# THE LIMIT AS $p \to \infty$ IN A NONLOCAL $p-$LAPLACIAN EVOLUTION EQUATION. A NONLOCAL APPROXIMATION OF A MODEL FOR SANDPILES

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Abstract. In this paper we study the nonlocal $\infty-$Laplacian type diffusion equation obtained as the limit as $p \to \infty$ to the nonlocal analogous to the $p-$Laplacian evolution,

$$u_t(t,x) = \int_{\mathbb{R}^N} J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))\,dy.$$

We prove existence and uniqueness of a limit solution that verifies an equation governed by the subdifferetial of a convex energy functional associated to the indicator function of the set $K = \{u \in L^2(\mathbb{R}^N) \,:\, |u(x)-u(y)| \le 1, \text{ when } x-y \in supp(J)\}$. We also find some explicit examples of solutions to the limit equation.

If the kernel $J$ is rescaled in an appropriate way, we show that the solutions to the corresponding nonlocal problems converge strongly in $L^\infty(0,T;L^2(\Omega))$ to the limit solution of the local evolutions of the $p-$laplacian, $v_t = \Delta_p v$. This last limit problem has been proposed as a model to describe the formation of a sandpile.

Moreover, we also analyze the collapse of the initial condition when it does not belong to $K$ by means of a suitable rescale of the solution that describes the initial layer that appears for $p$ large.

Finally, we give an interpretation of the limit problem in terms of Monge-Kantorovich mass transport theory.

## 1. Introduction

Our main purpose in this paper is to study a nonlocal $\infty-$Laplacian type diffusion equation obtained as the limit as $p \to \infty$ to the nonlocal analogous to the $p-$Laplacian evolution.

First, let us recall some known results on local evolution problems. In [25] (see also [3] and [24]) was investigated the limiting behavior as $p \to \infty$ of solutions to the quasilinear parabolic problem

$$P_p(u_0) \quad \begin{cases} v_{p,t} - \Delta_p v_p = f, & \text{in } ]0,T[\times\mathbb{R}^N, \\ v_p(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

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where $\Delta_p u = \mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$ and $f$ is nonnegative and represents a given source term, which is interpreted physically as adding material to an evolving system, within which mass particles are continually rearranged by diffusion.

We hereafter take the space $H = L^2(\mathbb{R}^N)$ and define for $1 < p < \infty$ the functional

$$F_p(v) = \begin{cases} \dfrac{1}{p}\displaystyle\int_{\mathbb{R}^N} |\nabla v(y)|^p \, dy, & \text{if } u \in L^2(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \\ +\infty & \text{if } u \in L^2(\mathbb{R}^N) \setminus W^{1,p}(\mathbb{R}^N). \end{cases}$$

Therefore, the PDE problem $P_p(u_0)$ has the standard reinterpretation

$$\begin{cases} f(t) - v_{p,t} = \partial F_p(v_p(t)), & \text{a.e. } t \in ]0, T[, \\ v_p(0, x) = u_0(x), & \text{in } \mathbb{R}^N. \end{cases}$$

In [25], assuming that $u_0$ is a Lipschitz function with compact support, satisfying

$$\|\nabla u_0\|_\infty \leq 1,$$

and for $f$ a smooth nonnegative function with compact support in $[0, T] \times \mathbb{R}^N$, it is proved that we can extract a sequence $p_i \to +\infty$ and obtain a limit function $v_\infty$, such that for each $T > 0$,

$$\begin{cases} v_{p_i} \to v_\infty, & \text{a.e. and in } L^2(\mathbb{R}^N \times (0, T)), \\ \nabla v_{p_i} \rightharpoonup \nabla v_\infty, \ v_{p_i,t} \rightharpoonup v_{\infty,t} & \text{weakly in } L^2(\mathbb{R}^N \times (0, T)). \end{cases}$$

Moreover, the limit function $v_\infty$ satisfies

$$P_\infty(u_0) \quad \begin{cases} f(t) - v_{\infty,t} \in \partial F_\infty(v_\infty(t)), & \text{a.e. } t \in ]0, T[, \\ v_\infty(0, x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$F_\infty(v) = \begin{cases} 0 & \text{if } v \in L^2(\mathbb{R}^N), \ |\nabla v| \leq 1, \\ +\infty & \text{in other case.} \end{cases}$$

This limit problem $P_\infty(u_0)$ explains the movement of a sandpile ($v_\infty(t, x)$ describes the amount of the sand at the point $x$ at time $t$), the main assumption being that the sandpile is stable when the slope is less or equal than one and unstable if not.

On the other hand, we have the following nonlocal nonlinear diffusion problem, which we call the *nonlocal p-Laplacian problem*,

$$P_p^J(u_0) \quad \begin{cases} u_{p,t}(t, x) = \displaystyle\int_{\mathbb{R}^N} J(x - y)|u_p(t, y) - u_p(t, x)|^{p-2}(u_p(t, y) - u_p(t, x))dy + f(t, x), \\ u_p(0, x) = u_0(x). \end{cases}$$

Here $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous radial function with compact support, $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)\,dx = 1$ (this last condition is not necessary to prove our results, it is imposed to simplify the exposition).

In [2] we have studied this problem when the integral is taken in a bounded domain $\Omega$ (hence dealing with homogeneous Neumann boundary conditions). We have obtained existence and uniqueness of solutions and, if the kernel $J$ is rescaled in an appropriate way, that the solutions to the corresponding nonlocal problems converge to the solution of the $p-$laplacian with homogeneous Neumann boundary conditions. We have also studied the asymptotic behaviour of the solutions as $t$ goes to infinity, showing the convergence to the mean value of the initial condition.

Let us note that the evolution problem $P_p^J(u_0)$ is the gradient flow associated to the functional

$$G_p^J(u) = \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u(y)-u(x)|^p\,dy\,dx,$$

which is the nonlocal analogous to the functional $F_p$ associated to the $p-$Laplacian.

Following [2], we obtain existence and uniqueness of a global solution for this nonlocal problem, see Section 2 for the precise statements and their proofs.

Our next result in this article concerns the limit as $p \to \infty$ in $P_p^J(u_0)$. We obtain that the limit functional is given by

$$G_\infty^J(u) = \begin{cases} 0 & \text{if } u \in L^2(\mathbb{R}^N), \ \ |u(x)-u(y)| \le 1, \text{ for } x-y \in \operatorname{supp}(J), \\ \\ +\infty & \text{in other case.} \end{cases}$$

Then, the nonlocal limit problem can be written as

$$P_\infty^J(u_0) \qquad \begin{cases} f(t,\cdot) - u_t(t,\cdot) \in \partial G_\infty^J(u(t)), & \text{a.e. } \ t \in ]0,T[, \\ \\ u(0,x) = u_0(x). \end{cases}$$

With these notations, we obtain the following result.

**Theorem 1.1.** *Let $T > 0$, $f \in BV(0,T;L^p(\mathbb{R}^N))$, $u_0 \in L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ such that $|u_0(x) - u_0(y)| \le 1$, for $x - y \in \operatorname{supp}(J)$, and $u_p$ the unique solution of $P_p^J(u_0)$. Then, if $u_\infty$ is the unique solution to $P_\infty^J(u_0)$,*

$$\lim_{p\to\infty} \sup_{t\in[0,T]} \|u_p(t,\cdot) - u_\infty(t,\cdot)\|_{L^2(\mathbb{R}^N)} = 0.$$

Our next step is to rescale the kernel $J$ appropriately and take the limit as the scaling parameter goes to zero.

In the sequel we assume that $\operatorname{supp}(J) = \overline{B}_1(0)$. For given $p > 1$ and $J$ we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right), \quad \text{where} \ \ C_{J,p}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|^p\,dz$$

is a normalizing constant in order to obtain the $p$-Laplacian in the limit instead a multiple of it. Associated to these kernels we have solutions $u_{p,\varepsilon}$ to the nonlocal problems $P_p^{J_{p,\varepsilon}}(u_0)$. Let us also consider the solution to the local problem $P_p(u_0)$. Working as in [2] again, we can prove the following result.

**Theorem 1.2.** *Let $p > N$ and assume $J(x) \geq J(y)$ if $|x| \leq |y|$. Let $T > 0$, $f \in BV(0,T; L^p(\mathbb{R}^N))$, $u_0 \in L^p(\mathbb{R}^N)$ and $u_{p,\varepsilon}$ the unique solution of $P_p^{J_{p,\varepsilon}}(u_0)$. Then, if $v_p$ is the unique solution of $P_p(u_0)$,*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_{p,\varepsilon}(t, \cdot) - v_p(t, \cdot)\|_{L^p(\mathbb{R}^N)} = 0.$$

Finally, let us rescale the limit problem $P_\infty^J(u_0)$ considering the functionals

$$G_\infty^\varepsilon(u) = \begin{cases} 0 & \text{if } u \in L^2(\mathbb{R}^N), \quad |u(x) - u(y)| \leq \varepsilon, \text{ for } |x - y| \leq \varepsilon, \\ \\ +\infty & \text{in other case,} \end{cases}$$

and the gradient flow associated to this functional,

$$P_\infty^\varepsilon(u_0) \qquad \begin{cases} f(t, \cdot) - u_t(t, \cdot) \in \partial G_\infty^\varepsilon(u(t)), & \text{a.e } t \in ]0, T[, \\ \\ u(0, x) = u_0(x). \end{cases}$$

We have the following theorem.

**Theorem 1.3.** *Let $T > 0$, $f \in L^1(0,T; L^2(\mathbb{R}^N))$, $u_0 \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that $\|\nabla u_0\|_\infty \leq 1$ and consider $u_{\infty,\varepsilon}$ the unique solution of $P_\infty^\varepsilon(u_0)$. Then, if $v_\infty$ is the unique solution of $P_\infty(u_0)$, we have*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_{\infty,\varepsilon}(t, \cdot) - v_\infty(t, \cdot)\|_{L^2(\mathbb{R}^N)} = 0.$$

Hence, we have approximated the sandpile model described in [3] and [25] by a nonlocal equation. In this nonlocal approximation a configuration of sand is stable when its height $u$ verifies $|u(x) - u(y)| \leq \varepsilon$ when $|x - y| \leq \varepsilon$. This is a sort of measure of how large is the size of irregularities of the sand; the sand can be completely irregular for sizes smaller than $\varepsilon$ but it has to be arranged for sizes greater than $\varepsilon$.

In [25] the authors studied the collapsing of the initial condition phenomena for the local problem $P_p(u_0)$ when the initial condition $u_0$ satisfies $\|\nabla u_0\|_\infty > 1$. They find that the limit of the solutions $v_p(t, x)$ to $P_p(u_0)$ is independent of time but does not coincide with $u_0$. They also describe the small layer in which the solution rapidly changes from being $u_0$ at $t = 0$ to something close to the final stationary limit for $t > 0$.

Now, our task is to perform a similar analysis for the nonlocal problem. To this end let us take $\varepsilon = 1$ and $f = 0$ and look for the limit as $p \to \infty$ of the solutions to the nonlocal problem $u_p$ when the initial condition $u_0$ does not verify that $|u_0(x) - u_0(y)| \leq 1$ for $x - y \in \text{supp}(J)$. We get that the nonlinear nature of the problem creates an initial short-time layer in which the solution changes very rapidly. We describe this layer by means of a limit evolution problem. We have the following result.

**Theorem 1.4.** *Let $u_p$ be the solution to $P_p^J(u_0)$ with initial condition $u_0 \in L^2(\mathbb{R}^N)$ such that*

$$1 < L = \sup_{|x-y| \in supp(J)} |u_0(x) - u_0(y)|.$$

*Then there exists the limit*

$$\lim_{p\to\infty} u_p(t,x) = u_\infty(x) \qquad \text{in } L^2(\mathbb{R}^N),$$

*which is a function independent of $t$ such that $|u_\infty(x) - u_\infty(y)| \le 1$ for $x - y \in supp(J)$. Moreover, $u_\infty(x) = v(1,x)$, where $v$ is the unique strong solution of the evolution equation*

$$\begin{cases} \dfrac{v}{t} - v_t \in \partial G_\infty^J(v), & t \in ]\tau, \infty[, \\ v(\tau, x) = \tau u_0(x), \end{cases}$$

*with $\tau = L^{-1}$.*

Remark that when $u_0$ verifies $|u_0(x) - u_0(y)| \le 1$ for $x - y \in \mathrm{supp}(J)$ then it is an immediate consequence of Theorem 1.1 that the limit exists and is given by

$$\lim_{p\to\infty} u_p(t,x) = u_0(x).$$

We can also give an interpretation of the limit problem $P_\infty(u_0)$ in terms of Monge-Kantorovich mass transport theory as in [25], [28] (see [33] for a general introduction to mass transportation problems). To this end let us consider the distance

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ [|x-y|] + 1 & \text{if } x \neq y. \end{cases}$$

Here $[\cdot]$ means the entire part of the number. Note that this function $d$ measures distances with jumps of length one. Then, given two measures (that for simplicity we will take absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^N$) $f_+$, $f_-$ in $\mathbb{R}^N$, and supposing the overall condition of mass balance

$$\int_{\mathbb{R}^N} f^+ \, dx = \int_{\mathbb{R}^N} f^- \, dy,$$

the Monge's problem associated to the distance $d$ is given by: minimize

$$\int d(x, s(x)) \, f_+(x) dx$$

among the set of maps $s$ that transport $f_+$ into $f_-$, which means

$$\int_{\mathbb{R}^N} h(s(x)) f^+(x) \, dx = \int_{\mathbb{R}^N} h(y) f^-(y) \, dy$$

for each continuous function $h : \mathbb{R}^N \to \mathbb{R}$. The dual formulation of this minimization problem, introduced by Kantorovich (see [24]), is given by

$$\max_{u \in K_\infty} \int_{\mathbb{R}^N} u(x)(f_+(x) - f_-(x))dx$$

where the set $K_\infty$ is given by

$$K_\infty := \{u \in L^2(\mathbb{R}^N) \ : \ |u(x) - u(y)| \leq 1, \ \text{for } |x - y| \leq 1\}.$$

We are assuming that $\mathrm{supp}(J) = \overline{B}_1(0)$ (in other case we have to redefine the distance $d$ accordingly).

With these definitions and notations we have the following result.

**Theorem 1.5.** *The solution $u_\infty(t, \cdot)$ of the limit problem $P_\infty^J(u_0)$ is a solution to the dual problem*

$$\max_{u \in K_\infty} \int_{\mathbb{R}^N} u(x)(f_+(x) - f_-(x))dx$$

*when the involved measures are the source term $f_+ = f(t, x)$ and the time derivative of the solution $f_- = u_t(t, x)$.*

Finally, let us observe that analogous results are also valid when we consider the Neumann problem in a bounded convex domain $\Omega$, that is, when all the involved integrals are taken in $\Omega$. See Section 8 for precise statements.

Let us end the introduction with some bibliographical discussion on nonlocal evolution problems. Nonlocal evolution equations of the form $u_t(t, x) = J * u - u(t, x)$, and variations of it, have been recently widely used to model diffusion processes, see [1], [2], [5], [7], [16], [17], [18], [20], [29] and [30].

As stated in [29], if $u(t, x)$ is thought of as the density of a single population at the point $x$ at time $t$, and $J(x - y)$ is thought of as the probability distribution of jumping from location $y$ to location $x$, then the convolution $(J * u)(t, x) = \int_{\mathbb{R}^N} J(y - x)u(t, y)\, dy$ is the rate at which individuals are arriving to position $x$ from all other places and $-u(t, x) = -\int_{\mathbb{R}^N} J(y-x)u(t, x)\, dy$ is the rate at which they are leaving location $x$ to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density $u$ satisfies the nonlocal equation. This equation is called a nonlocal diffusion equation since the diffusion of the density $u$ at a point $x$ and time $t$ does not only depend on $u(t, x)$, but on all the values of $u$ in a neighborhood of $x$ through the convolution term $J * u$. This equation shares many properties with the classical heat equation, $u_t = \Delta u$, such as bounded stationary solutions are constant, a maximum principle holds for both of them and perturbations propagate with infinite speed, [29]. However, there is no regularizing effect in general (see [17]).

Concerning scalings of the kernel that approximate different problems we refer to [2], [21] and [31], where usual diffusion equations where obtained taking limits similar to the ones considered here.

The rest of the paper is organized as follows: in Section 2 we collect some useful results that will be used in the proofs of the theorems, among them some technical tools from convex analysis; in Section 3 we consider the limit as $p \to \infty$ and prove Theorem 1.1; in Section 4 we deal with the limit as $\varepsilon \to 0$ and prove Theorem 1.3; in Section 5 we give a model for the collapse of a sandpile from an initially unstable configuration; in Section 6 we provide some examples of explicit solutions to $P_\infty^\varepsilon(u_0)$ and then take the limit as $\varepsilon \to 0$ recovering explicit solutions to the sandpile model; in Section 7 we deal with the interpretation of the limit equation in terms of a transport problem. Finally, in Section 8 we briefly explain what are the main results for the Neumann problem.

## 2. Preliminaries

To identify the limit of the solutions $u_p$ of problem $P_p^J(u_0)$ we will use the methods of convex analysis, and so we first recall some terminology (see [15], [12] and [4]).

If $H$ is a real Hilbert space with inner product $(\ ,\ )$ and $\Psi : H \to (-\infty, +\infty]$ is convex, then the subdifferential of $\Psi$ is defined as the multivalued operator $\partial\Psi$ given by

$$v \in \partial\Psi(u) \iff \Psi(w) - \Psi(u) \geq (v, w - u) \quad \forall\, w \in H.$$

The epigraph of $\Psi$ is defined by

$$\mathrm{Epi}(\Psi) = \{(u, \lambda) \in H \times \mathbb{R} \ :\ \lambda \geq \Psi(u)\}.$$

Given $K$ a closed convex subset of $H$, the indicator function of $K$ is defined by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

Then it is easy to see that the subdifferential is characterized as follows,

$$v \in \partial I_K(u) \iff u \in K \ \text{ and } \ (v, w - u) \leq 0 \quad \forall\, w \in K.$$

In case the convex functional $\Psi : H \to (-\infty, +\infty]$ is proper , lower-semicontinuous and $\min \Psi = 0$ , it is well known (see [12]) that the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\Psi(u(t)) \ni f(t), & \text{a.e } t \in ]0, T[, \\ u(0) = u_0, \end{cases}$$

has a unique strong solution for any $f \in L^2(0, T; H)$ and $u_0 \in \overline{D(\partial\Psi)}$.

The following convergence was studied by Mosco in [32] (see [4]). Suppose $X$ is a metric space and $A_n \subset X$. We define

$$\liminf_{n\to\infty} A_n = \{x \in X \ :\ \exists x_n \in A_n,\ x_n \to x\}$$

and

$$\limsup_{n\to\infty} A_n = \{x \in X \ : \ \exists x_{n_k} \in A_{n_k}, \ x_{n_k} \to x\}.$$

In the case $X$ is a normed space, we note by $s-\lim$ and $w-\lim$ the above limits associated respectively to the strong and to the weak topology of $X$.

Given a sequence $\Psi_n, \Psi : H \to (-\infty, +\infty]$ of convex lower-semicontinuous functionals, we say that $\Psi_n$ converges to $\Psi$ in the sense of Mosco if

$$w - \limsup_{n\to\infty} \mathrm{Epi}(\Psi_n) \subset \mathrm{Epi}(\Psi) \subset s - \liminf_{n\to\infty} \mathrm{Epi}(\Psi_n). \tag{2.1}$$

It is easy to see that (2.1) is equivalent to the two following conditions:

$$\forall\, u \in D(\Psi) \ \ \exists u_n \in D(\Psi_n) \ : \ u_n \to u \ \text{ and } \ \Psi(u) \geq \limsup_{n\to\infty} \Psi_n(u_n); \tag{2.2}$$

$$\text{for every subsequence } n_k, \text{ when } u_k \rightharpoonup u, \text{ it holds } \Psi(u) \leq \liminf_k \Psi_{n_k}(u_k). \tag{2.3}$$

As consequence of results in [14] and [4] we can write the following result.

**Theorem 2.1.** *Let $\Psi_n, \Psi : H \to (-\infty, +\infty]$ convex lower-semicontinuous functionals. Then the following statements are equivalent:*

(i) $\Psi_n$ *converges to* $\Psi$ *in the sense of Mosco.*

(ii) $(I + \lambda \partial \Psi_n)^{-1} u \to (I + \lambda \partial \Psi)^{-1} u, \quad \forall\, \lambda > 0, \ u \in H.$

*Moreover, any of these two conditions (i) or (ii) imply that*

(iii) *for every* $u_0 \in \overline{D(\partial\Psi)}$ *and* $u_{0,n} \in \overline{D(\partial\Psi_n)}$ *such that* $u_{0,n} \to u_0$*, and every* $f_n, f \in L^1(0,T;H)$ *with* $f_n \to f$*, if* $u_n(t)$*,* $u(t)$ *are the strong solutions of the abstract Cauchy problems*

$$\begin{cases} u_n'(t) + \partial\Psi_n(u_n(t)) \ni f_n, & a.e. \ \ t \in ]0,T[, \\ u_n(0) = u_{0,n}, \end{cases}$$

*and*

$$\begin{cases} u'(t) + \partial\Psi(u(t)) \ni f, & a.e. \ \ t \in ]0,T[, \\ u(0) = u_0, \end{cases}$$

*respectively, then*

$$u_n \to u \qquad in\ C([0,T] : H).$$

Let us also collect some preliminaries and notations concerning completely accretive operators that will be used afterwards (see [8]).

We denote by $J_0$ and $P_0$ the following sets of functions,

$$J_0 = \{j : \mathbb{R} \to [0, +\infty], \text{ such that } j \text{ is convex, lower semi-continuos and } j(0) = 0\},$$

$$P_0 = \{q \in C^\infty(\mathbb{R}) : 0 \le q' \le 1, \mathrm{supp}(q') \text{ is compact, and } 0 \notin \mathrm{supp}(q)\}.$$

Let $M(\mathbb{R}^N)$ denote the space of measurable functions from $\mathbb{R}^N$ into $\mathbb{R}$. We set $L(\mathbb{R}^N) := L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. Note that $L(\mathbb{R}^N)$ is a Banach space with the norm

$$\|u\|_{1+\infty} := \inf\{\|f\|_1 + \|g\|_\infty \ : f \in L^1(\mathbb{R}^N),\ g \in L^\infty(\mathbb{R}^N),\ f + g = u\}.$$

The closure of $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ in $L(\mathbb{R}^N)$ is denoted by $L_0(\mathbb{R}^N)$.

Given $u, v \in M(\mathbb{R}^N)$ we say

$$u \ll v \text{ if and only if } \int_{\mathbb{R}^N} j(u)\,dx \le \int_{\mathbb{R}^N} j(v)\,dx \quad \forall j \in J_0.$$

An operator $A \subset M(\mathbb{R}^N) \times M(\mathbb{R}^N)$ is said to be completely accretive if

$$u - \hat{u} \ll u - \hat{u} + \lambda(v - \hat{v}) \qquad \forall \lambda > 0 \text{ and } \forall (u, v), (\hat{u}, \hat{v}) \in A.$$

The following facts are proved in [8].

**Proposition 2.2.**

*(i) Let $u \in L_0(\mathbb{R}^N)$, $v \in L(\mathbb{R}^N)$, then*

$$u \ll u + \lambda v \quad \forall \lambda > 0 \quad \textit{if and only if} \quad \int_{\mathbb{R}^N} q(u)v \ge 0, \qquad \forall q \in P_0.$$

*(ii) If $v \in L_0(\mathbb{R}^N)$, then $\{u \in M(\mathbb{R}^N) : u \ll v\}$ is a weak sequentially compact subset of $L_0(\mathbb{R}^N)$.*

Concerning nonlocal models, in [2] we have studied the following nonlocal nonlinear diffusion problem, which we call the nonlocal $p$-Laplacian problem with homogeneous Neumann boundary conditions,

$$\begin{cases} u_t(t,x) = \displaystyle\int_\Omega J(x-y)|u(t,y) - u(t,x)|^{p-2}(u(t,y) - u(t,x))\,dy, \\ u(0,x) = u_0(x). \end{cases}$$

Using similar ideas and techniques we can deal with the nonlocal problem in $\mathbb{R}^N$.

Solutions to $P_p^J(u_0)$ are to be understood in the following sense.

**Definition 2.3.** *Let $1 < p < +\infty$. Let $f \in L^1(0,T; L^p(\mathbb{R}^N))$ and $u_0 \in L^p(\mathbb{R}^N)$. A solution of $P_p^J(u_0)$ in $[0,T]$ is a function $u \in C([0,T]; L^p(\mathbb{R}^N)) \cap W^{1,1}(]0,T[; L^p(\mathbb{R}^N))$ which satisfies $u(0,x) = u_0(x)$ a.e. $x \in \mathbb{R}^N$ and*

$$u_t(t,x) = \int_{\mathbb{R}^N} J(x-y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))\,dy + f(t,x) \quad \textit{a.e. in } ]0,T[\times\mathbb{R}^N.$$

Working as in [2], we can obtain the following result about existence and uniqueness of a global solution for this problem. Let us first define $B_p^J : L^p(\mathbb{R}^N) \to L^{p'}(\mathbb{R}^N)$ by

$$B_p^J u(x) = -\int_{\mathbb{R}^N} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, \qquad x \in \mathbb{R}^N.$$

Observe that, for every $u$, $v \in L^p(\mathbb{R}^N)$ and $T : \mathbb{R} \to \mathbb{R}$ such that $T(u-v) \in L^p(\mathbb{R}^N)$, it holds

$$\begin{aligned} &\int_{\mathbb{R}^N} (B_p^J u(x) - B_p^J v(x)) T(u(x)-v(x))dx = \\ &\qquad \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\,(T(u(y)-v(y)) - T(u(x)-v(x))) \times \\ &\qquad \times \left(|u(y)-u(x)|^{p-2}(u(y)-u(x)) - |v(y)-v(x)|^{p-2}(v(y)-v(x))\right)\,dy\,dx. \end{aligned} \tag{2.4}$$

Let us also define the operator

$$\mathcal{B}_p^J = \left\{(u,v) \in L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N) \ : \ v = B_p^J(u)\right\}.$$

It is easy to see that $\overline{\mathrm{Dom}(\mathcal{B}_p^J)} = L^p(\mathbb{R}^N)$ and $\mathcal{B}_p^J$ is positively homogeneous of degree $p-1$.

**Theorem 2.4.** *Let $1 < p < +\infty$. If $f \in BV(0,T;L^p(\mathbb{R}^N))$ and $u_0 \in D(\mathcal{B}_p^J)$ then there exists a unique solution to $P_p^J(u_0)$. If $f = 0$ then there exists a unique solution to $P_p^J(u_0)$ for all $u_0 \in L^p(\mathbb{R}^N)$.*

*Moreover, if $u_i(t)$ is a solution of $P_p^J(u_{i0})$ with $f = f_i$, $f_i \in L^1(0,T;L^p(\mathbb{R}^N))$ and $u_{i0} \in L^p(\mathbb{R}^N)$, $i = 1,2$, then, for every $t \in [0,T]$,*

$$\|(u_1(t)-u_2(t))^+\|_{L^p(\mathbb{R}^N)} \le \|(u_{10}-u_{20})^+\|_{L^p(\mathbb{R}^N)} + \int_0^t \|f_1(s)-f_2(s)\|_{L^p(\mathbb{R}^N)}ds.$$

*Proof.* Let us first show that $\mathcal{B}_p^J$ is completely accretive and verifies the following range condition

$$L^p(\mathbb{R}^N) = \mathrm{Ran}(I + \mathcal{B}_p^J). \tag{2.5}$$

Indeed, given $u_i \in \mathrm{Dom}(\mathcal{B}_p^J)$, $i = 1,2$ and $q \in P_0$, by (2.4) we have

$$\int_{\mathbb{R}^N} (B_p^J u_1(x) - B_p^J u_2(x))\, q(u_1(x)-u_2(x))\,dx \ge 0,$$

from where it follows that $\mathcal{B}_p^J$ is a completely accretive operator. To show that $\mathcal{B}_p^J$ satisfies the range condition we have to prove that for any $\phi \in L^p(\mathbb{R}^N)$ there exists $u \in \mathrm{Dom}(\mathcal{B}_p^J)$ such that $u = (I + B_p^J)^{-1}\phi$. Let us first take $\phi \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. For every $n \in \mathbb{N}$, let $\phi_n := \phi\chi_{B_n(0)}$. By the results in [2], the operator $B_{p,n}^J$ defined by

$$B_{p,n}^J u(x) = -\int_{B_n(0)} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, \qquad x \in B_n(0),$$

is $m$-completely accretive in $L^p(B_n(0))$. Then, there exists $u_n \in L^p(B_n(0))$, such that

$$u_n(x) + B_{p,n}^J u_n(x) = \phi_n(x), \qquad \text{a.e. in } \ B_n(0). \tag{2.6}$$

Moreover, $u_n \ll \phi_n$.

We denote by $\tilde{u}_n$ and $H_n$ the extensions

$$\tilde{u}_n(x) = \begin{cases} u_n(x) & \text{if } \ x \in B_n(0), \\ 0 & \text{if } \ x \in \mathbb{R}^N \setminus B_n(0), \end{cases}$$

and

$$H_n(x) = \begin{cases} B^J_{p,n} u_n(x), & \text{if } \ x \in B_n(0), \\ 0 & \text{if } \ x \in \mathbb{R}^N \setminus B_n(0). \end{cases}$$

Since, $u_n \ll \phi_n$, we have

$$\|\tilde{u}_n\|_q \leq \|\phi\|_q \qquad \text{for all } 1 \leq q \leq \infty, \forall\, n \in \mathbb{N}. \tag{2.7}$$

Hence, we can suppose

$$\tilde{u}_n \rightharpoonup u \quad \text{in } L^{p'}(\mathbb{R}^N). \tag{2.8}$$

On the other hand, multiplying (2.6) by $u_n$ and integrating, we get

$$\int_{B_n(0)} \int_{B_n(0)} J(x-y)|u_n(y) - u_n(x)|^p \, dydx \leq \|\phi\|_2 \quad \forall\, n \in \mathbb{N}, \tag{2.9}$$

which implies, by Hölder's inequality, that $\{H_n \ : \ n \in \mathbb{N}\}$ is bounded in $L^{p'}(\mathbb{R}^N)$. Therefore, we can assume that

$$H_n \rightharpoonup H \quad \text{in } L^{p'}(\mathbb{R}^N). \tag{2.10}$$

By (2.8) and (2.10), taking limit in (2.6), we get

$$u + H = \phi \qquad \text{a.e. in } \mathbb{R}^N. \tag{2.11}$$

Let us see that

$$H(x) = -\int_{\mathbb{R}^N} J(x-y)|u(y) - u(x)|^{p-2}(u(y) - u(x)) \, dy \quad \text{a.e. in } x \in \mathbb{R}^N. \tag{2.12}$$

In fact, multiplying (2.6) by $u_n$ and integrating, we get

$$\begin{aligned} &\int_{B(0,n)} B^J_{p,n} u_n \, u_n = \int_{B(0,n)} (\phi - u_n) u_n \\ &= \int_{B(0,n)} (\phi - u) u - \int_{B(0,n)} \phi(u - u_n) + \int_{B(0,n)} 2u(u - u_n) - \int_{B(0,n)} (u - u_n)(u - u_n). \end{aligned}$$

Therefore, by (2.11),

$$\limsup \int_{B_n(0)} B^J_{p,n} u_n \, u_n \leq \int_{\mathbb{R}^N} (\phi - u) u = \int_{\mathbb{R}^N} H \, u. \tag{2.13}$$

On the other hand, for any $v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, since

$$0 \leq \int_{B_n(0)} \left( B^J_{p,n} u_n - B^J_{p,n} v \right) (u_n - v),$$

we have that,

$$\int_{B_n(0)} B^J_{p,n} u_n \, u_n + \int_{B_n(0)} B^J_{p,n} v \, v \geq \int_{B_n(0)} B^J_{p,n} u_n \, v + \int_{B_n(0)} B^J_{p,n} v \, u_n.$$

Therefore, by (2.13),

$$\int_{\mathbb{R}^N} H \, u + \int_{\mathbb{R}^N} B^J_p v \, v \geq \int_{\mathbb{R}^N} H \, v + \int_{\mathbb{R}^N} B^J_p v \, u. \tag{2.14}$$

Taking now $v = u \pm \lambda w$, $\lambda > 0$ and $w \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and letting $\lambda \to 0$, we get

$$\int_{\mathbb{R}^N} H \, w = \int_{\mathbb{R}^N} B^J_p u \, w,$$

and consequently (2.12) is proved. Therefore, by (2.11), the range condition is satisfied for $\phi \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Let now $\phi \in L^p(\mathbb{R}^N)$. Take $\phi_n \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\phi_n \to \phi$ in $L^p(\mathbb{R}^N)$. Then, by our previous step, there exists $u_n = (I + \mathcal{B}^J_p)^{-1} \phi_n$. Since $\mathcal{B}^J_p$ is completely accretive, $u_n \to u$ in $L^p(\mathbb{R}^N)$, also $B^J_p u_n \to B^J_p u$ in $L^{p'}(\mathbb{R}^N)$ and we conclude that $u + \mathcal{B}^J_p u = \phi$.

Consequently (see [8] and [9]) we have that $\mathcal{B}^J_p$ is an $m$-accretive operator in $L^p(\mathbb{R}^N)$ and we get the existence of mild solution $u(t)$ of the abstract Cauchy problem

$$\begin{cases} u'(t) + \mathcal{B}^J_p u(t) = f(t), & t \in ]0, T[, \\ u(0) = u_0, \end{cases} \tag{2.15}$$

Now, by the Nonlinear Semigroup Theory (see [9], [22] or [23]), if $f \in BV(0, T; L^p(\mathbb{R}^N))$ and $u_0 \in D(\mathcal{B}^J_p)$, $u(t)$ is a strong solution of (2.15), that is, a solution of $P^J_p(u_0)$ in the sense of Definition 2.3. The same is true for all $u_0 \in L^p(\mathbb{R}^N)$ in the case $f = 0$ by the complete accretivity of $\mathcal{B}^J_p$, since $\mathrm{Dom}(\mathcal{B}^J_2) = L^2(\mathbb{R}^N)$ and for $p \neq 2$ the operator $\mathcal{B}^J_p$ is homogeneous of degree $p - 1$ (see [8]). Finally, the contraction principle follows from the general Nonlinear Semigroup Theory since the solutions $u_i$, $i = 1, 2$, are mild-solutions of (2.15). □

## 3. Limit as $p \to \infty$

Recall from the Introduction that the *nonlocal p-Laplacian evolution problem*

$$P^J_p(u_0) \quad \begin{cases} u_t(t,x) = \displaystyle\int_{\mathbb{R}^N} J(x-y) |u(t,y) - u(t,x)|^{p-2} (u(t,y) - u(t,x)) \, dy + f(t,x), \\ u(0,x) = u_0(x). \end{cases}$$

is the gradient flow associated to the functional

$$G^J_p(u) = \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y) |u(y) - u(x)|^p \, dy \, dx.$$

With a formal calculation, taking limit as $p \to \infty$, we arrive to the functional

$$G_\infty^J(u) = \begin{cases} 0 & \text{if } |u(x) - u(y)| \le 1, \text{ for } x - y \in \operatorname{supp}(J), \\ \\ +\infty & \text{in other case.} \end{cases}$$

Hence, if we define

$$K_\infty^J := \{u \in L^2(\mathbb{R}^N) \ : \ |u(x) - u(y)| \le 1, \text{ for } x - y \in \operatorname{supp}(J)\},$$

we have that the functional $G_\infty^J$ is given by the indicator function of $K_\infty^J$, that is, $G_\infty^J = I_{K_\infty^J}$. Then, the *nonlocal limit problem* can be written as

$$P_\infty^J(u_0) \quad \begin{cases} f(t,\cdot) - u_t(t) \in \partial I_{K_\infty^J}(u(t)), & \text{a.e. } t \in ]0,T[, \\ \\ u(0,x) = u_0(x). \end{cases}$$

**Proof of Theorem 1.1.** Let $T > 0$. Recall that we want to prove that, given $f \in BV(0,T;L^p(\mathbb{R}^N))$, $u_0 \in K_\infty^J \cap L^p(\mathbb{R}^N)$ and $u_p$ the unique solution of $P_p^J(u_0)$, if $u_\infty$ is the unique solution of $P_\infty^J(u_0)$, then

$$\lim_{p\to\infty} \sup_{t\in[0,T]} \|u_p(t,\cdot) - u_\infty(t,\cdot)\|_{L^2(\mathbb{R}^N)} = 0.$$

By Theorem 2.1, to prove the result it is enough to show that the functionals

$$G_p^J(u) = \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)|u(y) - u(x)|^p \, dy \, dx$$

converge to

$$G_\infty^J(u) = \begin{cases} 0 & \text{if } |u(x) - u(y)| \le 1, \text{ for } x - y \in \operatorname{supp}(J), \\ \\ +\infty & \text{in other case,} \end{cases}$$

as $p \to \infty$, in the sense of Mosco.

First, let us check that

$$\operatorname{Epi}(G_\infty^J) \subset s - \liminf_{p\to\infty} \operatorname{Epi}(G_p^J). \tag{3.1}$$

To this end let $(u,\lambda) \in \operatorname{Epi}(G_\infty^J)$. We can assume that $u \in K_\infty^J$ and $\lambda \ge 0$ (as $G_\infty^J(u) = 0$). Now take

$$v_p = u\chi_{B_{R(p)}(0)} \qquad \text{and} \qquad \lambda_p = G_p^J(u_p) + \lambda. \tag{3.2}$$

Then, as $\lambda \ge 0$ we have $(v_p, \lambda_p) \in \operatorname{Epi}(G_p^J)$. It is obvious that if $R(p) \to \infty$ as $p \to \infty$ we have

$$v_p \to u \quad \text{in} \quad L^2(\mathbb{R}^N),$$

and, if we choose $R(p) = p^{\frac{1}{4N}}$ we get

$$G_p^J(v_p) = \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J\left(x-y\right) \left|v_p(y) - v_p(x)\right|^p \, dy \, dx \le C \frac{R(p)^{2N}}{p} \to 0,$$

as $p \to \infty$, we get (3.1).

Finally, let us prove that

$$w - \limsup_{p\to\infty} \mathrm{Epi}(G_p^J) \subset \mathrm{Epi}(G_\infty^J). \tag{3.3}$$

To this end, let us consider a sequence $(u_{p_j}, \lambda_{p_j}) \in \mathrm{Epi}(G_{p_j}^J)$ $(p_j \to \infty)$, that is,

$$G_{p_j}^J(u_{p_j}) \le \lambda_{p_j},$$

with

$$u_{p_j} \rightharpoonup u, \qquad \text{and} \qquad \lambda_{p_j} \to \lambda.$$

Therefore we obtain that $0 \le \lambda$, since

$$0 \le G_{p_j}^J(u_{p_j}) \le \lambda_{p_j} \to \lambda.$$

On the other hand, we have that

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\left(x-y\right)\left|u_{p_j}(y)-u_{p_j}(x)\right|^{p_j}\,dy\,dx\right)^{1/p_j} \le (Cp_j)^{1/p_j}.$$

Now, fix a bounded domain $\Omega \subset \mathbb{R}^N$ and $q < p_j$. Then, by the above inequality,

$$\begin{aligned}
&\left(\int_{\Omega}\int_{\Omega} J\left(x-y\right)\left|u_{p_j}(y)-u_{p_j}(x)\right|^{q}\,dy\,dx\right)^{1/q} \\
&\le \left(\int_{\Omega}\int_{\Omega} J\left(x-y\right)\,dy\,dx\right)^{(p_j-q)/p_jq} \\
&\qquad \times \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\left(x-y\right)\left|u_{p_j}(y)-u_{p_j}(x)\right|^{p_j}\,dy\,dx\right)^{1/p_j} \\
&\le \left(\int_{\Omega}\int_{\Omega} J\left(x-y\right)\,dy\,dx\right)^{(p_j-q)/p_jq} (Cp_j)^{1/p_j}.
\end{aligned}$$

Hence, we can extract a subsequence (if necessary) and let $p_j \to \infty$ to obtain

$$\left(\int_{\Omega}\int_{\Omega} J\left(x-y\right)\left|u(y)-u(x)\right|^{q}\,dy\,dx\right)^{1/q} \le \left(\int_{\Omega}\int_{\Omega} J\left(x-y\right)\,dy\,dx\right)^{1/q}.$$

Now, just taking $q \to \infty$, we get

$$|u(x)-u(y)| \le 1 \qquad \text{a.e. } (x,y) \in \Omega\times\Omega,\ x-y \in \mathrm{supp}(J).$$

As $\Omega$ is arbitrary we conclude that

$$u \in K_\infty^J.$$

This ends the proof. $\square$

## 4. Limit as $\varepsilon \to 0$

In this section, to simplify, we assume that $\mathrm{supp}(J) = \overline{B}_1(0)$. For given $p > 1$ and $J$, we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right), \quad \text{where } \; C_{J,p}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|^p \, dz$$

is a normalizing constant in order to obtain the $p$-Laplacian in the limit instead a multiple of it, that is, we want the limit problem to be

$$N_p(u_0) \quad \begin{cases} u_t = \Delta_p u + f, & \text{in } ]0,T[\times\mathbb{R}^N, \\ u(0,x) = u_0(x), & \text{in } \mathbb{R}^N. \end{cases}$$

We will use the following result from [2] which is a variant of [13, Theorem 4] (the first statement is given in [2] for bounded domains $\Omega$ but it also holds for general open sets).

**Proposition 4.1.** *Let $\Omega$ an open subset of $\mathbb{R}^N$. Let $1 \le p < +\infty$. Let $\rho : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous radial function with compact support, non-identically zero, and $\rho_n(x) := n^N \rho(nx)$. Let $\{f_n\}$ be a sequence of functions in $L^p(\Omega)$ such that*

$$\int_\Omega \int_\Omega |f_n(y) - f_n(x)|^p \rho_n(y-x) \, dx \, dy \le M \frac{1}{n^p}.$$

*(1) If $\{f_n\}$ is weakly convergent in $L^p(\Omega)$ to $f$, then $f \in W^{1,q}(\Omega)$ and moreover*

$$(\rho(z))^{1/p} \chi_\Omega \left(x + \frac{1}{n}z\right) \frac{f_n\left(x + \frac{1}{n}z\right) - f_n(x)}{1/n} \rightharpoonup (\rho(z))^{1/p} \, z \cdot \nabla f$$

*weakly in $L^p(\Omega) \times L^p(\Omega)$.*

*(2) If we further assume $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ and $\rho(x) \ge \rho(y)$ if $|x| \le |y|$ then $\{f_n\}$ is relatively compact in $L^p(\Omega)$, and consequently, there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ in $L^p(\Omega)$ with $f \in W^{1,p}(\Omega)$.*

Using the above proposition we can take the limit as $\varepsilon \to 0$ for a fixed $p > N$.

**Proof of Theorem 1.2.** Recall that we have $p > N$ and $J(x) \ge J(y)$ if $|x| \le |y|$, $T > 0$, $f \in L^1(0,T;L^p(\mathbb{R}^N))$, $u_0 \in L^p(\mathbb{R}^N)$ and $u_{p,\varepsilon}$ the unique solution of $P_p^{J_{p,\varepsilon}}(u_0)$. We want to show that, if $u_p$ is the unique solution of $N_p(u_0)$, then

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_{p,\varepsilon}(t,\cdot) - u_p(t,\cdot)\|_{L^p(\mathbb{R}^N)} = 0. \tag{4.1}$$

Since $\mathcal{B}_p^J$ is $m$-accretive, to get (4.1) it is enough to see (see [9] or [22])

$$\left(I + \mathcal{B}_p^{J_{p,\varepsilon}}\right)^{-1} \phi \to \left(I + \mathcal{B}_p\right)^{-1} \phi \quad \text{in } L^p(\mathbb{R}^N) \text{ as } \varepsilon \to 0$$

for any $\phi \in C_c(\mathbb{R}^N)$.

Let $\phi \in C_c(\mathbb{R}^N)$ and $u_\varepsilon := \left(I + \mathcal{B}_p^{J_{p,\varepsilon}}\right)^{-1} \phi$. Then,

$$\begin{aligned}\int_{\mathbb{R}^N} u_\varepsilon v - \frac{C_{J,p}}{\varepsilon^{p+N}} \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right) |u_\varepsilon(y) - u_\varepsilon(x)|^{p-2} \times \\ \times (u_\varepsilon(y) - u_\varepsilon(x))\, dy\, v(x)\, dx = \int_{\mathbb{R}^N} \phi v\end{aligned} \tag{4.2}$$

for every $v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Changing variables, we get

$$\begin{aligned}&-\frac{C_{J,p}}{\varepsilon^{p+N}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right) |u_\varepsilon(y) - u_\varepsilon(x)|^{p-2} (u_\varepsilon(y) - u_\varepsilon(x))\, dy\, v(x)\, dx \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{C_{J,p}}{2} J(z) \left| \frac{u_\varepsilon(x+\varepsilon z) - u_\varepsilon(x)}{\varepsilon} \right|^{p-2} \frac{u_\varepsilon(x+\varepsilon z) - u_\varepsilon(x)}{\varepsilon} \times \\ &\qquad \times \frac{v(x+\varepsilon z) - v(x)}{\varepsilon}\, dx\, dz.\end{aligned} \tag{4.3}$$

So we can rewrite (4.2) as

$$\begin{aligned}&\int_{\mathbb{R}^N} \phi(x) v(x)\, dx - \int_{\mathbb{R}^N} u_\varepsilon(x) v(x)\, dx \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{C_{J,p}}{2} J(z) \left| \frac{u_\varepsilon(x+\varepsilon z) - u_\varepsilon(x)}{\varepsilon} \right|^{p-2} \frac{u_\varepsilon(x+\varepsilon z) - u_\varepsilon(x)}{\varepsilon} \times \\ &\qquad \times \frac{v(x+\varepsilon z) - v(x)}{\varepsilon}\, dx\, dz.\end{aligned} \tag{4.4}$$

We shall see that there exists a sequence $\varepsilon_n \to 0$ such that $u_{\varepsilon_n} \to u$ in $L^p(\mathbb{R}^N)$, $u \in W^{1,p}(\mathbb{R}^N)$ and $u = (I + B_p)^{-1} \phi$, that is,

$$\int_{\mathbb{R}^N} uv + \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla v = \int_{\mathbb{R}^N} \phi v \quad \text{for every } v \in W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

Since $u_\varepsilon \ll \phi$, there exists a sequence $\varepsilon_n \to 0$ such that

$$u_{\varepsilon_n} \rightharpoonup u, \quad \text{weakly in } L^p(\mathbb{R}^N), \quad u \ll \phi.$$

Observe that $\|u_{\varepsilon_n}\|_{L^\infty(\mathbb{R}^N)}, \|u\|_{L^\infty(\mathbb{R}^N)} \le \|\phi\|_{L^\infty(\mathbb{R}^N)}$. Taking $\varepsilon = \varepsilon_n$ and $v = u_{\varepsilon_n}$ in (4.4) and applying Young's inequality, we get

$$\begin{aligned}&\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{2} \frac{C_{J,p}}{{\varepsilon_n}^N} J\left(\frac{x-y}{\varepsilon_n}\right) \left| \frac{u_{\varepsilon_n}(y) - u_{\varepsilon_n}(x)}{\varepsilon_n} \right|^p dx\, dy \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{C_{J,p}}{2} J(z) \left| \frac{u_{\varepsilon_n}(x+\varepsilon_n z) - u_{\varepsilon_n}(x)}{\varepsilon_n} \right|^p dx\, dz \le M := \frac{1}{2} \int_{\mathbb{R}^N} |\phi(x)|^2\, dx.\end{aligned} \tag{4.5}$$

Therefore, by Proposition 4.1,

$$u \in W^{1,p}(\mathbb{R}^N),$$
$$u_{\varepsilon_n} \to u \text{ in } L^p_{\text{loc}}(\mathbb{R}^N)$$

and

$$\left(\frac{C_{J,p}}{2}J(z)\right)^{1/p}\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n} \rightharpoonup \left(\frac{C_{J,p}}{2}J(z)\right)^{1/p} z\cdot\nabla u(x) \tag{4.6}$$

weakly in $L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N)$. Let us prove now the tightness of $\{u_{\varepsilon_n}\}$, which is to say, that no mass moves to infinity as $p\to+\infty$. For this, assume $\mathrm{supp}(\phi)\subset B_R(0)$ and fix $S>2R$. Select a smooth function $\varphi\in C^\infty(\mathbb{R}^N)$ such that $0\le\varphi\le 1$, $\varphi\equiv 0$ on $B_R(0)$, $\varphi\equiv 1$ on $\mathbb{R}^N\setminus B_S(0)$ and $|\nabla\varphi|\le\frac{2}{S}$. Taking in (4.4) $\varphi|u_{\varepsilon_n}|^{p-2}u_{\varepsilon_n}$, and having in mind that $\|u_{\varepsilon_n}\|_{L^\infty(\mathbb{R}^N)}\le\|\phi\|_{L^\infty(\mathbb{R}^N)}$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}|u_{\varepsilon_n}|^p(x)\varphi(x)\,dx\\
&\le\frac{C_{J,p}}{2{\varepsilon_n}^p}\int_{\mathbb{R}^N}\int_{B_1(0)}J(z)|u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)|^{p-1}|u_{\varepsilon_n}(x+\varepsilon_n z)|^{p-1}\\
&\qquad\qquad\times|\varphi(x+\varepsilon_n z)-\varphi(x)|\,dzdx\\
&\le\frac{C_{J,p}\|\phi\|_{L^\infty}^{p-1}}{S}\int_{\{|x|\le S+1\}}\int_{B_1(0)}J(z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^{p-1}dy\,dx\\
&\le\frac{C_{J,p}\|\phi\|_{L^\infty}^{p-1}}{S}\left(\int_{\{|x|\le S+1\}}\int_{B_1(0)}J(z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^{p}dy\right)^{\frac{1}{p'}}\\
&\qquad\times\left(\int_{\{|x|\le S+1\}}\int_{B_1(0)}J(z)dz\right)^{\frac1p}dx\\
&=O(S^{-1+\frac{N}{p}}),
\end{aligned}$$

the last equality being true by (4.5) and since

$$\left(\int_{\{|x|\le S+1\}}\int_{B_1(0)}J(z)dz\right)^{\frac1p}dx\le C(S+1)^{\frac{N}{p}}.$$

Consequently,

$$\int_{\{|x|\ge S\}}|u_{\varepsilon_n}|^p(x)\,dx=O(S^{-1+\frac{N}{p}})$$

uniformly in $\varepsilon_n$. Therefore,

$$u_{\varepsilon_n}\to u \text{ in } L^p(\mathbb{R}^N).$$

Moreover, from (4.5), we can also assume that

$$\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^{p-2}\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\rightharpoonup\chi(x,z)$$

weakly in $L^{p'}(\mathbb{R}^N)\times L^{p'}(\mathbb{R}^N)$. Therefore, passing to the limit in (4.4) for $\varepsilon=\varepsilon_n$, we get

$$\int_{\mathbb{R}^N}uv+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot\nabla v(x)\,dx\,dz=\int_{\mathbb{R}^N}\phi v \tag{4.7}$$

for every $v$ smooth and by approximation for every $v\in W^{1,p}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$.

From now on we follow closely the arguments in [2], but we include some details here for the sake of completeness.

Let us see now that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)z\cdot\nabla v(x)\,dx\,dz=\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\cdot\nabla v. \tag{4.8}$$

In fact, taking $v=u_{\varepsilon_n}$ in (4.4), by (4.7) we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^n}\frac{C_{J,p}}{2}J(z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\,dz=\int_{\mathbb{R}^N}\phi u_{\varepsilon_n}-\int_{\mathbb{R}^N}u_{\varepsilon_n}u_{\varepsilon_n}\\&=\int_{\mathbb{R}^N}\phi u-\int_{\mathbb{R}^n}uu-\int_{\mathbb{R}^N}\phi(u-u_{\varepsilon_n})+\int_{\mathbb{R}^N}2u(u-u_{\varepsilon_n})-\int_{\mathbb{R}^N}(u-u_{\varepsilon_n})(u-u_{\varepsilon_n})\\&\le\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot\nabla u(x)\,dx\,dz-\int_{\mathbb{R}^N}\phi(u-u_{\varepsilon_n})+\int_{\mathbb{R}^nN}2u(u-u_{\varepsilon_n}).\end{aligned}$$

Consequently,

$$\begin{aligned}&\limsup_n\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\,dz\\&\qquad\le\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot\nabla u(x)\,dx\,dz.\end{aligned}\tag{4.9}$$

Now, by the monotonicity property (2.4), for every $\rho$ smooth,

$$\begin{aligned}&-\frac{C_{J,p}}{{\varepsilon_n}^{p+N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}J\left(\frac{x-y}{\varepsilon_n}\right)|\rho(y)-\rho(x)|^{p-2}(\rho(y)-\rho(x))\,dy\,(u_{\varepsilon_n}(x)-\rho(x))\,dx\\&\le-\frac{C_{J,p}}{{\varepsilon_n}^{p+N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}J\left(\frac{x-y}{\varepsilon_n}\right)|u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x)|^{p-2}(u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x))\,dy\,(u_{\varepsilon_n}(x)-\rho(x))\,dx.\end{aligned}$$

Using the change of variable (4.3) and taking limits, on account of (4.6) and (4.9), we obtain for every $\rho$ smooth,

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)|z\cdot\nabla\rho|^{p-2}z\cdot\nabla\rho\,z\cdot(\nabla u-\nabla\rho)\\&\le\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot(\nabla u(x)-\nabla\rho(x))\,dx\,dz,\end{aligned}$$

and then, by approximation, for every $\rho\in W^{1,p}(\mathbb{R}^N)$. Taking now, $\rho=u\pm\lambda v$, $\lambda>0$ and $v\in W^{1,p}(\mathbb{R}^N)$, and letting $\lambda\to 0$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)z\cdot\nabla v(x)\,dx\,dz\\&=\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\int_{\Omega}|z\cdot\nabla u(x)|^{p-2}(z\cdot\nabla u(x))\,(z\cdot\nabla v(x))\,dx\,dz.\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)z\cdot\nabla v(x)\,dx\,dz=C_{J,p}\int_{\mathbb{R}^N}\mathbf{a}(\nabla u)\cdot\nabla v\quad\text{ for every } v\in W^{1,p}(\mathbb{R}^N),$$

where

$$\mathbf{a}_j(\xi) = C_{J,p} \int_{\mathbb{R}^N} \frac{1}{2} J(z)\, |z \cdot \xi|^{p-2} \; z \cdot \xi \, z_j \, dz.$$

Hence, if we prove that

$$\mathbf{a}(\xi) = |\xi|^{p-2}\xi, \tag{4.10}$$

then (4.8) is true and $u = (I + B_p)^{-1} \phi$. So, to finish the proof we only need to show that (4.10) holds. Obviously, $\mathbf{a}$ is positively homogeneous of degree $p-1$, that is,

$$\mathbf{a}(t\xi) = t^{p-1}\mathbf{a}(\xi) \qquad \text{for all } \; \xi \in \mathbb{R}^N \quad \text{and all } t > 0.$$

Therefore, in order to prove (4.10) it is enough to see that

$$\mathbf{a}_i(\xi) = \xi_i \quad \text{for all } \xi \in \mathbb{R}^N, |\xi| = 1, \qquad i = 1, \ldots, N.$$

Now, let $R_{\xi,i}$ be the rotation such that $R^t_{\xi,i}(\xi) = \mathbf{e}_i$, where $\mathbf{e}_i$ is the vector with components $(\mathbf{e}_i)_i = 1$, $(\mathbf{e}_i)_j = 0$ for $j \neq i$, being $R^t_{\xi,i}$ the transpose of $R_{\xi,i}$. Observe that

$$\xi_i = \xi \cdot \mathbf{e}_i = R^t_{\xi,i}(\xi) \cdot R^t_{\xi,i}(\mathbf{e}_i) = \mathbf{e}_i \cdot R^t_{\xi,i}(\mathbf{e}_i).$$

On the other hand, since $J$ is radial, $C^{-1}_{J,p} = \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_i|^p \, dz$ and

$$\mathbf{a}(\mathbf{e}_i) = \mathbf{e}_i \quad \text{for every } i.$$

Making the change of variables $z = R_{\xi,i}(y)$, since $J$ is a radial function, we obtain

$$\begin{aligned} \mathbf{a}_i(\xi) \; &= C_{J,p} \int_{\mathbb{R}^N} \frac{1}{2} J(z)|z \cdot \xi|^{p-2} z \cdot \xi \; z \cdot \mathbf{e}_i \, dz \\ &= C_{J,p} \int_{\mathbb{R}^N} \frac{1}{2} J(y)|y \cdot \mathbf{e}_i|^{p-2} y \cdot \mathbf{e}_i \; y \cdot R^t_{\xi,i}(\mathbf{e}_i) \, dy \\ &= \mathbf{a}(\mathbf{e}_i) \cdot R^t_{\xi,i}(\mathbf{e}_i) = \mathbf{e}_i \cdot R^t_{\xi,i}(\mathbf{e}_i) = \xi_i, \end{aligned}$$

and the proof finishes. □

For $\varepsilon > 0$, we rescale the functional $G^J_\infty$ as follows

$$G^\varepsilon_\infty(u) = \begin{cases} 0 & \text{if } \; |u(x) - u(y)| \leq \varepsilon, \; \text{for } |x - y| \leq \varepsilon, \\ +\infty & \text{in other case.} \end{cases}$$

In other words, $G^\varepsilon_\infty = I_{K_\varepsilon}$, where

$$K_\varepsilon := \{u \in L^2(\mathbb{R}^N) \; : \; |u(x) - u(y)| \leq \varepsilon, \; \text{for } |x - y| \leq \varepsilon\}.$$

Consider the gradient flow associated to the functional $G^\varepsilon_\infty$

$$P^\varepsilon_\infty(u_0) \quad \begin{cases} f(t,\cdot) - u_t(t,\cdot) \in \partial I_{K_\varepsilon}(u(t)), & \text{a.e. } t \in ]0,T[, \\ u(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

and the problem

$$P_\infty(u_0) \quad \begin{cases} f(t,\cdot) - u_{\infty,t} \in \partial I_{K_0}(u_\infty), & \text{a.e. } t \in ]0,T[, \\ u_\infty(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$K_0 := \left\{ u \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N) \ : \ |\nabla u| \le 1 \right\}.$$

Observe that if $u \in K_0$, $|\nabla u| \le 1$. Hence, $|u(x) - u(y)| \le |x-y|$, from where it follows that $u \in K_\varepsilon$, that is, $K_0 \subset K_\varepsilon$.

With all these definitions and notations, we can proceed with the limit as $\varepsilon \to 0$ for the sandpile model ($p = \infty$).

**Proof of Theorem 1.3.** We have $T > 0$, $f \in L^1(0,T; L^2(\mathbb{R}^N))$, $u_0 \in K_0$ and $u_{\infty,\varepsilon}$ the unique solution of $P_\infty^\varepsilon(u_0)$. We have to show that if $v_\infty$ is the unique solution of $P_\infty(u_0)$, then

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \| u_{\infty,\varepsilon}(t,\cdot) - v_\infty(t,\cdot) \|_{L^2(\mathbb{R}^N)} = 0.$$

Since $u_0 \in K_0$, $u_0 \in K_\varepsilon$ for all $\varepsilon > 0$, and consequently there exists $u_{\infty,\varepsilon}$ the unique solution of $P_\infty^\varepsilon(u_0)$.

By Theorem 2.1 to prove the result it is enough to show that $I_{K_\varepsilon}$ converges to $I_{K_0}$ in the sense of Mosco. It is easy to see that

$$K_{\varepsilon_1} \subset K_{\varepsilon_2}, \quad \text{if } \ \varepsilon_1 \le \varepsilon_2. \tag{4.11}$$

Since $K_0 \subset K_\varepsilon$ for all $\varepsilon > 0$, we have

$$K_0 \subset \bigcap_{\varepsilon > 0} K_\varepsilon.$$

On the other hand, if

$$u \in \bigcap_{\varepsilon > 0} K_\varepsilon,$$

we have

$$\frac{|u(y) - u(x)|}{|y-x|} \le 1, \quad \text{a.e } \ x, y \in \mathbb{R}^N,$$

from where it follows that $u \in K_0$. Therefore, we have

$$K_0 = \bigcap_{\varepsilon > 0} K_\varepsilon. \tag{4.12}$$

Note that

$$\mathrm{Epi}(I_{K_0}) = K_0 \times [0,\infty[, \quad \mathrm{Epi}(I_{K_\varepsilon}) = K_\varepsilon \times [0,\infty[ \ \ \forall\, \varepsilon > 0. \tag{4.13}$$

By (4.12) and (4.13), we have

$$\mathrm{Epi}(I_{K_0}) \subset s - \liminf_{\varepsilon \to 0} \mathrm{Epi}(I_{K_\varepsilon}). \tag{4.14}$$

On the other hand, given $(u,\lambda) \in w - \limsup_{\varepsilon\to 0} \mathrm{Epi}(I_{K_\varepsilon})$ there exists $(u_{\varepsilon_k},\lambda_k) \in K_{\varepsilon_k} \times [0,\infty[$, such that $\varepsilon_k \to 0$ and

$$u_{\varepsilon_k} \rightharpoonup u \quad \text{in } L^2(\mathbb{R}^N), \quad \lambda_k \to \lambda \ \text{ in } \mathbb{R}.$$

By (4.11), given $\varepsilon > 0$, there exists $k_0$, such that $u_{\varepsilon_k} \in K_\varepsilon$ for all $k \geq k_0$. Then, since $K_\varepsilon$ is a closed convex set, we get $u \in K_\varepsilon$, and, by (4.12), we obtain that $u \in K_0$. Consequently,

$$w - \limsup_{n\to\infty} \mathrm{Epi}(I_{K_\varepsilon}) \subset \mathrm{Epi}(I_{K_0}). \tag{4.15}$$

Finally, by (4.14) and (4.15), and having in mind (2.1), we obtain that $I_{K_\varepsilon}$ converges to $I_{K_0}$ in the sense of Mosco. □

## 5. Collapse of the initial condition

Recall that we have mentioned in the Introduction that Evans, Feldman and Gariepy in [25] study the behavior of the solution $v_p$ of the initial value problem

$$\begin{cases} v_{p,t} - \Delta_p v_p = 0, & t \in ]0,T[, \\ v_p(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

in the "infinitely fast diffusion" limit $p \to \infty$, that is, when the initial condition $u_0$ is a Lipschitz function with compact support, satisfying

$$\text{ess}\ \sup_{\mathbb{R}^N} |\nabla u_0| = L > 1.$$

They prove that for each time $t > 0$

$$v_p(t,\cdot) \to v_\infty(\cdot), \quad \text{uniformly as } \ p \to +\infty,$$

where $v_\infty$ is independent of time and satisfies

$$\text{ess}\ \sup_{\mathbb{R}^N} |\nabla v_\infty| \leq 1.$$

Moreover, $v_\infty(x) = v(1,x)$, $v$ solving the nonautonomous evolution equation

$$\begin{cases} \dfrac{v}{t} - v_t \in \partial I_{K_0}(v), & t \in ]\tau,\infty[ \\ v(\tau,x) = \tau u_0(x), \end{cases}$$

where $\tau = L^{-1}$. They interpreted this as a crude model for the collapse of a sandpile from an initially unstable configuration. The proof of this result is based in a scaling argument, which was extended by Bénilan, Evans and Gariepy in [10], to cover general nonlinear evolution equations governed by homogeneous accretive operators. Here, using this general result, we prove similar results for our nonlocal model.

We look for the limit as $p \to \infty$ of the solutions to the nonlocal problem $P_p^J(u_0)$ when the initial datum $u_0$ satisfies

$$1 < L = \sup_{x-y\in \mathrm{supp}(J)} |u_0(x) - u_0(y)|.$$

For $p > 2$, we consider in the Banach space $X = L^2(\mathbb{R}^N)$ the operators $\partial G_p^J$. Then, $\partial G_p^J$ are $m$-accretive operators in $L^2(\mathbb{R}^n)$ and also positively homogeneous of degree $p-1$. Moreover, the solution $u_p$ to the nonlocal problem $P_p^J(u_0)$ coincides with the strong solution of the abstract Cauchy problem

$$\begin{cases} -u_t(t,x) \in \partial G_p^J(u(t)), & \text{a.e } t \in ]0,T[, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

Let

$$C := \left\{ u \in L^2(\mathbb{R}^N) \ : \ \exists (u_p, v_p) \in \partial G_p^J \text{ with } u_p \to u, \ v_p \to 0 \text{ as } p \to \infty \right\}.$$

It is easy to see that

$$C = K_\infty^J = \left\{ u \in L^2(\mathbb{R}^N) \ : \ |u(x) - u(y)| \le 1, \text{ for } x - y \in \operatorname{supp}(J) \right\}.$$

Then,

$$X_0 := \overline{\bigcup_{\lambda > 0} \lambda C}^{L^2(\mathbb{R}^N)} = L^2(\mathbb{R}^N).$$

**Lemma 5.1.** *For $f \in L^2(\mathbb{R}^N)$ and $p > N$, let $u_p := (I + \partial G_p^J)^{-1} f$. Then, the set of functions $\{u_p \ : \ p > N\}$ is precompact in $L^2(\mathbb{R}^N)$.*

*Proof.* First assume that $f$ is bounded and the support of $f$ lies in the ball $B_R(0)$. Since the operator $\partial G_p^J$ is completely accretive (observe that $\partial G_p^J = \overline{\mathcal{B}_p^J \cap (L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N))}^{L^2(\mathbb{R}^N)}$), we have the estimates

$$\|u_p\|_{L^\infty} \le \|f\|_{L^\infty}, \quad \|u_p\|_{L^2} \le \|f\|_{L^2}$$

and

$$\|u_p(\cdot) - u_p(\cdot + h)\|_{L^2} \le \|f(\cdot) - f(\cdot + h)\|_{L^2}$$

for each $h \in \mathbb{R}^N$. Consequently, $\{u_p \ : \ p > N\}$ is precompact in $L^2(K)$ for each compact set $K \subset \mathbb{R}^N$. We must show that $\{u_p \ : \ p > N\}$ is tight. For this, fix $S > 2R$ and select a smooth function $\varphi \in C^\infty(\mathbb{R}^N)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 0$ on $B_R(0)$, $\varphi \equiv 1$ on $\mathbb{R}^N \setminus B_S(0)$ and $|\nabla \varphi| \le \frac{2}{S}$.

We have

$$u_p(x) = \int_{\mathbb{R}^N} J(x-y)|u_p(y) - u_p(x)|^{p-2}(u_p(y) - u_p(x))\, dy + f(x).$$

Then, multiplying by $\varphi u_p$ and integrating, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} u_p^2(x)\varphi(x)\,dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))u_p(x)\varphi(x)\,dydx \\
&= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))(u_p(y)\varphi(y)-u_p(x)\varphi(x))\,dydx \\
&\le -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))u_p(y)(\varphi(y)-\varphi(x))\,dydx.
\end{aligned}$$

Now, since $|\nabla\varphi| \le \frac{2}{S}$, by Hölder's inequality we obtain

$$\begin{aligned}
&\left|\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))u_p(y)(\varphi(y)-\varphi(x))\,dydx\right| \\
&\le \frac{\|f\|_{L^\infty}}{S}\int_{\{|x|\le S+1\}}\left(\int_{B_1(x)} J(x-y)|u_p(y)-u_p(x)|^{p-1}\,dy\right)dx \\
&\le \frac{\|f\|_{L^\infty}}{S}\left(\int_{\{|x|\le S+1\}}\int_{B_1(x)} J(x-y)|u_p(y)-u_p(x)|^{p}dy\right)^{\frac{1}{p'}} \\
&\qquad\times\left(\int_{\{|x|\le S+1\}}\int_{B_1(x)} J(x-y)dy\right)^{\frac{1}{p}}dx \\
&\le M(S+1)^{\frac{N}{p}-1} = O(S^{-1+\frac{N}{p}}),
\end{aligned}$$

the last inequality being true since $\int\int J(x-y)|u_p(y)-u_p(x)|^p$ is bounded uniformly in $p$. Hence,

$$\int_{\{|x|\ge S\}} u_p^2(x)\,dx = O(S^{-1+\frac{N}{p}})$$

uniformly in $p > N$. This proves tightness and we have established compactness in $L^2(\mathbb{R}^N)$ provided $f$ is bounded and has compact support. The general case follows, since such functions are dense in $L^2(\mathbb{R}^N)$. □

**Proof of Theorem 1.4.** By the above Lemma, given $f \in L^2(\mathbb{R}^N)$ if $u_p := (I+\partial G_p^J)^{-1}f$, there exists a sequence $p_j \to +\infty$, such that $u_{p_j} \to v$ in $L^2(\mathbb{R}^N)$ as $j\to\infty$. In the proof of Theorem 1.1 we have established that the functionals $G_p^J$ converge to $I_{K_\infty^J}$, as $p\to\infty$, in the sense of Mosco. Then, by Theorem 2.1, we have $v = (I + I_{K_\infty^J})^{-1}f$. Therefore, the limit

$$Pf := \lim_{p\to\infty}(I+\partial G_p^J)^{-1}f$$

exists in $L^2(\mathbb{R}^N)$, for all $f \in X_0 = L^2(\mathbb{R}^N)$, and $Pf = f$ if $f \in C = K_\infty^J$. Moreover,

$$P^{-1} - I = \partial I_{K_\infty^J}$$

and $u = Pf$ is the unique solution of

$$u + \partial I_{K^J_\infty} u \ni f.$$

Therefore, as consequence of the main result of [10], we have obtained Theorem 1.4. $\square$

## 6. Explicit solutions

In this section we show some explicit examples of solutions to

$$P^\varepsilon_\infty(u_0) \quad \begin{cases} f(t,x) - u_t(t,x) \in \partial G^\varepsilon_\infty(u(t)), & \text{a.e. } t \in ]0,T[, \\ u(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$G^\varepsilon_\infty(u) = \begin{cases} 0 & \text{if } u \in L^2(\mathbb{R}^N),\ |u(x) - u(y)| \le \varepsilon,\ \text{for } |x-y| \le \varepsilon, \\ +\infty & \text{in other case.} \end{cases}$$

In order to verify that a function $u(t,x)$ is a solution to $P^\varepsilon_\infty(u_0)$ we need to check that

$$G^\varepsilon_\infty(v) \ge G^\varepsilon_\infty(u) + \langle f - u_t,\ v - u \rangle, \qquad \text{for all } v \in L^2(\mathbb{R}^N). \tag{6.1}$$

To this end we can assume that $v \in K_\varepsilon$ (otherwise $G^\varepsilon_\infty(v) = +\infty$ and then (6.1) becomes trivial). Therefore,

$$u(t,\cdot) \in K_\varepsilon \tag{6.2}$$

and (6.1) can be rewritten as

$$0 \ge \int_{\mathbb{R}} (f(t,x) - u_t(t,x))(v(x) - u(t,x))\, dx \tag{6.3}$$

for every $v \in K_\varepsilon$.

**Example 1.** Let us consider, in one space dimension, as source an approximation of a delta function

$$f(t,x) = f_\eta(t,x) = \frac{1}{\eta} \chi_{[-\frac{\eta}{2},\frac{\eta}{2}]}(x), \quad 0 < \eta \le 2\varepsilon,$$

and as initial datum

$$u_0(x) = 0.$$

Now, let us find the solution by looking at its evolution between some critical times.

First, for small times, the solution to $P^\varepsilon_\infty(u_0)$ is given by

$$u(t,x) = \frac{t}{\eta} \chi_{[-\frac{\eta}{2},\frac{\eta}{2}]}(x), \tag{6.4}$$

for

$$t \in [0, \eta\varepsilon).$$

Remark that $t_1 = \eta\varepsilon$ is the first time when $u(t,x) = \varepsilon$ and hence it is immediate that $u(t,\cdot) \in K_\varepsilon$. Moreover, as $u_t(t,x) = f(t,x)$ then (6.3) holds.

For times greater than $t_1$ the support of the solution is greater than the support of $f$. Indeed the solution can not be larger than $\varepsilon$ in $[-\frac{\eta}{2}, \frac{\eta}{2}]$ without being larger than zero in the adjacent intervals of size $\varepsilon$, $[\frac{\eta}{2}, \frac{\eta}{2}+\varepsilon]$ and $[-\frac{\eta}{2}-\varepsilon, -\frac{\eta}{2}]$.

We have

$$u(t,x) = \begin{cases} \varepsilon + k_1(t-t_1) & \text{for } x \in [-\frac{\eta}{2}, \frac{\eta}{2}], \\ k_1(t-t_1) & \text{for } x \in [-\frac{\eta}{2}-\varepsilon, \frac{\eta}{2}+\varepsilon] \setminus [-\frac{\eta}{2}, \frac{\eta}{2}], \\ 0 & \text{for } x \notin [-\frac{\eta}{2}-\varepsilon, \frac{\eta}{2}+\varepsilon], \end{cases} \tag{6.5}$$

for times $t$ such that

$$t \in [t_1, t_2)$$

where

$$k_1 = \frac{1}{2\varepsilon + \eta} \quad \text{and} \quad t_2 = t_1 + \frac{\varepsilon}{k_1} = 2\varepsilon^2 + 2\varepsilon\eta.$$

Note that $t_2$ is the first time when $u(t,x) = 2\varepsilon$ for $x \in [-\frac{\eta}{2}, \frac{\eta}{2}]$. Again it is immediate to see that $u(t,\cdot) \in K_\varepsilon$, since for $|x-y| < \varepsilon$ the maximum of the difference $u(t,x) - u(t,y)$ is exactly $\varepsilon$. Now let us check (6.3).

Using the explicit formula for $u(t,x)$ given in (6.5), we obtain

$$\begin{aligned} &\int_{\mathbb{R}} (f(t,x) - u_t(t,x))(v(x) - u(t,x))\,dx = \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \left(\frac{1}{\eta} - u_t(t,x)\right)(v(x) - u(t,x))\,dx \\ &\quad + \int_{\frac{\eta}{2}}^{\frac{\eta}{2}+\varepsilon} (-u_t(t,x))(v(x) - u(t,x))\,dx + \int_{-\frac{\eta}{2}-\varepsilon}^{-\frac{\eta}{2}} (-u_t(t,x))(v(x) - u(t,x))\,dx \\ &= \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \left(\frac{1}{\eta} - k_1\right)(v(x) - (\varepsilon + k_1(t-t_1)))\,dx + \int_{\frac{\eta}{2}}^{\frac{\eta}{2}+\varepsilon} (-k_1)(v(x) - (k_1(t-t_1)))\,dx \\ &\quad + \int_{-\frac{\eta}{2}-\varepsilon}^{-\frac{\eta}{2}} (-k_1)(v(x) - (k_1(t-t_1)))\,dx \\ &= \left(-\eta\left(\frac{1}{\eta} - k_1\right) + 2\varepsilon k_1\right) k_1(t-t_1) - \varepsilon\eta\left(\frac{1}{\eta} - k_1\right) + \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \left(\frac{1}{\eta} - k_1\right) v(x)\,dx \\ &\qquad - \int_{\frac{\eta}{2}}^{\frac{\eta}{2}+\varepsilon} k_1 v(x)\,dx - \int_{-\frac{\eta}{2}-\varepsilon}^{-\frac{\eta}{2}} k_1 v(x)\,dx. \end{aligned} \tag{6.6}$$

From our choice of $k_1$ we get

$$-\eta\left(\frac{1}{\eta} - k_1\right) + 2\varepsilon k_1 = 0$$

and, since $v \in K_\varepsilon$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} (f(t,x) - u_t(t,x))(v(x) - u(t,x))\, dx \\
&= -2\varepsilon^2 k_1 + \frac{2\varepsilon k_1}{\eta} \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} v(x)\, dx - k_1 \int_{\frac{\eta}{2}}^{\frac{\eta}{2}+\varepsilon} v(x)\, dx - k_1 \int_{-\frac{\eta}{2}-\varepsilon}^{-\frac{\eta}{2}} v(x)\, dx \le 0.
\end{aligned}
\tag{6.7}
$$

In fact, without loss of generality we can suppose that

$$
\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} v(x)\, dx = 0.
$$

Then

$$
\int_0^{\eta/2} (-v) = a, \quad \int_{-\eta/2}^{0} (-v) = -a. \tag{6.8}
$$

Consequently,

$$
-v \le \frac{2}{\eta} a + \varepsilon \quad \text{in } [0,\varepsilon]. \tag{6.9}
$$

Indeed, if (6.9) does not hold, then $-v > \frac{2}{\eta} a$ in $[0,\varepsilon]$ which contradicts (6.8).

Now, by (6.8), since $v \in K_\varepsilon$,

$$
\begin{aligned}
\int_{\varepsilon}^{\varepsilon+\eta/2} (-v(x))dx &= \int_0^{\eta/2} (-v(y+\varepsilon))dy \\
&= \int_0^{\eta/2} (-v(y+\varepsilon) + v(y))dy + \int_0^{\eta/2} (-v(y))dy \\
&\le \varepsilon \frac{\eta}{2} + a.
\end{aligned}
\tag{6.10}
$$

Therefore, by (6.9) and (6.10),

$$
\begin{aligned}
\int_{\eta/2}^{\varepsilon+\eta/2} (-v) &= \int_{\eta/2}^{\varepsilon} (-v) + \int_{\varepsilon}^{\varepsilon+\eta/2} (-v) \\
&\le \left( \frac{2}{\eta} a + \varepsilon \right) \left( \varepsilon - \frac{\eta}{2} \right) + \varepsilon \frac{\eta}{2} + a = \frac{2}{\eta} a\varepsilon + \varepsilon^2.
\end{aligned}
\tag{6.11}
$$

Similarly,

$$
\int_{-\varepsilon-\eta/2}^{-\eta/2} (-v) \le -\frac{2}{\eta} a\varepsilon + \varepsilon^2. \tag{6.12}
$$

Consequently, by (6.11) and (6.12),

$$\int_{\frac{\eta}{2}}^{\frac{\eta}{2}+\varepsilon}(-v)+\int_{-\frac{\eta}{2}-\varepsilon}^{-\frac{\eta}{2}}(-v)\leq 2\varepsilon^2.$$

Now, it is easy to generalize and verify the following general formula that describes the solution for every $t \geq 0$. For any given integer $l \geq 0$ we have

(6.13)

$$u(t,x)=\begin{cases} l\varepsilon+k_l(t-t_l), & x\in[-\frac{\eta}{2},\frac{\eta}{2}],\\ (l-1)\varepsilon+k_l(t-t_l), & x\in[-\frac{\eta}{2}-\varepsilon,\frac{\eta}{2}+\varepsilon]\setminus[-\frac{\eta}{2},\frac{\eta}{2}],\\ \dots & \\ k_l(t-t_l), & x\in[-\frac{\eta}{2}-l\varepsilon,\frac{\eta}{2}+l\varepsilon]\setminus[-\frac{\eta}{2}-(l-1)\varepsilon,\frac{\eta}{2}+(l-1)\varepsilon],\\ 0, & x\notin[-\frac{\eta}{2}-l\varepsilon,\frac{\eta}{2}+l\varepsilon],\end{cases}$$

for

$$t\in[t_l,t_{l+1}),$$

where

$$k_l=\frac{1}{2l\varepsilon+\eta}\quad\text{and}\quad t_{l+1}=t_l+\frac{\varepsilon}{k_l},\quad t_0=0.$$

From formula (6.13) we get, taking the limit as $\eta\to 0$, that the expected solution to (6.1) with $f=\delta_0$ is given by, for any given integer $l\geq 1$,

$$u(t,x)=\begin{cases} (l-1)\varepsilon+k_l(t-t_l), & x\in[-\varepsilon,\varepsilon],\\ (l-2)\varepsilon+k_l(t-t_l), & x\in[-2\varepsilon,2\varepsilon]\setminus[-\varepsilon,\varepsilon],\\ \dots & \\ k_l(t-t_l), & x\in[-l\varepsilon,l\varepsilon]\cup[-(l-1)\varepsilon,(l-1)\varepsilon],\\ 0, & x\notin[-l\varepsilon,l\varepsilon],\end{cases}\tag{6.14}$$

for

$$t\in[t_l,t_{l+1})$$

where

$$k_l=\frac{1}{2l\varepsilon},\quad t_{l+1}=t_l+\frac{\varepsilon}{k_l},\quad t_1=0.$$

Remark that, since the space of functions $K_\varepsilon$ is not contained into $C(\mathbb{R})$, the formulation (6.3) with $f=\delta_0$ does not make sense. Hence the function $u(t,x)$ described by (6.14) is to be understood as a generalized solution to (6.1) (it is obtained as a limit of solutions to approximating problems).

Note that the function $u(t_l, x)$ is a "regular and symmetric pyramid" composed by squares of side $\varepsilon$.

**Recovering the sandpile model as $\varepsilon \to 0$.** Now, to recover the sandpile model, let us fix

$$l\varepsilon = L,$$

and take the limit as $\varepsilon \to 0$ in the previous example. We get that $u(t,x) \to v(t,x)$, where

$$v(t,x) = (L - |x|)_+, \qquad \text{for } t = L^2,$$

that is exactly the evolution given by the sandpile model with initial datum $u_0 = 0$ and a point source $\delta_0$, see [3].

Therefore, this concrete example illustrates the general convergence result Theorem 1.3.

**Example 2.** The explicit formula (6.13) can be easily generalized to the case in where the source depends on $t$ in the form

$$f(t,x) = \varphi(t)\chi_{[-\frac{\eta}{2},\frac{\eta}{2}]}(x),$$

with $\varphi$ a nonnegative integrable function and $0 < \eta \leq \varepsilon$. We arrive to the following formulas, setting

$$g(t) = \int_0^t \varphi(s)ds,$$

for any given integer $l \geq 0$,

$$u(t,x) = \begin{cases} l\varepsilon + \hat{k}_l\left(g(t) - g(t_l)\right), & x \in [-\frac{\eta}{2}, \frac{\eta}{2}], \\ (l-1)\varepsilon + \hat{k}_l\left(g(t) - g(t_l)\right), & x \in [-\frac{\eta}{2} - \varepsilon, \frac{\eta}{2} + \varepsilon] \setminus [-\frac{\eta}{2}, \frac{\eta}{2}], \\ \dots & \\ \hat{k}_l\left(g(t) - g(t_l)\right), & x \in [-\frac{\eta}{2} - l\varepsilon, \frac{\eta}{2} + l\varepsilon] \setminus [-\frac{\eta}{2} - (l-1)\varepsilon, \frac{\eta}{2} + (l-1)\varepsilon] \\ 0, & x \notin [-\frac{\eta}{2} - l\varepsilon, \frac{\eta}{2} + l\varepsilon], \end{cases}$$

for

$$t \in [t_l, t_{l+1}),$$

where

$$\hat{k}_l = \frac{\eta}{\eta + 2l\varepsilon} \quad \text{and} \quad g(t_{l+1}) - g(t_l) = \frac{\varepsilon}{\hat{k}_l}, \quad t_0 = 0.$$

Observe that $t_l$ is the first time at which the solution reaches level $l\varepsilon$.

We can also consider $\varphi$ changing sign. In this case the solution increases if $\varphi(t)$ is positive in every interval of size $\varepsilon$ (around the support of the source $[-\frac{\eta}{2}, \frac{\eta}{2}]$) for which $u(x) - u(y) = i\varepsilon$ with $|x - y| = i\varepsilon$ for some $x \in [-\frac{\eta}{2}, \frac{\eta}{2}]$ (here $i$ is any integer). While if $\varphi(t)$ is negative the solution decreases in every interval of size $\varepsilon$ for which $u(x) - u(y) = -i\varepsilon$ with $|x - y| = i\varepsilon$ for some $x \in [-\frac{\eta}{2}, \frac{\eta}{2}]$.

**Example 3.** Observe that if $\eta > 2\varepsilon$, then $u(t,x)$ given in (6.5) does not satisfy (6.3) for a test function $v \in K_\varepsilon$ whose values in $[-\frac{\eta}{2}-\varepsilon, \frac{\eta}{2}+\varepsilon]$ are

$$v(x) = \begin{cases} -\beta\dfrac{\varepsilon}{2} + 2\varepsilon & \text{for } x \in [-\frac{\eta}{2}+\varepsilon, \frac{\eta}{2}-\varepsilon], \\ -\beta\dfrac{\varepsilon}{2} + \varepsilon & \text{for } x \in [-\frac{\eta}{2}, \frac{\eta}{2}] \setminus [-\frac{\eta}{2}+\varepsilon, \frac{\eta}{2}-\varepsilon], \\ -\beta\dfrac{\varepsilon}{2} & \text{for } x \in [-\frac{\eta}{2}-\varepsilon, \frac{\eta}{2}+\varepsilon] \setminus [-\frac{\eta}{2}, \frac{\eta}{2}], \end{cases}$$

for $\beta = 4(1-\varepsilon/\eta)$ which is greater that 2.

At this point one can ask what happens in the previous situation when $\eta > 2\varepsilon$. In this case the solution begins to grow as before with constant speed in the support of $f$ but after the first time when it reaches level $\varepsilon$ the situation changes. Consider, for example, that the source is given by

$$f(t,x) = \frac{1}{\varepsilon}\chi_{[-2\varepsilon,2\varepsilon]}(x).$$

In this case the solution to our nonlocal problem with $u_0(x) = 0$, $u(t,x)$, can be described as follows. Firstly we have

$$u(t,x) = \frac{t}{\varepsilon}\chi_{[-2\varepsilon,2\varepsilon]}(x),$$

for

$$t \in [0, \varepsilon^2).$$

Remark that $t_1 = \varepsilon^2$ is the first time when $u(t,x) = \varepsilon$ and hence it is immediate that $u(t,\cdot) \in K_\varepsilon$. Moreover, as $u_t(t,x) = f(t,x)$ then (6.3) holds.

For times greater that $t_1$ we have

$$u(t,x) = \begin{cases} \varepsilon + \dfrac{1}{\varepsilon}(t-t_1) & \text{for } x \in [-\varepsilon, \varepsilon], \\ \varepsilon + k_1(t-t_1) & \text{for } x \in [-2\varepsilon, -\varepsilon] \cup [\varepsilon, 2\varepsilon], \\ k_1(t-t_1) & \text{for } x \in [-3\varepsilon, -2\varepsilon] \cup [2\varepsilon, 3\varepsilon], \\ 0 & \text{for } x \notin [-3\varepsilon, 3\varepsilon], \end{cases}$$

for

$$t \in [t_1, t_2),$$

where

$$k_1 = \frac{1}{2\varepsilon} \quad \text{and} \quad t_2 = \varepsilon^2 + 2\varepsilon^2 = 3\varepsilon^2.$$

With this expression of $u(t,x)$ it is easy to see that it verifies (6.3).

For times greater than $t_2$ an expression similar to (6.13) holds. We leave the details to the reader.

**Example 4.** For two or more dimensions we can get similar formulas. Given a bounded domain $\Omega_0 \subset \mathbb{R}^N$ let us define inductively

$$\Omega_1 = \left\{ x \in \mathbb{R}^N \ : \ \exists y \in \Omega_0, \text{ with } |x-y| < \varepsilon \right\}$$

and

$$\Omega_j = \left\{ x \in \mathbb{R}^N \ : \ \exists y \in \Omega_{j-1}, \text{ with } |x-y| < \varepsilon \right\}.$$

In the sequel, for simplicity, we consider the two dimensional case $N = 2$. Let us take as source

$$f(t,x) = \chi_{\Omega_0}(x), \quad \Omega_0 = B(0, \varepsilon/2),$$

and, as initial datum,

$$u_0(x) = 0.$$

In this case, for any integer $l \geq 0$, the solution to (6.1) is given by

$$u(t,x) = \begin{cases} l\varepsilon + \hat{k}_l(t-t_l), & x \in \Omega_0, \\ (l-1)\varepsilon + \hat{k}_l(t-t_l), & x \in \Omega_1 \setminus \Omega_0, \\ \dots & \\ \hat{k}_l(t-t_l), & x \in \Omega_l \setminus \bigcup_{j=1}^{l-1} \Omega_j, \\ 0, & x \notin \Omega_l, \end{cases} \tag{6.15}$$

for

$$t \in [t_l, t_{l+1}),$$

where

$$\hat{k}_l = \frac{|\Omega_0|}{|\Omega_l|}, \qquad t_{l+1} = t_l + \frac{\varepsilon}{\hat{k}_l}, \quad t_0 = 0.$$

Note that the solution grows in strips of width $\varepsilon$ around the set $\Omega_0$ where the source is localized.

As in the previous examples, the result is evident for $t \in [0, t_1)$. Let us see it for $t \in [t_1, t_2)$, a similar argument works for later times. It is clear that $u(t,\cdot) \in K_\varepsilon$, let us check (6.3). Working as in Example 1, we must show that

$$(1-\hat{k}_1)\int_{\Omega_0} v - \hat{k}_1 \int_{\Omega_1 \setminus \Omega_0} v \leq (1-\hat{k}_1)\varepsilon|\Omega_0| \quad \forall v \in K_\varepsilon,$$

where $\Omega_1 = B(0, 3\varepsilon/2)$. Since $\hat{k}_1 = |\Omega_0|/|\Omega_1|$, the last inequality is equivalent to

$$\left| \frac{1}{|\Omega_0|} \int_{\Omega_0} v - \frac{1}{|\Omega_1 \setminus \Omega_0|} \int_{\Omega_1 \setminus \Omega_0} v \right| \leq \varepsilon \quad \forall v \in K_\varepsilon. \tag{6.16}$$

By density, it is enough to prove (6.16) for any $v \in K_\varepsilon$ continuous.

Let us now divide $\Omega_0 = \{r(\cos\theta, \sin\theta) : 0 \le \theta \le 2\pi, 0 \le r < \frac{\varepsilon}{2}\}$ and $\Omega_1 \setminus \Omega_0 = \{r(\cos\theta, \sin\theta) : 0 \le \theta \le 2\pi, \varepsilon \le r < \frac{3}{2}\varepsilon\}$ as follows. Consider the partitions

$$0 = \theta_0 < \theta_1 < .... < \theta_N = 2\pi,$$

with $\theta_i - \theta_{i-1} = 2\pi/N$, $N \in \mathbb{N}$,

$$0 = r_0 < r_1 < .... < r_N = \varepsilon/2$$

and

$$\varepsilon/2 = \tilde{r}_0 < \tilde{r}_1 < .... < \tilde{r}_N = 3\varepsilon/2,$$

such that the measure of

$$B_{ij} = \{r(\cos\theta, \sin\theta) : \theta_{i-1} < \theta < \theta_i \ r_{j-1} < r < r_j\}$$

is constant, that is, $|B_{ij}| = |\Omega_0|/N^2$, and the measure of

$$A_{ij} = \{r(\cos\theta, \sin\theta) : \theta_{i-1} < \theta < \theta_i \ \tilde{r}_{j-1} < r < \tilde{r}_j\}$$

is also constant, that is, $|A_{ij}| = |\Omega_1 \setminus \Omega_0|/N^2$. In this way we have partitioned $\Omega_0$ and $\Omega_1 \setminus \Omega_0$ as a disjoint family of $N^2$ sets such that

$$\left| \Omega_0 \setminus \bigcup_{i,j=1}^{N} B_{ij} \right| = 0, \qquad \left| (\Omega_1 \setminus \Omega_0) \setminus \bigcup_{i,j=1}^{N} A_{ij} \right| = 0.$$

By construction, if we take

$$x_{ij} = r_j(\cos\theta_{i-1}, \sin\theta_{i-1}) \in B_{ij}, \quad \tilde{x}_{ij} = \tilde{r}_{j-1}(\cos\theta_{i-1}, \sin\theta_{i-1}) \in A_{ij},$$

then $|x_{ij} - \tilde{x}_{ij}| \le \varepsilon$ for all $i, j = 1, \ldots N$.

Given a continuous function $v \in K_\varepsilon$, by uniform continuity of $v$, for $\delta > 0$, there exists $\rho > 0$ such that

$$|v(x) - v(y)| \le \frac{\delta}{2} \qquad \text{if } \ |x - y| \le \rho.$$

Hence, if we take $N$ big enough such that diameter$(B_{ij}) \le \rho$ and diameter$(A_{ij}) \le \rho$, we have

$$\left| \int_{\Omega_0} v(x) - \sum_{i,j=1}^{N} v(x_{ij})|B_{ij}| \right| \le \frac{\delta|\Omega_0|}{2}$$

and

$$\left| \int_{\Omega_1\setminus\Omega_0} v(x) - \sum_{i,j=1}^{N} v(\tilde{x}_{ij})|A_{ij}| \right| \le \frac{\delta|\Omega_1 \setminus \Omega_0|}{2}.$$

Since $v \in K_\varepsilon$ and $|x_{ij} - \tilde{x}_{ij}| \le \varepsilon$, $|v(x_{ij}) - v(\tilde{x}_{ij})| \le \varepsilon$. Consequently,

$$\begin{aligned}
&\left|\frac{1}{|\Omega_0|}\int_{\Omega_0} v - \frac{1}{|\Omega_1\setminus\Omega_0|}\int_{\Omega_1\setminus\Omega_0} v\right| \\
\leq &\left|\frac{1}{|\Omega_0|}\sum_{i,j=1}^{N} v(x_{ij})|B_{ij}| - \frac{1}{|\Omega_1\setminus\Omega_0|}\sum_{i,j=1}^{N} v(\tilde{x}_{ij})|A_{ij}|\right| + \delta \\
= &\left|\frac{1}{N^2}\sum_{i,j=1}^{N} v(x_{ij}) - \frac{1}{N^2}\sum_{i,j=1}^{N} v(\tilde{x}_{ij})\right| + \delta \\
\leq &\ \varepsilon + \delta.
\end{aligned}$$

Therefore, since $\delta > 0$ is arbitrary, (6.16) is obtained.

Again the explicit formula (6.15) can be easily generalized to the case where the source depends on $t$ in the form

$$f(t,x) = \varphi(t)\chi_{\Omega_0}(x).$$

**An estimate of the support of $u_t$.** Taking a source $f \geq 0$ supported in a set $A$, let us see where the material is added (places where $u_t$ is positive). Let us compute a set that we will call $\Omega^*(t)$ as follows. Let

$$\Omega_0(t) = A,$$

and define inductively

$$\Omega_1(t) = \left\{x \in \mathbb{R}^N \setminus \Omega_0(t)\ :\ \exists y \in \Omega_0(t) \text{ with } |x-y| < \varepsilon \text{ and } u(t,y) - u(t,x) = \varepsilon\right\}$$

and

$$\Omega_j(t) = \left\{x \in \mathbb{R}^N \setminus \Omega_{j-1}(t)\ :\ \exists y \in \Omega_{j-1}(t) \text{ with } |x-y| < \varepsilon \text{ and } u(t,y) - u(t,x) = \varepsilon\right\}.$$

With these sets $\Omega_i(t)$ (observe that there exists a finite number of such sets, since $u(t,x)$ is bounded) let

$$\Omega^*(t) = \bigcup_i \Omega_i(t).$$

We have that

$$u_t(t,x) = 0, \qquad \text{for } x \notin \Omega^*(t).$$

Indeed, this can be easily deduced using an appropriate test function $v$ in (6.3). Just take $v(x) = u(x,t)$ but for a small neighborhood near $x \notin \Omega^*(t)$.

**Example 5.** Finally, note that an analogous description like in the above examples can be made for an initial condition that is of the form

$$u_0(x) = \sum_{i=-K}^{K} a_i \chi_{[i\varepsilon,(i+1)\varepsilon]}(x),$$

with

$$|a_i - a_{i\pm1}| \leq \varepsilon, \qquad a_{-K} = a_K = 0,$$

(this last condition is needed just to imply that $u_0 \in K_\varepsilon$) together with the sum of a finite number of delta functions placed at points $x_l = l\varepsilon$ (or a finite sum of functions of time times the characteristic functions of some intervals of the form $[l\varepsilon, (l+1)\varepsilon])$ as the source term.

For example, let us consider a source placed in just one interval, $f(t,x) = \chi_{[0,\varepsilon]}(x)$. Initially, $u(0,x) = l\varepsilon$ for $x \in [0,\varepsilon]$. Let us take $w_1(x)$ the regular and symmetric pyramid centered at $[0,\varepsilon]$ of height $(l+1)\varepsilon$ (and base of length $(2l-1)\varepsilon$). With this pyramid and the initial condition let us consider the set

$$\Lambda_1 = \{ j \, : \, w_1(x) > u(0,x) \text{ for } x \in (j\varepsilon, (j+1)\varepsilon) \} .$$

This set contains the indexes of the intervals in which the sand is being added in the first stage. During this first stage, $u(t,x)$ is given by

$$u(t,x) = u(0,x) + \frac{t}{\operatorname{Card}(\Lambda_1)} \sum_{j \in \Lambda_1} \chi_{[j\varepsilon,(j+1)\varepsilon]}(x),$$

for $t \in [0,t_1]$, where $t_1 = \operatorname{Card}(\Lambda_1)\varepsilon$ is the first time at which $u$ is of size $(l+1)\varepsilon$ in the interval $[0,\varepsilon]$.

From now on the evolution follows the same scheme. In fact,

$$u(t,x) = u(t_i,x) + \frac{t - t_i}{\operatorname{Card}(\Lambda_i)} \sum_{j \in \Lambda_i} \chi_{[j\varepsilon,(j+1)\varepsilon]}(x),$$

for

$$t \in [t_i, t_{i+1}], \qquad t_{i+1} - t_i = \operatorname{Card}(\Lambda_i)\varepsilon.$$

Where, from the pyramid $w_i$ of height $(l+i)\varepsilon$, we obtain

$$\Lambda_i = \{ j \, : \, w_i(x) > u(t_i,x) \text{ for } x \in (j\varepsilon, (j+1)\varepsilon) \} .$$

Remark that eventually the pyramid $w_k$ is bigger than the initial condition, from this time on the evolution is the same as described for $u_0 = 0$ in the first example.

In case we have two sources, the pyramids $w_i$, $\tilde{w}_i$ corresponding to the two sources eventually intersect. In the interval where the intersection takes place, $u_t$ is given by the greater of the two possible speeds (that correspond to the different sources). If both possible speeds are the same this interval has to be computed as corresponding to both sources simultaneously.

**Recovering the sandpile model.** Note that any initial condition $w_0$ with $|\nabla w_0| \leq 1$ can be approximated by an $u_0$ like the one described above. Hence we can obtain an explicit solution of the nonlocal model that approximates the solutions constructed in [3].

**Compact support of the solutions.** Note also that when the source $f$ and the initial condition $u_0$ are compactly supported and bounded then also the solution is compactly supported and bounded for all positive times. This property has to be contrasted with the fact that solutions to the nonlocal $p-$laplacian $P_p^J(u_0)$ are not compactly supported even if $u_0$ is.

## 7. A mass transport interpretation

In [25], [28] or [24], a mass transfer interpretation of the limit problem $P_\infty(u_0)$ is described. Our aim in this section is to give an alternative explanation of our limit problem $G^J_\infty(u_0)$ in a transport setting, as we mentioned in the Introduction.

**Proof of Theorem 1.5.** Let

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ [|x-y|] + 1 & \text{if } x \neq y. \end{cases}$$

Here $[\cdot]$ means the entire part of the number. Note that this function $d$ measures distances with jumps of length one.

Then, given two measures (that for simplicity we will take absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^N$) $f_+$, $f_-$ in $\mathbb{R}^N$, and supposing the overall condition of mass balance

$$\int_{\mathbb{R}^N} f^+ \, dx = \int_{\mathbb{R}^N} f^- \, dy,$$

the Monge's problem associated to the distance $d$ is given by: minimize

$$\int d(x, s(x))\, f_+(x) dx \tag{7.1}$$

among maps $s$ that transport $f_+$ into $f_-$.

The dual formulation of this problem is given by

$$\max_{u \in K_\infty} \int_{\mathbb{R}^N} u(x)(f_+(x) - f_-(x)) dx \tag{7.2}$$

where, as before, $K_\infty$ is given by

$$K_\infty := \left\{ u \in L^2(\mathbb{R}^N) \ : \ |u(x) - u(y)| \leq 1, \text{ for } |x-y| \leq 1 \right\}.$$

We are assuming that $\operatorname{supp}(J) = \overline{B}_1(0)$ (in other case we may redefine the distance $d$ accordingly).

Then it is easy to obtain that the solution $u_\infty(t,\cdot)$ of the limit problem $G^J_\infty(u_0)$ is a solution to the dual problem (7.2) when the involved measures are the source $f(t,x)$ and the time derivative of the solution $u_{\infty,t}(t,x)$. In fact, we have

$$G^J_\infty(v) \geq G^J_\infty(u_\infty) + \langle f - u_{\infty,t},\ v - u_\infty \rangle, \qquad \text{for all } v \in L^2(\mathbb{R}^N).$$

That is equivalent to

$$u_\infty(t,\cdot) \in K_\infty$$

and

$$0 \geq \int_{\mathbb{R}^N} (f(t,x) - u_{\infty,t}(t,x))(v(x) - u_\infty(t,x))\, dx \tag{7.3}$$

for every $v \in K_\infty$. Now, we just observe that (7.3) is

$$\int_{\mathbb{R}^N} (f(t,x) - u_{\infty,t}(t,x))u_\infty(t,x)\,dx \geq \int_{\mathbb{R}^N} (f(t,x) - u_{\infty,t}(t,x))v(x)\,dx.$$

Therefore, we have that $u_\infty(t,\cdot)$ is a solution to the dual mass transport problem.

Consequently, we conclude that the mass of sand added by the source $f(t,\cdot)$ is transported (via $u(t,\cdot)$ as the transport potential) to $u_{\infty,t}(t,\cdot)$ at each time $t$. □

This mass transport interpretation of the problem can be clearly observed looking at the concrete examples in Section 6.

## 8. Neumann boundary conditions

Analogous calculations can be done with solutions to the Neumann boundary value problem for the nonlocal $p-$Laplacian.

Let $\Omega$ be a convex domain in $\mathbb{R}^N$. As we have mentioned, in [2] we have studied the evolution problem

$$P_p^{J,\Omega}(u_0) \begin{cases} u_{p,t}(t,x) = \displaystyle\int_\Omega J(x-y)|u_p(t,y) - u_p(t,x)|^{p-2}(u_p(t,y) - u_p(t,x))dy + f(t,x), \\ u_p(0,x) = u_0(x), \quad \text{in } \Omega. \end{cases}$$

The associated functional being

$$G_p^{J,\Omega}(u) = \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|u(y) - u(x)|^p\,dy\,dx.$$

This is the nonlocal analogous to the Neumann problem for the $p-$Laplacian since in this evolution problem, we have imposed a zero flux condition across the boundary of $\Omega$, see [2].

Also, let us consider the rescaled problems with $J_\varepsilon$, that we call $P_p^{J_\varepsilon,\Omega}(u_0)$, and the corresponding limit problems

$$P_\infty^{\varepsilon,\Omega}(u_0) \quad \begin{cases} f(t,\cdot) - u_t(t,\cdot) \in \partial G_\infty^{J,\Omega}(u(t)), & \text{a.e. } t \in ]0,T[, \\ u(0,x) = u_0(x), & \text{in } \Omega. \end{cases}$$

With associated functionals

$$G_\infty^{\varepsilon,\Omega}(u) = \begin{cases} 0 & \text{if } \ |u(x) - u(y)| \leq \varepsilon, \text{ for } |x-y| \leq \varepsilon; \quad x,y \in \Omega, \\ +\infty & \text{in other case.} \end{cases}$$

The limit problem of the local $p-$Laplacians being

$$P_\infty^{\Omega}(u_0) \quad \begin{cases} f(t) - v_{\infty,t} \in \partial F_\infty^{\Omega}(v_\infty(t)), & \text{a.e. } t \in ]0,T[, \\ v_\infty(0,x) = g(x), & \text{in } \Omega, \end{cases}$$

where the functional $F_\infty^\Omega$ is defined in $L^2(\Omega)$ by

$$F_\infty^\Omega(v) = \begin{cases} 0 & \text{if } |\nabla v| \le 1, \\ \\ +\infty & \text{in other case.} \end{cases}$$

In these limit problems we assume that the material is confined in a domain $\Omega$, thus we are looking at models for sandpiles inside a container, see also [28].

Working as in the previous sections we can prove that

**Theorem 8.1.** *Let $\Omega$ be a convex domain in $\mathbb{R}^N$.*

*(1) Let $T > 0$, $u_0 \in L^2(\Omega)$ such that $|u_0(x) - u_0(y)| \le 1$, for $x - y \in \Omega \cap \operatorname{supp}(J)$ and $u_p$ the unique solution of $P_p^{J,\Omega}(u_0)$. Then, if $u_\infty$ is the unique solution to $P_\infty^{J,\Omega}(u_0)$,*

$$\lim_{p\to\infty} \sup_{t\in[0,T]} \|u_p(t,\cdot) - u_\infty(t,\cdot)\|_{L^2(\Omega)} = 0.$$

*(2) Let $p > 1$ be and assume $J(x) \ge J(y)$ if $|x| \le |y|$. Let $T > 0$, $u_0 \in L^p(\Omega)$ and $u_{p,\varepsilon}$ the unique solution of $P_p^{J_\varepsilon,\Omega}(u_0)$. Then, if $v_p$ is the unique solution of $P_p(u_0)$,*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_{p,\varepsilon}(t,\cdot) - v_p(t,\cdot)\|_{L^p(\Omega)} = 0.$$

*(3) Let $T > 0$, $u_0 \in L^2(\Omega) \cap W^{1,\infty}(\Omega)$ such that $|\nabla u_0| \le 1$ and consider $u_{\infty,\varepsilon}$ the unique solution of $P_\infty^{\varepsilon,\Omega}(u_0)$. Then, if $v_\infty$ is the unique solution of $P_\infty^{\Omega}(u_0)$, we have*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_{\infty,\varepsilon}(t,\cdot) - v_\infty(t,\cdot)\|_{L^2(\Omega)} = 0.$$

Part *(2)* was proved in [2], the other statements follows just by considering, as we did before, the Mosco convergence of the associated functionals. We leave the details to the reader.

**Example 6.** In this case, let us also compute an explicit solution to the limit problem $P_\infty^{1,\Omega}(u_0)$ (to simplify we have considered $\varepsilon = 1$ in this example). Let us consider a recipient $\Omega = (0,l)$ with $l$ an integer greater than 1, $u_0 = 0$ and a source given by $f(t,x) = \chi_{[0,1]}(x)$. Then the solution is given by

$$u(t,x) = t\chi_{[0,1]}(x),$$

for times $t \in [0,1]$. For $t \in [1,3]$ we get

$$u(t,x) = \begin{cases} 1 + \dfrac{t-1}{2} & x \in [0,1), \\ \dfrac{t-1}{2} & x \in [1,2), \\ 0 & x \notin [0,2). \end{cases}$$

In general we have, until the recipient is full, for any $k = 1, ..., l$ and for $t \in [t_{k-1}, t_k)$

$$u(t,x) = \begin{cases} k-1+\dfrac{t-t_{k-1}}{k} & x \in [0,1), \\ k-2+\dfrac{t-t_{k-1}}{k} & x \in [1,2) \\ \dots & \\ \dfrac{t-t_{k-1}}{k} & x \in [k-1,k), \\ 0 & x \notin [0,k) \end{cases}$$

Here $t_k = t_{k-1} + k$ is the first time when the solution reaches level $k$, that is $u(t_k, 0) = k$.

For times even greater, $t \geq t_l$, the solution turns out to be

$$u(t,x) = \begin{cases} l+\dfrac{t-t_l}{l} & x \in [0,1), \\ l-1+\dfrac{t-t_l}{l} & x \in [1,2), \\ \dots & \\ 1+\dfrac{t-t_l}{l} & x \in [l-1,l). \end{cases}$$

Hence, when the recipient is full the solution grows with speed $1/l$ uniformly in $(0, l)$.

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