb{R}^d)$ -norm of  $v$ , defined by (5.15), can be bounded from above by the  $L^p(\mathbb{R}^d)$ -norm of  $u$  as follows:

$$(5.18) \quad \|v\|_{L^p(\mathbb{R}^d)} \leq C(p, d) \|u\|_{L^p(\mathbb{R}^d)}.$$

Using this estimate and interpolation inequalities we obtain that  $v$ , the low frequency projection of  $u$ , satisfies

$$\begin{aligned}
(5.19) \quad \|v\|_{L^2(\mathbb{R}^d)}^2 &\leq \left( \|v\|_{L^{1+\varepsilon}(\mathbb{R}^d)}^{1-\beta(\varepsilon)} \|v\|_{L^{2^*}(\mathbb{R}^d)}^{\beta(\varepsilon)} \right)^2 \leq \left( c(\varepsilon, d) \|u\|_{L^{1+\varepsilon}(\mathbb{R}^d)}^{1-\beta(\varepsilon)} \|v\|_{L^{2^*}(\mathbb{R}^d)}^{\beta(\varepsilon)} \right)^2 \\
&\leq c^2(\varepsilon, d) c(\delta)^{-\beta(\varepsilon)} \|u\|_{L^{1+\varepsilon}(\mathbb{R}^d)}^{2(1-\beta(\varepsilon))} \langle A_2 u, u \rangle^{\beta(\varepsilon)},
\end{aligned}$$

where  $c(\varepsilon, d)$  is given by applying (5.18) with  $p = 1 + \varepsilon$  and  $\beta(\varepsilon)$  by

$$\frac{1}{2} = \frac{1 - \beta(\varepsilon)}{1 + \varepsilon} + \frac{\beta(\varepsilon)}{2^*},$$

that is,

$$\beta(\varepsilon) = \frac{(1 - \varepsilon)d}{d + 2 - \varepsilon(d - 2)}.$$

Combining (5.16), (5.17) and (5.19) we obtain

$$(5.20) \quad \|u\|_{L^2(\mathbb{R}^d)}^2 \leq c(\varepsilon, \delta, d) \|u\|_{L^{1+\beta(\varepsilon)}(\mathbb{R}^d)}^{2(1-\beta(\varepsilon))} \langle A_2 u, u \rangle^{\alpha(\varepsilon)} + \langle A_2 u, u \rangle.$$

The proof is now finished.  $\square$

We end this section with a crucial but simple result concerning the decay of solutions to a differential inequality.

LEMMA 36. *Let  $t_0 \geq 0$  and  $\psi : [0, \infty) \rightarrow (0, \infty)$  such that for all  $t > t_0$*

$$(5.21) \quad \psi_t + \alpha\psi^\beta t^\gamma \leq 0$$

*holds for some constants  $\alpha > 0$ ,  $\beta > 1$  and  $\gamma$ . Then there exists a positive constant  $c(\alpha, \beta)$  such that*

$$\psi(t) \leq c(\alpha, \beta, \gamma)(t^{\gamma+1} - (t_0)^{\gamma+1})^{-\frac{1}{\beta-1}}$$

*holds for all  $t > t_0$ .*

PROOF. Inequality (5.21) gives us

$$\psi_t \psi^{-\beta} + \alpha t^\gamma \leq 0.$$

Integrating on  $[t_0, t]$  we find that for any  $t \geq t_0$

$$\frac{\psi^{1-\beta}(t)}{1-\beta} - \frac{\psi^{1-\beta}(t_0)}{1-\beta} + \alpha \frac{(t^{\gamma+1} - (t_0)^{\gamma+1})}{\gamma+1} \leq 0.$$

Then

$$\alpha(t^{\gamma+1} - (t_0)^{\gamma+1})(\beta-1)(\gamma+1)^{-1} \leq \psi^{1-\beta}(t)$$

and hence

$$\psi(t) \leq c(\alpha, \beta, \gamma)(t^{\gamma+1} - (t_0)^{\gamma+1})^{-\frac{1}{\beta-1}}$$

where  $c(\alpha, \beta, \gamma)$  is a constant.  $\square$

### Decay estimates for the linear diffusion problem with a nonlinear source.

In this section we will obtain the long time behavior of the solutions  $u$  to the following equation

$$(5.22) \quad u_t(x, t) = \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t)) dy + f(u)(x, t)$$

under suitable assumptions on the kernel  $J$  and the nonlinearity  $f$ . Our goal is to obtain here a proof of the decay rate of the solution  $u$  to (5.22) by using energy methods.

The main result of this section is the following theorem.

THEOREM 37. *Let  $J(x, y)$  be a symmetric nonnegative kernel satisfying HJ1) as in Theorem 26 and  $f$  be a locally Lipschitz function with  $f(s)s \leq 0$ . For any  $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  there exists a unique solution to equation (5.22) which satisfies*

$$(5.23) \quad \|u(t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)} \quad \text{and} \quad \|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}$$

*for every  $t > 0$ .*

*Moreover, if  $d \geq 3$  and  $J$  also satisfies HJ2) then the following holds:*

$$(5.24) \quad \|u(t)\|_{L^q(\mathbb{R}^d)} \leq C(q, d)\|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{q})}$$

*for all  $q \in [1, \infty)$  and for all  $t$  sufficiently large.*

REMARK 38. *The proof uses the results of Theorem 26 and Corollary 31 in the particular case  $p = 2$ . In order to apply Corollary 31 we need to assume  $d > 2$ , i.e.  $d \geq 3$ .*

The following lemma will be used in the proof of Theorem 37.

LEMMA 39. *Let  $d > 2$  and  $u$  such that  $u(t) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$  for all  $t \geq 0$  satisfying:*

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^2(x, t) dx + \langle A_2 u(t), u(t) \rangle \leq 0, \quad \text{for all } t > 0,$$

with  $J$  as in Theorem 26. Assuming that

$$(5.25) \quad \|u(t)\|_{L^1(\mathbb{R}^d)} \leq \|u(0)\|_{L^1(\mathbb{R}^d)}, \quad \text{for all } t > 0,$$

there exists a constant  $c(d, J)$  such that

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq c(d, J) \|u(0)\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}$$

holds for all  $t$  large enough.

REMARK 40. *Under the same hypotheses we can replace the initial time  $t = 0$  with any positive time  $t_0$ , the result being the same for large time  $t$ ,*

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq c(d) \|u(t_0)\|_{L^1(\mathbb{R}^d)} (t - t_0)^{-\frac{d}{2}(1-\frac{1}{2})}.$$

PROOF OF LEMMA 39. By Corollary 31 and property (5.25) we obtain

$$\begin{aligned} \|u(t)\|_{L^2(\mathbb{R}^d)}^2 &\leq C_1(J) \|u(t)\|_{L^1(\mathbb{R}^d)}^{2(1-\alpha(2))} \langle A_2 u(t), u(t) \rangle^{\alpha(2)} + C_2(J) \langle A_2 u(t), u(t) \rangle \\ &\leq C_1(J) \|u(0)\|_{L^1(\mathbb{R}^d)}^{2(1-\alpha(2))} \langle A_2 u(t), u(t) \rangle^{\alpha(2)} + C_2(J) \langle A_2 u(t), u(t) \rangle \end{aligned}$$

where  $\alpha(2) = d/(d+2)$  is given by (5.9). To simplify the presentation we will assume without loss of generality that  $C_1(J) = C_2(J) = 1$  (otherwise one can track the constants that appear in each step of the proof). Then for any  $t > 0$ ,  $\langle A_2 u(t), u(t) \rangle$  satisfies

$$H^{-1}(\|u(t)\|_{L^2(\mathbb{R}^d)}^2) \leq \langle A_2 u(t), u(t) \rangle$$

where

$$H(x) = \|u(0)\|_{L^1(\mathbb{R}^d)}^{2(1-\alpha(2))} x^{\alpha(2)} + x.$$

Analyzing the function  $H_{a,\beta}(x) = ax^\beta + x$ ,  $a > 0$ ,  $\beta \in (0, 1)$ , we observe that

$$H_{a,\beta}(x) \leq \begin{cases} 2x & x > a^{\frac{1}{1-\beta}}, \\ 2ax^\beta, & x < a^{\frac{1}{1-\beta}} \end{cases}$$

and then

$$(5.26) \quad H_{a,\beta}^{-1}(y) \geq \begin{cases} \frac{y}{2}, & y > 2a^{\frac{1}{1-\beta}}, \\ (\frac{y}{2a})^{\frac{1}{\beta}}, & y < 2a^{\frac{1}{1-\beta}}. \end{cases}$$



Applying this property to  $a = \|u(0)\|_{L^1(\mathbb{R}^d)}^{1-\alpha(2)}$ ,  $\beta = \alpha(2)$  we find that  $\langle A_2 u(t), u(t) \rangle$  verifies:

$$\langle A_2 u(t), u(t) \rangle \geq \begin{cases} \frac{1}{2} \|u(t)\|_{L^2(\mathbb{R}^d)}^2, & \|u(t)\|_{L^2(\mathbb{R}^d)}^2 > 2 \|u(0)\|_{L^1(\mathbb{R}^d)}^2, \\ \left( \frac{\|u(t)\|_{L^2(\mathbb{R}^d)}^2}{2 \|u(0)\|_{L^1(\mathbb{R}^d)}^{2(1-\alpha(2))}} \right)^{\frac{1}{\alpha(2)}}, & \|u(t)\|_{L^2(\mathbb{R}^d)}^2 < 2 \|u(0)\|_{L^1(\mathbb{R}^d)}^2. \end{cases}$$

Then,  $\phi(t) = \|u(t)\|_{L^2(\mathbb{R}^d)}^2$  satisfies the following differential inequality for all  $t \geq 0$ :

$$\phi_t(t) + \frac{\phi(t)}{2} \chi_{\{\phi(t) > 2 \|u(0)\|_{L^1(\mathbb{R}^d)}^2\}} + \left( \frac{\phi(t)}{2 \|u(0)\|_{L^1(\mathbb{R}^d)}^{2(1-\alpha(2))}} \right)^{\frac{1}{\alpha(2)}} \chi_{\{\phi(t) < 2 \|u(0)\|_{L^1(\mathbb{R}^d)}^2\}} \leq 0.$$

This implies the existence of a time  $t_0$  such that  $\phi(t_0) < 2 \|u(0)\|_{L^1(\mathbb{R}^d)}^2$ . If not, then for all time  $t \geq t_0$  we get  $\phi(t) > 2 \|u(0)\|_{L^1(\mathbb{R}^d)}^2$  and

$$\phi_t(t) + \frac{1}{2} \phi(t) \leq 0.$$

Integrating the above inequality on  $(t_0, t)$  we obtain  $\phi(t) \leq e^{-(t-t_0)/2} \phi(t_0)$  which contradicts our assumption. Thus, there exists  $t_0$  such that  $\phi(t_0) < 2 \|u(0)\|_{L^1(\mathbb{R}^d)}^2$ . Using that  $\phi_t(t) \leq 0$  we obtain that  $\phi(t) < 2 \|u(0)\|_{L^1(\mathbb{R}^d)}^2$  holds for all  $t \geq t_0$  and  $\phi(t)$  satisfies the following differential inequality for all  $t \geq t_0$ :

$$\phi_t(t) + \left( \frac{\phi(t)}{2 \|u(0)\|_{L^1(\mathbb{R}^d)}^{2(1-\alpha(2))}} \right)^{\frac{1}{\alpha(2)}} \leq 0.$$

Integrating it on  $(t_0, t)$  we get by Lemma 36 with  $\gamma = 0$  that  $\phi$  satisfies

$$\phi(t) \leq C \|u(0)\|_{L^1(\mathbb{R}^d)}^2 (t - t_0)^{-d(1-\frac{1}{2})}, \quad t > t_0,$$

in other words

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq C \|u(0)\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}$$

holds for all time  $t$  large enough.  $\square$

**PROOF OF THEOREM 37. Step I. Global existence and uniqueness.** First, let us prove the existence and uniqueness of a local solution. To this end we use a fixed point argument.

Let us consider the space

$$X = C^0([0, T]; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$$

with the norm

$$\|u\|_X = \max_{t \in [0, T]} \left\{ \|u(t)\|_{L^1(\mathbb{R}^d)} + \|u(t)\|_{L^\infty(\mathbb{R}^d)} \right\}.$$

We observe that the operator  $A : X \rightarrow X$  defined by

$$Au(x) = \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t))dy$$

is continuous since using HJ1) and the symmetry of  $J$  we get

$$\|Au\|_{L^\infty(\mathbb{R}^d)} \leq 2\|u\|_{L^\infty(\mathbb{R}^d)} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)dy \leq 2C\|u\|_{L^\infty(\mathbb{R}^d)}$$

and

$$\|Au\|_{L^1(\mathbb{R}^d)} \leq 2\|u\|_{L^1(\mathbb{R}^d)} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)dy \leq 2C\|u\|_{L^1(\mathbb{R}^d)}.$$

Since the map  $u \rightarrow f(u)$  is Lipschitz continuous on bounded subsets of  $X$  (as a consequence of the properties of  $f$ ) classical results on semilinear evolution problems guarantees the existence of a unique local solution  $u$ .

We now prove (5.23) which guarantee the global existence of solutions to equation (5.22). We multiply equation (5.22) with  $\text{sgn}(u)$  and integrate on  $\mathbb{R}^d$ :

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |u(x, t)|dx &= \int_{\mathbb{R}^d} u_t(x, t) \text{sgn}(u(x, t))dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t)) \text{sgn}(u(x, t))dxdy \\ &\quad + \int_{\mathbb{R}^d} f(u(x, t)) \text{sgn}(u(x, t))dx. \end{aligned}$$

Using Lemma 25 and the fact that  $f(s)s \leq 0, s \in \mathbb{R}$ , we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |u(x, t)|dx &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t)) \text{sgn}(u(x, t))dydx \\ &= -\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t))(\text{sgn}(u(y, t)) - \text{sgn}(u(x, t)))dxdy \\ &\leq 0. \end{aligned}$$

From here it follows that

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}.$$

Now, multiplying the equation by  $(u(x, t) - M)_+$ , where  $M = \|u_0\|_{L^\infty(\mathbb{R}^d)}$ , and integrating on  $\mathbb{R}^d$  we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \frac{(u(x, t) - M)_+^2}{2} dx &= \int_{\mathbb{R}^d} u_t(x, t)(u(x, t) - M)_+ dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t))(u(x, t) - M)_+ dydx \\ &\quad + \int_{\mathbb{R}^d} f(u(x, t))(u(x, t) - M)_+ dx. \end{aligned}$$

Using Lemma 25, the sign property of the function  $f$  and the fact that for any two real numbers  $a$  and  $b$  we have

$$|a_+ - b_+|^2 \leq (a - b)(a_+ - b_+),$$

it follows that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \frac{(u(x, t) - M)_+^2}{2} dx &= \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t))(u(x, t) - M)_+ dy dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)(u(y, t) - u(x, t))((u(y, t) - M)_+ - (u(x, t) - M)_+) dx dy \\ &\leq -\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y)|(u(y, t) - M)_+ - (u(x, t) - M)_+|^2 dx dy. \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^d} \frac{(u(x, t) - M)_+^2}{2} dx = 0$$

and we obtain that  $u(x, t) \leq M$  for all  $t \geq 0$  and a.e.  $x \in \mathbb{R}^d$ .

In a similar way we get  $u(x, t) \geq -M$  for all  $t \geq 0$  and a.e.  $x \in \mathbb{R}^d$ .

We conclude that  $\|u\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}$  and that the solution  $u$  is global.

**Step II. Proof of the long time behaviour.** We divide the proof in several steps.

**Step II a). The case  $p = 2$ .** Multiplying equation (5.22) by  $\text{sgn}(u)$  and  $u$  we obtain

$$(5.27) \quad \frac{d}{dt} \int_{\mathbb{R}^d} |u(t, x)| dx \leq 0$$

and

$$(5.28) \quad \frac{d}{dt} \int_{\mathbb{R}^d} u^2(t) dx + \langle A_2 u(t), u(t) \rangle \leq 0.$$

Inequality (5.27) implies that (5.25) holds.

Inequalities (5.27) and (5.28) allow us to apply Lemma 39. Thus we obtain that

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}.$$

holds for large enough  $t$ . This gives us, by interpolation, the long time behaviour of the solution  $u$  in any  $L^q(\mathbb{R}^d)$ -norm when  $1 \leq q \leq 2$ .

**Step II b). The case  $p = 2^{n+1}$ .** We use an iterative argument to prove that once the result is assumed for  $p = 2^n$  we get the result for  $p = 2^{n+1}$ .

Assume that it holds for  $p = 2^n$ . Then

$$\|u(t)\|_{L^{2^n}(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2^n})}$$

holds for all  $t$  large enough.

Let us fix  $r = 2^{n+1}$ . We multiply equation (5.22) with  $u^{r-1}$  to obtain

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} \int_{\mathbb{R}^d} u^r(x, t) dx &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y) (u(x, t) - u(y, t)) u^{r-1}(x, t) dx dy \\ &= -\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y) (u(x, t) - u(y, t)) (u^{r-1}(x, t) - u^{r-1}(y, t)) dx dy \\ &\leq -c(r) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x, y) (u^{r/2}(x, t) - u^{r/2}(y, t))^2 dx dy. \end{aligned}$$

Then  $v = u^{r/2}$  verifies:

$$\frac{d}{dt} \int_{\mathbb{R}^d} v^2(x, t) dx + c(r) \langle A_2 v(t), v(t) \rangle \leq 0, \quad t > 0.$$

By Lemma 39 and Remark 40 we obtain that for large time  $t$  the following holds:

$$\|v(t)\|_{L^2(\mathbb{R}^d)} \leq \|v(t/2)\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}.$$

Then

$$\|u^{r/2}(t)\|_{L^2(\mathbb{R}^d)} \leq \|u^{r/2}(t/2)\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2})}$$

and using that  $r = 2^{n+1}$ :

$$\|u(t)\|_{L^{2^{n+1}}(\mathbb{R}^d)}^{2^n} \leq C(d, n) \|u(t/2)\|_{L^{2^n}(\mathbb{R}^d)}^{2^n} t^{-\frac{d}{2}(1-\frac{1}{2})}.$$

Using the hypothesis on the  $L^{2^n}(\mathbb{R}^d)$ -norm of  $u$  we get

$$\begin{aligned} \|u(t)\|_{L^{2^{n+1}}(\mathbb{R}^d)} &\leq C(d, n) \|u(t/2)\|_{L^{2^n}(\mathbb{R}^d)} t^{-\frac{d}{2}(\frac{1}{2^n} - \frac{1}{2^{n+1}})} \\ &\leq C(d, n) \|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2^n})} t^{-\frac{d}{2}(\frac{1}{2^n} - \frac{1}{2^{n+1}})} \\ &\leq C(d, n) \|u_0\|_{L^1(\mathbb{R}^d)} t^{-\frac{d}{2}(1-\frac{1}{2^{n+1}})}. \end{aligned}$$

The proof is now finished since we can interpolate between the cases  $r = 2^n$  and  $r = 2^{n+1}$ ,  $n \geq 0$  an integer. Indeed, given  $q \in (1, \infty)$  we can find a positive integer  $n$  such that  $2^n \leq q < 2^{n+1}$ . Then

$$\|u\|_{L^q(\mathbb{R}^d)} \leq \|u\|_{L^{2^n}(\mathbb{R}^d)}^a \|u\|_{L^{2^{n+1}}(\mathbb{R}^d)}^{(1-a)},$$

where  $a = a(q, n)$  is given by

$$\frac{1}{q} = \frac{a}{2^n} + \frac{1-a}{2^{n+1}}$$

and the general case follows.

The proof of decay property (5.24) is now finished.  $\square$

**Examples of exponential decay.** In this section we present a simple example of  $J(x, y)$  for which we obtain exponential decay of the solutions to the linear problem

$$(5.29) \quad u_t(x, t) = \int_{\mathbb{R}} J(x, y) (u(y, t) - u(x, t)) dy.$$

Note that, to simplify, we restrict ourselves to one space dimension.

LEMMA 41. *Let  $a : \mathbb{R} \rightarrow \mathbb{R}$  be a diffeomorphism. Assume that*

$$J(x, y) \geq \frac{1}{2} \quad \text{on} \quad |y - a(x)| \leq 1,$$

where the function  $a$  satisfies

$$\sup_{\mathbb{R}} |(a^{-1})_x| < 1 \quad \text{or} \quad \inf_{\mathbb{R}} |(a^{-1})_x| > 1$$

then there exists a positive constant  $C$  such that

$$\langle A_2 u, u \rangle \geq C \|u\|_{L^2(\mathbb{R})}^2.$$

PROOF. Using the symmetry of the function  $J$  we get

$$J(x, y) \geq \frac{1}{2} \quad \text{on} \quad |x - a(y)| < 1.$$

Thus

$$(5.30) \quad J(x, y) \geq \frac{1}{4} \chi_{\{|x-a(y)|<1\}} + \frac{1}{4} \chi_{\{|y-a(x)|<1\}}.$$

Let us consider  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  a smooth positive function, supported on  $(-1, 1)$ . Then

$$2\|\psi\|_{L^\infty(\mathbb{R})} J(x, y) \geq \rho(x, y) := \psi(x - a(y)) + \psi(y - a(x))$$

and

$$(5.31) \quad 2\|\psi\|_{L^\infty(\mathbb{R})} \langle A_2 u, u \rangle \geq \iint_{\mathbb{R}^2} \rho(x, y) (u(x) - u(y))^2 dx dy.$$

Let be  $\theta$  a positive constant which will be fixed latter. Using the elementary inequality

$$(b - c)^2 = b^2 + c^2 - 2bc \geq b^2 + c^2 - \theta b^2 - \frac{1}{\theta} c^2 = (1 - \theta)(b^2 - \frac{c^2}{\theta})$$

we get

$$\begin{aligned} \iint_{\mathbb{R}^2} \rho(x, y) (u(x) - u(y))^2 dx dy &\geq (1 - \theta) \iint_{\mathbb{R}^2} \psi(y - a(x)) \left( u^2(x) - \frac{u^2(y)}{\theta} \right) dx dy \\ &= (1 - \theta) \left( \int_{\mathbb{R}} u^2(x) dx \int_{\mathbb{R}} \psi(y) dy - \frac{1}{\theta} \int_{\mathbb{R}} u^2(y) \int_{\mathbb{R}} \psi(y - a(x)) dx dy \right) \\ &= (1 - \theta) \int_{\mathbb{R}} u^2(x) \left( \int_{\mathbb{R}} \psi(y) dy - \frac{1}{\theta} \int_{\mathbb{R}} \psi(x - a(y)) dy \right) dx \\ &= (1 - \theta) \int_{\mathbb{R}} u^2(x) \left( \int_{\mathbb{R}} \psi(y) dy - \frac{1}{\theta} \int_{\mathbb{R}} \psi(x - y) |(a^{-1})_x(y)| dy \right) dx \\ &= \frac{1 - \theta}{\theta} \int_{\mathbb{R}} \psi(y) dy \int_{\mathbb{R}} u^2(x) \left( \theta - \frac{(\psi * |(a^{-1})_x|)(x)}{\int_{\mathbb{R}} \psi(y) dy} \right) dx. \end{aligned}$$

Then

$$\begin{aligned} & \iint_{\mathbb{R}^2} \rho(x, y) (u(x) - u(y))^2 dx dy \\ & \geq \begin{cases} \frac{1-\theta}{\theta} \int_{\mathbb{R}} \psi(y) dy \int_{\mathbb{R}} u^2(x) \left( \theta - \frac{\sup_{x \in \mathbb{R}} \psi * |(a^{-1})_x|}{\int_{\mathbb{R}} \psi(y) dy} \right) dx, & \theta < 1, \\ \frac{1-\theta}{\theta} \int_{\mathbb{R}} \psi(y) dy \int_{\mathbb{R}} u^2(x) \left( \theta - \frac{\inf \psi * (a^{-1})_x}{\int_{\mathbb{R}} \psi(y) dy} \right) dx, & \theta > 1. \end{cases} \end{aligned}$$

If

$$\sup_{x \in \mathbb{R}} |(a^{-1})_x(x)| < 1$$

then

$$\sup_{x \in \mathbb{R}} (\psi * |(a^{-1})_x|)(x) < \int_{\mathbb{R}} \psi(y) dy.$$

We choose  $\theta$  satisfying

$$\frac{\sup_{\mathbb{R}} \psi * |(a^{-1})_x|}{\int_{\mathbb{R}} \psi(y) dy} < \theta < 1$$

and thus by (5.31)

$$2\|\psi\|_{L^\infty(\mathbb{R})} \langle A_2 u, u \rangle \geq C(\theta, \psi, a) \|u\|_{L^2(\mathbb{R})}^2.$$

The other case

$$\inf_{x \in \mathbb{R}} |(a^{-1})_x(x)| > 1$$

can be treated in a similar way. Here we have

$$\inf_{x \in \mathbb{R}} (\psi * |(a^{-1})_x|)(x) > \int_{\mathbb{R}} \psi(y) dy$$

and then we choose  $\theta$  satisfying

$$\frac{\inf_{\mathbb{R}} \psi * |(a^{-1})_x|}{\int_{\mathbb{R}} \psi(y) dy} > \theta > 1$$

and thus

$$2\|\psi\|_{L^\infty(\mathbb{R})} \langle A_2 u, u \rangle \geq C(\theta, \psi, a) \|u\|_{L^2(\mathbb{R})}^2.$$

□

**THEOREM 42.** *Let  $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . Then the solution to (5.29) verifies*

$$\|u(\cdot, t)\|_{L^2(\mathbb{R})} \leq C e^{-Ct}$$

for all  $t > t_0$ .

PROOF. Multiplying equation (5.29) by  $u$  we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^2(t) dx + \langle A_2 u(t), u(t) \rangle \leq 0,$$

and using our previous estimate (Lemma 41) we get

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^2(t) dx + C \int_{\mathbb{R}^d} u^2(t) dx \leq 0,$$

from where the result follows. □





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