Simultaneous vs. non simultaneous blow-up

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Abstract

In this paper we study the possibility of simultaneous blow-up for positive solutions of a system of two heat equations, $u_t = \Delta u$, $v_t = \Delta v$, in a bounded smooth domain Ω, with boundary conditions $\frac{\partial u}{\partial \eta} = u^{p_{11}} v^{p_{12}}$, $\frac{\partial v}{\partial \eta} = u^{p_{21}} v^{p_{22}}$. We prove that if u blows up then v can fail to blow up if and only if $p_{11} > 1$ and $p_{21} > p_{11} - 1$.

1 Introduction.

In this note we study blowing up solutions of the following parabolic system,

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0,T), \\ v_t = \Delta v & \text{in } \Omega \times (0,T), \end{cases} \tag{1.1}$$

with boundary conditions,

$$\begin{cases} \frac{\partial u}{\partial \eta} = u^{p_{11}} v^{p_{12}} & \text{on } \partial\Omega \times (0,T), \\ \frac{\partial v}{\partial \eta} = u^{p_{21}} v^{p_{22}} & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1.2}$$

and initial data,

$$\begin{cases} u(x,0) = u_0(x) & \text{in } \Omega, \\ v(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{1.3}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^N$ and $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative. Throughout this paper we assume that $p_{ij} \geq 0$ and $u_0, v_0 > 0$. Under the hypothesis, $p_{11} > 1$, $p_{22} > 1$ or $(p_{11} - 1)(p_{22} - 1) - p_{12}p_{21} < 0$ it is proved

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in [10] (see also [12]) that the solution (u,v) blows up in finite time, T. At that time T we have, by the result of [10],

$$\limsup_{t\nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} = +\infty.$$

We observe that, a priori, there is no reason that guarantee that both functions, u and v, go to infinity simultaneously at time T. In this work we address this problem and characterize the simultaneous blow-up in the following sense: suppose that u goes to infinity at time T then v may remain bounded up to this time T if and only if $p_{11} > 1$ (this guarantee blow-up for u) and $p_{21} < p_{11} - 1$ (this implies that the coupling between u and v is "weak").

We prove:

Theorem 1.1 *Assume that u blows up at time T and that v remains bounded up to time T, then $p_{11} > 1$ and $p_{21} < p_{11} - 1$.*

Theorem 1.2 *If $p_{11} > 1$ and $p_{21} < p_{11} - 1$ then there exists initial data u_0 , v_0 such that u has blow-up but v remains bounded.*

To prove these theorems we have to assume that the blow-up rate for a single equation

$$\begin{cases} w_t = \Delta w & \text{in } \Omega \times (0,T), \\ \frac{\partial w}{\partial \eta} \geq c w^q & \text{on } \partial\Omega \times (0,T), \\ w(x,0) = w_0(x) & \text{in } \Omega, \end{cases} \tag{1.4}$$

with $q > 1$, is given by

$$\max_{\overline{\Omega}} w(\cdot,t) \leq \frac{C}{(T-t)^{\frac{1}{2(q-1)}}} \tag{1.5}$$

This estimate was proved in [4] and [5] under the hypothesis $q < \frac{N}{N-2}$ (for $N = 1, 2$ this requirement is not necessary) and some assumptions on the initial data. For the blow-up rate for related problems see [1], [3], [11] and [14].

2 Proof of the main results

We begin by Theorem 1.1. Assume that (u,v) has finite time blow-up T and that v remains bounded up to time T. Therefore, u must blow-up at time T. As u is a solution of

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} = u^{p_{11}} v^{p_{12}} \leq C u^{p_{11}} & \text{on } \partial\Omega \times (0,T), \\ w(x,0) = w_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

that has finite time blow-up, we must have $p_{11} > 1$ (see [13], [7]).

Now we want to prove that $p_{21} < p_{11} - 1$. Suppose not and let $\Gamma(x,t)$ be the fundamental solution of the heat equation, namely

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

Now for $x \in \partial\Omega$, using Green's identity and the jump relation, with $1 - \alpha/2 < \mu < 1$ (see [2], [5], [8], [14]) we have

$$\frac{1}{2}v(x,t) = \int_\Omega \Gamma(x-y,t-z)v(y,z)\ dy + \int_z^t \int_{\partial\Omega} \frac{\partial v}{\partial \eta}(y,\tau)\Gamma(x-y,t-\tau)\ dS_y\,d\tau - \\ - \int_z^t \int_{\partial\Omega} \frac{\partial \Gamma}{\partial \eta}(x-y,t-\tau)v(y,\tau)\ dS_y\,d\tau. \tag{2.2}$$

Now we set $V(t) = \sup_{\Omega\times[z,t]} v$. Since Ω is smooth, for instance $\partial\Omega \in C^{1+\alpha}$, Γ satisfies (see [2], [5], [8], [14])

$$\left|\frac{\partial \Gamma}{\partial \eta}(x-y,t-\tau)\right| \le \frac{C}{(t-\tau)^\mu |x-y|^{N+1-2\mu-\alpha}}$$

As $\frac{\partial v}{\partial \eta} = u^{p_{21}} v^{p_{22}} \ge C u^{p_{21}}$,

$$\frac{1}{2}V(t) \ge C\int_z^t \int_{\partial\Omega} u^{p_{21}}(y,\tau)\Gamma(x-y,t-\tau)\ dS_y\,d\tau - CV(t)(T-z)^{1-\mu}.$$

Now, we choose z such that $C(T-z)^{1-\mu} < 1/2$ then

$$V(t) \ge C\int_z^t \int_{\partial\Omega} u^{p_{21}}(y,\tau)\Gamma(x-y,t-\tau)\ dS_y\,d\tau.$$

From [9] we have that the blow-up set for u is contained in $\partial\Omega$, so let $x_0 \in \partial\Omega$ be a blow-up point for u. It is proved in [5] that under the hypothesis (1.5) the blow-up limit is nontrivial, that is

$$\liminf_{t\to T} \inf_{\|x\|\le K} u(x_0 + x\sqrt{T-t}, t)(T-t)^{\frac{1}{2(p_{11}-1)}} \neq 0.$$

Then there exists a constant c such that

$$u(x_0 + x\sqrt{T-t}, t) \ge c\frac{1}{(T-t)^{\frac{1}{2(p_{11}-1)}}} \qquad \forall\ \|x\| \le K.$$

With this bound for u we obtain,

$$V(t) \ge C\int_z^t \frac{1}{(T-\tau)^{\frac{p_{21}}{2(p_{11}-1)}}} \int_{\partial\Omega\cap\{\|y-x_0\|\le K(T-t)^{1/2}\}} \Gamma(x-y,t-\tau)\ dS_y\,d\tau$$

$$\geq C\int_z^t \frac{1}{(T-\tau)^{\frac{p_{21}}{2(p_{11}-1)}}(t-\tau)^{1/2}}\,d\tau.$$

One can check that the integral in the right hand side diverges as $t \nearrow T$ if $\frac{p_{21}}{2(p_{11}-1)} \geq \frac{1}{2}$, and hence v must blow up at time T, a contradiction.

Now we prove Theorem 1.2. We want to choose u_0, v_0 in order to obtain a blowing up solution (u,v) with v bounded. First assume that $p_{22} > 1$. We choose $0 < v_0(x) < 1/4$ and observe that, as $p_{11} > 1$, every positive solution has finite time blow-up and u satisfies (1.4), so, since we assume that (1.5) holds, we have

$$u(x,t) \leq \frac{C}{(T-t)^{\frac{1}{2(p_{11}-1)}}}.$$

From this we observe that if u_0 is large with v_0 fixed then T becomes small.

As before, we want to use the representation formula obtained from the fundamental solution. As $\frac{\partial v}{\partial \eta} = u^{p_{21}}v^{p_{22}} \leq Cv^{p_{22}}(T-t)^{-\frac{p_{21}}{2(p_{11}-1)}}$, using that $V(t)$ is increasing, from (2.2) we obtain,

$$\frac{1}{2}V(t) \leq V(z) + CV(t)^{p_{22}}\int_z^t \frac{1}{(T-\tau)^{\frac{p_{21}}{2(p_{11}-1)}}(t-\tau)^{1/2}}\,d\tau + CV(t)(T-z)^{1-\mu}.$$

We choose u_0 large enough in order to get T small such that $C(T-z)^{1-\mu} < 1/8$. Since $\frac{p_{21}}{2(p_{11}-1)} < 1/2$, we can assume that the integral is less than $1/8$ if T is small (u_0 large), hence, choosing $z = 0$,

$$\frac{3}{8}V(t) \leq V(0) + \frac{1}{8}V(t)^{p_{22}}.$$

We claim that $V(t)$ is less that one $(0 < t < T)$. To prove this, suppouse not and let $0 < t_0 < T$ be the first time such that $V(t_0) = 1$ (i.e $V(t) < 1$ for all $0 < t < t_0$). For $0 \leq t \leq t_0$ we have $V(t)^{p_{22}} \leq V(t)$ and then

$$\frac{1}{4}V(t_0) \leq V(0) < \frac{1}{4}$$

a contradiction with $V(t_0) = 1$. We can conclude that v remains bounded up to time T.

Next, assume that $p_{22} \leq 1$, we choose $v_0 > 1$ in order to obtain $V(t) > 1$ and with the same arguments as before we obtain,

$$\frac{1}{4}V(t) - \frac{1}{8}V(t)^{p_{22}} \leq V(z).$$

Now $V(t)^{p_{22}} \leq V(t)$ and then,

$$\frac{1}{8}V(t) \leq V(z).$$

The right hand side of the last inequality is bounded uniformly in t as we wanted to prove. $\square$

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