# The behaviour of the $p(x)-$Laplacian eigenvalue problem as $p(x) \to \infty$


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**Abstract**

In this paper we study the behaviour of the solutions to the eigenvalue problem corresponding to the $p(x)-$Laplacian operator

$$\begin{array}{ll} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \Lambda_{p(x)}|u|^{p(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{array}$$

as $p(x) \to \infty$. We consider a sequence of functions $p_n(x)$ that goes to infinity uniformly in $\overline{\Omega}$. Under adequate hypotheses on the sequence $p_n$, namely that the limits

$$\nabla \ln p_n(x) \to \xi(x), \qquad \text{and} \qquad \frac{p_n}{n}(x) \to q(x),$$

exist, we prove that the corresponding eigenvalues $\Lambda_{p_n}$ and eigenfunctions $u_{p_n}$ verify that

$$(\Lambda_{p_n})^{1/n} \to \Lambda_\infty, \qquad u_{p_n} \to u_\infty \text{ uniformly in } \overline{\Omega},$$

where $\Lambda_\infty$, $u_\infty$ is a nontrivial viscosity solution of the following problem

$$\begin{array}{ll} \min\{-\Delta_\infty u_\infty - \log(|\nabla u_\infty|)\langle \xi, \nabla u_\infty\rangle, \, |\nabla u_\infty|^q - \Lambda_\infty u_\infty^q\} = 0, & \text{in } \Omega \\ u_\infty = 0, & \text{on } \partial\Omega. \end{array}$$



**2000 AMS Subject Classification:** 35P30, 35J60, 35J20

**Keywords and phrases:** $\infty$-Laplacian, $p(x)$-laplacian, eigenvalue problems.


## 1 Introduction

In this work we analyze the behaviour of the solutions to the eigenvalue problem corresponding to the $p(x)-$Laplacian operator as $p(x) \to \infty$. More precisely, we consider the following problems

$$\left\{ \begin{array}{ll} -\mathrm{div}(|\nabla u|^{p_n(x)-2}\nabla u) = \Lambda_{p_n}|u|^{p_n(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

with $\Omega \subset \mathbb{R}^N$ being a bounded smooth domain, and the sequence of functions $p_n : \overline{\Omega} \to \mathbb{R}$ such that $p_n \in C(\overline{\Omega})$ and $p_n(x) > 1$, for every $n \geq 1$ and every $x \in \overline{\Omega}$.

For $n$ fixed, solutions to the eigenvalue problem (1.1) have been analyzed in [10]. Our purpose in this work is to study how the solutions to (1.1) behave when we consider a sequence of functions such that $p_n(x) \to \infty$ for every $x \in \overline{\Omega}$, as $n \to \infty$.

To give some motivation for this study, let us recall briefly what happens when $p$ is constant in $\Omega$. In this case, the limit of (1.1) as $p \to \infty$ has been studied in [4], [14], [15], see also the survey [2], and leads naturally to the infinity Laplacian eigenvalue problem

$$\min \left\{ |\nabla u|(x) - \Lambda_\infty u(x), \ -\Delta_\infty u(x) \right\} = 0, \tag{1.2}$$

where the infinity Laplacian, $\Delta_\infty$, is given by,

$$\Delta_\infty u := \left( D^2 u \, \nabla u \right) \cdot \nabla u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i x_j}.$$

In fact, it is proved there that the limit as $p \to \infty$ exists both for the eigenfunctions, $u_p \to u_\infty$ uniformly, and for the eigenvalues $(\Lambda_p)^{1/p} \to \Lambda_\infty$, where the pair $u_\infty$, $\Lambda_\infty$ is a nontrivial solution to (1.2).

Solutions to $-\Delta_\infty u = 0$ (that are called infinity harmonic functions) solve the optimal Lipschitz extension problem (see [1] and the survey paper [2]) and are used in several applications, for example, in optimal transportation, image processing and tug-of-war games (see, e.g., [9], [11], [5], [23], [24] and the references therein). On the other hand, problems related to PDEs involving variable exponents became popular a recently due to applications in elasticity and the modelling of electrorheological fluids. The functional analytical tools needed for the analysis have been extensively developed, see [17] and [8] and also the recent survey [12] and references therein. Although a natural extension of the theory, the problem addressed here is a natural continuation of recent papers. In [21], the authors treat the case of a variable exponent that equals infinity in a subdomain of $\Omega$ and in [19], [22], the limit of $p(x)$-harmonic functions is studied, that is, the limit as $p(x) \to \infty$ of solutions to $\Delta_{p(x)} u = 0$ with $u = g$ on $\partial\Omega$.

Here we will assume that $p_n(x)$ is a sequence of $C^1$ functions in $\Omega$ such that

$$p_n(x) \to +\infty, \quad \text{uniformly in } \Omega; \tag{1.3}$$

$$\nabla \ln p_n(x) \to \xi(x), \quad \text{uniformly in } \Omega, \tag{1.4}$$

$$\frac{p_n}{n}(x) \to q(x), \quad \text{uniformly in } \Omega. \tag{1.5}$$

For the limit functions $\xi$ and $q$ we assume that $\xi \in C(\Omega : \mathbb{R}^N)$ and that $q \in C(\Omega : \mathbb{R})$ is strictly positive.

Under these assumptions we have the following result.

**Theorem 1.1** *For any sequence $p_n(x)$ satisfying (1.3) – (1.5) let $\Lambda_{p_n}$ and $u_{p_n}$ be the corresponding first eigenvalues and eigenfunctions of the problem $-\Delta_{p_n(x)} u_{p_n} = \Lambda_{p_n} |u_{p_n}|^{p_n(x)-2} u_{p_n}$ in $\Omega$ with Dirichlet boundary conditions, $u_{p_n}|_{\partial\Omega} = 0$, normalized by $\int_\Omega \frac{|u|^{p_n(x)}}{p_n(x)}\, dx = 1$. Then, there is a subsequence such that*

$$u_{p_i} \to u_\infty \qquad \text{in } C^\beta(\overline{\Omega}), \quad \text{for some } 0 < \beta < 1,$$

*and*

$$(\Lambda_{p_i})^{1/n_i} \to \Lambda_\infty,$$

*where $u_\infty$ is nontrivial and $u_\infty$, $\Lambda_\infty$ verify, in the viscosity sense,*

$$\begin{cases} \min\{-\Delta_\infty u_\infty - \log(|\nabla u_\infty|)\langle \xi, \nabla u_\infty\rangle,\ |\nabla u_\infty|^q - \Lambda_\infty u_\infty^q\} = 0, & \text{in } \Omega \\ u_\infty = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

**Remark 1.1** *Comparing the limit problem* (1.6) *with* (1.2)*, we note the dependence in $x$ of the sequence $p_n$. In fact, two limits play a role here, $\nabla \ln p_n(x) \to \xi(x)$ and $\frac{p_n}{n}(x) \to q(x)$.*

We now present some examples of possible sequences $p_n(x)$. We are specially interested in understanding (1.4) and (1.5) and hope the examples shed some light on the meaning of this assumption.

1. $p_n(x) = n$; we have $\xi = 0$ and $q = 1$.

2. $p_n(x) = p(x) + n$; we get again $\xi = 0$ and $q = 1$.

3. $p_n(x) = np(x)$; now we get a nontrivial vector field $\xi(x) = \nabla(\ln(p(x)))$ and a nontrivial $q$, $q(x) = p(x)$.

4. $p_n(x) = n^a p(x/n)$ [scaling in $x$]; in this case, we have

$$\nabla(\ln p_n(x)) = \frac{\nabla p}{p}(x/n)\frac{1}{n} \to 0$$

   and so $\xi = 0$. Moreover, we have $q(x) = p(0)$ if and only if $a = 1$.

   This calculations also hold for $p_n(x) = n + p(x/n)$, we have $\xi = 0$ and $q(x) = 1$.

5. $p_n(x) = n^a p(nx)$; we get

$$\nabla(\ln p_n(x)) = \frac{n\nabla p}{p}(nx),$$

   which does not have a limit as $n \to \infty$. The same happens with $p_n(x) = n + p(nx)$, for which

$$\nabla(\ln p_n(x)) = \frac{n\nabla p(nx)}{n + p(nx)},$$

   that does not have a uniform limit (although it is bounded).

6. We can modify the previous example to get a nontrivial limit. Assume that $r = r(\theta)$ is a function of the angular variable and that $0 \notin \Omega$; then consider $p_n(x) = n + r(nx)$ to obtain

$$\nabla(\ln p_n(x)) = \frac{n\nabla r(nx)}{n + r(nx)} \to \nabla r(\theta).$$

   In this case we get $q(x) = 1$.

7. Finally, we can combine examples (3) and (6). Let $p_n(x) = np(x) + r(nx)$, with $q$ and $\Omega$ as in (6). We get

$$\nabla(\ln p_n(x)) = \frac{n\nabla p(x) + n\nabla r(nx)}{np(x) + r(nx)} \to \frac{\nabla p(x) + \nabla r(\theta)}{p(x)}.$$

   In this case $q(x) = p(x)$.

## 2 Preliminaries

We introduce now some notation and preliminary results. See [7], [8], [10], [17] and the survey [12] for more details. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as follows

$$L^{p(x)}(\Omega) = \left\{ u \text{ such that } u : \Omega \to \mathbb{R} \text{ is measureable and } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \right\},$$

and is endowed with the norm

$$|u|_{p(x)} = \inf \left\{ \tau > 0 \text{ such that } \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \le 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is given by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with the norm

$$\|u\| = \inf \left\{ \tau > 0 \text{ such that } \int_\Omega \left| \frac{\nabla u(x)}{\tau} \right|^{p(x)} + \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \le 1 \right\}.$$

Let us denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The following result holds.

**Proposition 2.1** *i) The spaces* $\left(L^{p(x)}(\Omega), |\cdot|_{p(x)}\right)$, $\left(W^{1,p(x)}(\Omega), \|\cdot\|\right)$ *and* $\left(W_0^{1,p(x)}(\Omega), \|\cdot\|\right)$ *are separable, reflexive and uniformly convex Banach spaces.*

*ii) Hölder inequality holds, namely*

$$\int_\Omega |uv|\, dx \le 2|u|_{p(x)}|v|_{q(x)}, \quad \forall u \in L^{p(x)}(\Omega), \forall v \in L^{q(x)}(\Omega),$$

*where* $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

*iii) If* $q \in C(\overline{\Omega})$ *and* $0 < q(x) < p^*(x)$ *for every* $x \in \overline{\Omega}$, *then the imbedding from* $W^{1,p(x)}(\Omega)$ *to* $L^{q(x)}(\Omega)$ *is compact and continuous, where* $p^*(x)$ *is given by*

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & p(x) < N, \\ \infty, & p(x) \ge N. \end{cases}$$

*iv) There exists a constant* $C > 0$ *such that*

$$|u|_{p(x)} \le C|\nabla u|_{p(x)}, \qquad \text{for every } u \in W_0^{1,p(x)}(\Omega).$$

*Therefore,* $|\nabla u|_{p(x)}$ *and* $\|u\|$ *are equivalent norms on* $W_0^{1,p(x)}(\Omega)$.

Let us introduce now some results concerning to problem (1.1) for fixed $n$. Namely, we consider the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \Lambda_{p(x)}|u|^{p(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

**Definition 2.1** *Let $\Lambda_{p(x)} \in \mathbb{R}$ and $u \in W_0^{1,p(x)}(\Omega)$. We say that $(\Lambda_{p(x)}, u)$ is a solution to the eigenvalue problem (2.1) if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx = \Lambda_{p(x)} \int_\Omega |u|^{p(x)-2} uv \, dx, \quad \forall v \in W_0^{1,p(x)}(\Omega).$$

*As usual, we call $\Lambda_{p(x)}$ an eigenvalue of (2.1) and $u$ an eigenfunction corresponding to $\Lambda_{p(x)}$.*

Let us denote $X = W_0^{1,p(x)}(\Omega)$. We define the following functionals $F, G : X \to \mathbb{R}$ by

$$F(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx, \quad G(u) = \int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx,$$

and, for $\alpha > 0$, the $C^1$-submanifold of $X$,

$$M_\alpha = \{u \in X \text{ such that } G(u) = \alpha\}.$$

It is well known that $(\Lambda_{p(x)}, u)$ solves problem (2.1) if and only if $u$ is a critical point of the functional $\widetilde{F} := F_{|M_\alpha} : M_\alpha \to \mathbb{R}$. In order to determine the critical points of this functional let us introduce the following sets

$$\Sigma = \{A \subset X \setminus \{0\} \text{ such that } A \text{ is compact and } A = -A\},$$

$$\Sigma_k = \{A \in \Sigma \text{ such that } \gamma(A) \geq k\},$$

where $\gamma(A)$ denotes the genus of $A$. The values defined by

$$c_{k,\alpha} = \sup_{A\in\Sigma_k \ , A\subset M_\alpha} \inf_{u\in A} F(u), \ k = 1, 2, ...$$

are critical values of $F$ on $M_\alpha$ verifying $c_{1,\alpha} \geq c_{2,\alpha} \geq \cdots \geq c_{k,\alpha} \geq c_{k+1,\alpha} \geq \cdots$ and $c_{k,\alpha} \to 0$ as $k \to \infty$. Then, if $u_k \in M_\alpha$ is a critical point of $F$, its corresponding eigenvalue is given by

$$\Lambda_{p(x),k} = \frac{\displaystyle\int_\Omega |\nabla u_k|^{p(x)} dx}{\displaystyle\int_\Omega |u_k|^{p(x)} dx} \geq \frac{p^-\alpha}{p^+ c_{k,\alpha}},$$

where

$$p^- = \min_{x\in\overline{\Omega}} p(x), \qquad p^+ = \max_{x\in\overline{\Omega}} p(x). \tag{2.2}$$

If we denote $\Lambda = \{\Lambda_{p(x)} \in \mathbb{R} \text{ such that } \Lambda_{p(x)} \text{ is an eigenvalue of (2.1)}\}$, we have that $\Lambda$ is a nonempty infinite set such that $\sup \Lambda = +\infty$. It is also known that in general $\inf \Lambda = 0$, unless the function $p$ is monotone in at least one direction, in which case $\inf \Lambda > 0$, see [10].

# 3 The limit problem as $p_n(x) \to \infty$

Our interest in this section is to analyze the behaviour of the first eigenvalue (and its corresponding eigenfunctions) of problem (1.1) when $p_n(x) \to +\infty$. To this end, we note that from the previous section we have that the first eigenvalue for $p_n(x)$ is given by

$$\Lambda_{p_n} = \frac{\displaystyle\int_\Omega |\nabla u_{p_n}|^{p_n(x)} dx}{\displaystyle\int_\Omega |u_{p_n}|^{p_n(x)} dx}. \tag{3.1}$$

The function $u_{p_n}$ is the critical point for

$$c_{1,1}^n = \sup_{A \in \Sigma_1\ , A \subset M_1} \inf_{u \in A} F(u),$$

where we have fixed the parameter $\alpha = 1$. Note that the definition above is equivalent to

$$c_{1,1}^n = \inf_{u \in B} \int_\omega \frac{|\nabla u|^{p_n(x)}}{p_n(x)}\, dx, \quad \text{with } B = \left\{ u \in X : \int_\Omega \frac{|u|^{p_n(x)}}{p_n(x)}\, dx = 1 \right\}. \tag{3.2}$$

It is known (see [10] for details) that for each $n$ fixed $u_{p_n}(x) > 0$ for every $x \in \Omega$ or $u_{p_n}(x) < 0$ for every $x \in \Omega$. In the sequel we will consider for each $n$ the positive solution

$$u_{p_n}(x) > 0, \text{ for every } x \in \Omega. \tag{3.3}$$

Our purpose is to study the pair $(u_{p_n}, \Lambda_{p_n})$, given by (3.1) and (3.2), as the function $p_n(x)$ goes to infinity as $n \to \infty$. Next, we introduce the following notation: we define

$$p_n^- = \min_{x \in \overline{\Omega}} p_n(x), \qquad p_n^+ = \max_{x \in \overline{\Omega}} p_n(x). \tag{3.4}$$

By (1.5) it is clear that there exist the limits

$$\lim_{n \to \infty} \frac{p_n^-}{n} = q^-, \qquad \lim_{n \to \infty} \frac{p_n^+}{n} = q^+, \tag{3.5}$$

for some $q^-$, $q^+$.

Our next aim is to find an upper bound for $(\Lambda_{p_n})^{1/n}$.

**Lemma 3.1** *Let $\Lambda_{p_n}$ the first eigenvalue of problem* (1.1) *given in* (3.1). *There exists a positive constant $K$, independent of $n$, such that*

$$(\Lambda_{p_n})^{1/n} \leq K. \tag{3.6}$$

**Proof.** We begin with a uniform bound for $(c_{1,1}^n)^{1/n}$. Let us consider the function $u(x) = a\delta(x)$, with $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ and the constant $a > 0$ such that $u \in B$, that is, we chose $a$ verifying

$$\int_\Omega \frac{(a\delta(x))^{p_n(x)}}{p_n(x)}\, dx = 1.$$

Let us show that $a$ is uniformly bounded. Let us denote $\Omega_1 = \{x \in \Omega : \varepsilon \le \delta(x) \le 1\}$ and $\Omega_2 = \{x \in \Omega : \delta(x) > 1\}$. Then, taking into account the definitions (3.4) and (3.5) we have

$$\begin{aligned} 1 \ &\geq \left( \int_{\Omega_1 \cup \Omega_2} \frac{(a\delta(x))^{p_n(x)}}{p_n(x)} \, dx \right)^{1/n} \geq \left( \max\{a^{p_n^+}, a^{p_n^-}\} \mu(\Omega) \frac{\varepsilon^{p_n^+} + 1}{p_n^+} \right)^{1/n} \\ &\geq \max\{a^{q^+ - \varepsilon}, a^{q^- + \varepsilon}\} \left( \frac{1}{p_n^+} \right)^{1/n} \geq \frac{1}{2} \max\{a^{q^+ - \varepsilon}, a^{q^- + \varepsilon}\}, \end{aligned}$$

for $n$ sufficiently large and $\varepsilon > 0$ small, and the uniform bound on $a$ follows.

Using $u$ as test function in

$$c_{1,1}^n = \inf_{u \in B} \int_\omega \frac{|\nabla u|^{p_n(x)}}{p_n(x)} \, dx, \quad \text{with } B = \left\{ u \in X : \int_\Omega \frac{|u|^{p_n(x)}}{p_n(x)} \, dx = 1 \right\}$$

we get that

$$(c_{1,1}^n)^{1/n} \le \left( \int_\Omega \frac{a^{p_n(x)}}{p_n(x)} \, dx \right)^{1/n} \le \left( \frac{\max\{a^{p_n^+}, a^{p_n^-}\}}{p_n^-} \mu(\Omega) \right)^{1/n} \le \max\{a^{q^+ + \varepsilon}, a^{q^- - \varepsilon}\} \left( \frac{\mu(\Omega)}{p_n^-} \right)^{1/n}.$$

Since $\left( \frac{\mu(\Omega)}{p_n^-} \right)^{1/n} \to 1$ as $n \to \infty$, it holds that $(c_{1,1}^n)^{1/n} \le C$ for $n$ large.

We proceed now with the bound on the first eigenvalue. Let $u_{p_n}$ be the point at which $c_{1,1}^n$ reaches its infimum. We observe that

$$\left( \int_\Omega |\nabla u_{p_n}|^{p_n(x)} dx \right)^{1/n} \le \left( p_n^+ \int_\Omega \frac{|\nabla u_{p_n}|^{p_n(x)}}{p_n(x)} \, dx \right)^{1/n} \le 2(c_{1,1}^n)^{1/n} \le 2C. \tag{3.7}$$

On the other hand

$$\begin{aligned} \left( \int_\Omega |\nabla u_{p_n}|^{p_n(x)} dx \right)^{1/n} &= (\Lambda_{p_n})^{1/n} \left( \int_\Omega |u_{p_n}|^{p_n(x)} dx \right)^{1/n} \\ &\geq (\Lambda_{p_n})^{1/n} \left( p_n^- \int_\Omega \frac{|u_{p_n}|^{p_n(x)}}{p_n(x)} dx \right)^{1/n} \geq c(\Lambda_{p_n})^{1/n}, \end{aligned}$$

which together with (3.7) gives the uniform bound on the first eigenvalue (3.6). $\square$

The previous result allows us to consider a subsequence $n_i \to \infty$ such that $(\Lambda_{p_{n_i}})^{1/n_i} \to \Lambda_\infty$ and, as we see in the next lemma, we can also extract a subsequence $u_{p_{n_i}} \to u_\infty$ in $C^\beta(\Omega)$.

**Lemma 3.2** *There exists a subsequence $\{u_{p_{n_i}}\}$ converging to some nontrivial function $u_\infty$ in $C^\beta(\Omega)$, for some $0 < \beta < 1$.*

**Proof.** Let us take $m < n$. Then by (3.7) we get

$$\int_\Omega \left( |\nabla u_{p_n}|^{\frac{p_n(x)}{n}} \right)^m dx \le \left( \int_\Omega |\nabla u_{p_n}|^{p_n(x)} \, dx \right)^{m/n} (\mu(\Omega))^{1-m/n} \le K,$$

thus $|\nabla u_{p_n}|^{\frac{p_n(x)}{n}}$ is uniformly bounded in $L^m(\Omega)$, which implies that $|\nabla u_{p_n}|$ is uniformly bounded in $L^{\frac{mp_n(x)}{n}}(\Omega) \subset L^{m(q^-(x)-\varepsilon)}(\Omega)$, by Hölder inequality (we take $\varepsilon$ such that $q^-(x) - \varepsilon > 1$, $\forall x \in \Omega$). If we take now $m$ such that $m(q^-(x) - \varepsilon) \geq N$, then by the continuous embedding in iii) of Proposition 2.1 we have that $W_0^{1,m(q^-(x)-\varepsilon)}(\Omega) \subset C^\beta(\Omega)$, $0 < \beta < 1$. Therefore, there exists a subsequence $\{u_{p_{n_i}(x)}\}$ such that

$$u_{p_{n_i}(x)} \rightharpoonup u_\infty, \text{ weakly in } W^{1,m(q^-(x)-\varepsilon)}(\Omega) \quad \text{and} \quad u_{p_{n_i}(x)} \to u_\infty, \text{ strongly in } C^\beta(\Omega). \tag{3.8}$$

Note that we have the normalization

$$\left( \int_\Omega \frac{1}{p_n(x)} |u_{p_n}|^{p_n(x)} dx \right)^{1/n} = 1,$$

hence

$$\left( \frac{1}{p_n^-} \right)^{1/n} \left( \int_\Omega |u_{p_n}|^{p_n(x)} dx \right)^{1/n} \geq 1,$$

and then we have that

$$\left( \frac{\mu(\Omega)}{p_n^-} \right)^{1/n} \max\{(\|u_{p_n}\|_\infty)^{p_n^+}, (\|u_{p_n}\|_\infty)^{p_n^-}\}^{1/n} \geq 1.$$

If we pass to the limit as $n \to \infty$ in the previous estimate, taking into account (1.5) and (3.8) we get that

$$\max\{(\|u_\infty\|_\infty)^{q^+}, (\|u_\infty\|_\infty)^{q^-}\} \geq q^+,$$

and thus $u_\infty$ is nontrivial. $\square$

In order to identify the limit problem satisfied by any cluster point $u_\infty$ we introduce the concept of viscosity solutions to problem (1.1). Assuming that $u_{p_n}$ are smooth enough to differentiate (1.1), we get

$$\begin{aligned} &-|\nabla u_{p_n}|^{p_n(x)-2} \left( \Delta u_{p_n} + \log(|\nabla u_{p_n}|) \sum_{i=1}^N \frac{\partial u_{p_n}}{\partial x_i} \frac{\partial p_n(x)}{\partial x_i} \right) \\ &-(p_n(x) - 2)|\nabla u_{p_n}|^{p_n(x)-4} \sum_{i,j=1}^N \frac{\partial u_{p_n}}{\partial x_i} \frac{\partial u_{p_n}}{\partial x_j} \frac{\partial^2 u_{p_n}}{\partial x_i \partial x_j} = \Lambda_{p_n} |u_{p_n}|^{p_n(x)-2} u_{p_n}. \end{aligned} \tag{3.9}$$

We recall that the last operator involving the second derivatives is denoted as $\Delta_\infty$, that is

$$\Delta_\infty u = \sum_{i,j=1}^N \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Equation (3.9) is nonlinear but elliptic (degenerate), thus it makes sense to consider viscosity subsolutions and supersolutions of it. Let $y \in \mathbb{R}$, $z$, $\theta \in \mathbb{R}^N$, and $S$ a real symmetric matrix. We define the following continuous function

$$\begin{aligned} H_{p_n(x)}(y, z, \theta, S) &= -|z|^{p_n(x)-2} \Big( \operatorname{trace}(S) + \log(|z|) \langle z, \theta \rangle \Big) \\ &\quad -(p_n(x) - 2)|z|^{p_n(x)-4} \langle S \cdot z, z \rangle - \Lambda_{p_n} |y|^{p_n(x)-2} y. \end{aligned} \tag{3.10}$$

To define the notion of viscosity solution we are interested in viscosity super and subsolutions of the partial differential equation

$$\begin{cases} H_{p_n(x)}(u_{p_n}, \nabla u_{p_n}, \nabla p_n, D^2 u_{p_n}) = 0, & \text{in } \Omega, \\ u_{p_n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

**Definition 3.1** *An upper semicontinuous function $u$ defined in $\Omega$ is a viscosity subsolution of* (3.11) *if, $u|_{\partial\Omega} \leq 0$ and, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that*

*i)* $u(x_0) = \phi(x_0)$,

*ii)* $u(x) < \phi(x)$, *if* $x \neq x_0$,

*then*

$$H_{p_n(x)}(\phi(x_0), \nabla\phi(x_0), \nabla p_n(x_0), D^2\phi(x_0)) \leq 0.$$

**Definition 3.2** *A lower semicontinuous function $u$ defined in $\Omega$ is a viscosity supersolution of* (3.11) *if, $u|_{\partial\Omega} \geq 0$ and, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that*

*i)* $u(x_0) = \phi(x_0)$,

*ii)* $u(x) > \phi(x)$, *if* $x \neq x_0$,

*then*

$$H_{p_n(x)}(\phi(x_0), \nabla\phi(x_0), \nabla p_n(x_0), D^2\phi(x_0)) \geq 0.$$

We observe that in both of the above definitions the second condition is required just in a neigbourhood of $x_0$ and the strict inequality can be relaxed. We refer to [6] for more details about general theory of viscosity solutions, and [13], [16] for viscosity solutions related to the $\infty-$Laplacian and the $p-$Laplacian operators. The following result can be shown as in [15], we include the proof for convenience of the reader.

**Lemma 3.3** *A continuous weak solution to equation* (1.1) *is a viscosity solution to* (3.11).

**Proof.** The proof is analogous to the one of Proposition 2.4 in [21]. We reproduce it here for the sake of completeness and readability.

We omit the subscript $n$ in this proof. Let us show that if $u$ is continuous weak supersolution then, it is a viscosity supersolution. Let $x_0 \in \Omega$ and a let $\phi$ be a test function such that $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict minimum at $x_0$. We want to show that

$$\begin{aligned} -\Delta_{p(x_0)}\phi(x_0) &= -|\nabla\phi(x_0)|^{p(x_0)-2}\Delta\phi(x_0) \\ &\quad -(p(x_0)-2)|\nabla\phi(x_0)|^{p(x_0)-4}\Delta_\infty\phi(x_0) \\ &\quad -|\nabla\phi(x_0)|^{p(x_0)-2}\ln(|\nabla\phi|)(x_0)\,\langle\nabla\phi(x_0), \nabla p(x_0)\rangle \\ &\geq \Lambda_{p(x)}|\phi|^{p(x)-2}\phi(x_0). \end{aligned}$$

Assume, *ad contrarium*, that this is not the case; then there exists a radius $r > 0$ such that $B(x_0, r) \subset \Omega$ and

$$\begin{aligned} -\Delta_{p(x)}\phi(x) &= -|\nabla\phi(x)|^{p(x)-2}\Delta\phi(x) \\ &\quad -(p(x)-2)|\nabla\phi(x)|^{p(x)-4}\Delta_\infty\phi(x) \\ &\quad -|\nabla\phi(x)|^{p(x)-2}\ln(|\nabla\phi|)(x)\langle\nabla\phi(x), \nabla p(x)\rangle \\ &< \Lambda_{p(x)}|\phi|^{p(x)-2}\phi(x), \end{aligned}$$

for every $x \in B(x_0, r)$. Set

$$m = \inf_{|x-x_0|=r}(u-\phi)(x)$$

and let $\Phi(x) = \phi(x) + m/2$.

This function $\Phi$ verifies $\Phi(x_0) > u(x_0)$, $\Phi < u$ on $\partial B(x_0, r)$ and

$$-\Delta_{p(x)}\Phi = -\mathrm{div}(|\nabla\Phi|^{p(x)-2}\nabla\Phi) < \Lambda_{p(x)}|\phi|^{p(x)-2}\phi, \qquad \text{in } \ B(x_0, r). \tag{3.12}$$

Multiplying (3.12) by $(\Phi - u)^+$, which vanishes on the boundary of $B(x_0, r)$, we get

$$\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla(\Phi-u)\,dx < \int_{B(x_0,r)\cap\{\Phi>u\}} \Lambda_{p(x)}|\phi|^{p(x)-2}\phi(\Phi-u)\,dx.$$

On the other hand, taking $(\Phi - u)^+$, extended by zero outside $B(x_0, r)$, as test function in the weak formulation of the eigenvalue problem, we obtain

$$\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla(\Phi-u)\,dx = \int_{B(x_0,r)\cap\{\Phi>u\}} \Lambda_{p(x)}|u|^{p(x)-2}u(\Phi-u)\,dx.$$

Upon subtraction and using a well know inequality, we conclude

$$\begin{aligned} 0 &> \int_{B(x_0,r)\cap\{\Phi>u\}} \Big(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(\Phi-u)\,dx \\ &\geq c\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla\Phi - \nabla u|^{p(x)}\,dx, \end{aligned}$$

a contradiction.

This proves that $u$ is a viscosity supersolution. The proof that $u$ is a viscosity subsolution runs as above and we omit the details. □

We have all the ingredients to compute the limit of the equation

$$H_{p_n(x)}(u_{p_n}, \nabla u_{p_n}, \nabla p_n, D^2u_{p_n}) = 0$$

as $p_n(x) \to \infty$ in the viscosity sense, that is to identify the limit equation verified by any $u_\infty$ as in (3.8).

In the sequel we assume that we have a subsequence $p_{n_i}(x) \to \infty$ with the assumptions stated in the introduction such that

$$\lim_{i\to\infty} u_{p_{n_i}} = u_\infty$$

uniformly in $\Omega$ and $(\Lambda_{p_{n_i}})^{1/n_i} \to \Lambda_\infty$. We denote as $u_{p_n}$ and $\Lambda_{p_n}$ such subsequences for readable reasons.

We define for $y \in \mathbb{R}$ $z, \theta \in \mathbb{R}^N$ and $S$ a symmetric real matrix,

$$H_\infty(y, z, q, \theta, S) = \min\{-\langle S \cdot z, z\rangle - \log(|z|)\langle \theta, z\rangle, |z|^q - \Lambda_\infty y^q\}. \tag{3.13}$$

Note that $H_\infty(u, \nabla u, q, \xi, D^2u) = 0$ is the equation that appears in (1.6).

**Theorem 3.1** *A function $u_\infty$ obtained as a limit of a subsequence of $\{u_{p_n}\}$ is a viscosity solution of the equation*

$$H_\infty(u, \nabla u, q, \xi, D^2u) = 0,$$

*with $H_\infty$ defined in* (3.13)*, and $\xi$ and $q$ given by* (1.4) *and* (1.5) *respectively.*

**Proof.** Consider $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and $u_\infty(x) > \phi(x)$ for every $x \in B(x_0, R)$, $x \neq x_0$, with $R > 0$ fixed and verifying that $B(x_0, 2R) \subset \Omega$. For $0 < r < R$ it holds that

$$\inf\{u_\infty - \phi \text{ in } B(x_0, R) \setminus B(x_0, r)\} > 0.$$

Since $u_{p_n} \to u_\infty$ uniformly in $\overline{B(x_0, R)}$, for $n \geq n_0$ the function $u_{p_n} - \phi$ attains its minimum value in $B(x_0, r)$. Let us denote by $x_n \in B(x_0, r)$ such a point. By letting $r \to 0$ we get a subsequence such that $x_{n_r} \to x_0$ as $n_r \to \infty$. To simplify we denote such subindexes by $x_n$ and $u_{p_n}$.

On the other hand we have that $u_{p_n}$ is a viscosity supersolution of (3.11). Then,

$$\begin{aligned} &-|\nabla\phi(x_n)|^{p_n(x_n)-2}\Big(\Delta\phi(x_n) + \log(|\nabla\phi(x_n)|)\langle\nabla p_n(x_n), \nabla\phi(x_n)\rangle\Big) \\ &-(p_n(x_n) - 2)|\nabla\phi(x_n)|^{p_n(x_n)-4}\langle\nabla\phi(x_n)D^2\phi(x_n), \nabla\phi(x_n)^t\rangle \\ &\geq \Lambda_{p_n}|\phi(x_n)|^{p_n(x_n)-2}\phi(x_n). \end{aligned} \tag{3.14}$$

We observe that, at the point $x_n$

$$\Lambda_{p_n}|\phi(x_n)|^{p_n(x_n)-2}\phi(x_n) = \Lambda_{p_n}|u_{p_n}(x_n)|^{p_n(x_n)-2}u_{p_n}(x_n) > 0,$$

if we assume that $u_\infty(x_0) > 0$. In consequence, by (3.14) we deduce that $|\nabla\phi(x_n)| > 0$ and we can multiply this inequality by $(p_n(x_n) - 2)^{-1}|\nabla\phi(x_n)|^{-(p_n(x_n)-4)}$, to obtain that

$$\begin{aligned} &\frac{-|\nabla\phi(x_n)|^2\Big(\Delta\phi(x_n) - \log(|\nabla\phi(x_n)|)\langle\nabla p_n(x_n), \nabla\phi(x_n)\rangle\Big)}{p_n(x_n) - 2} - \langle\nabla\phi(x_n)D^2\phi(x_n), \nabla\phi(x_n)^t\rangle \\ &\geq \left(\frac{\Lambda_{p_n}^{1/n}\phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}}\right)^n \frac{|\nabla\phi(x_n)|^4\phi(x_n)}{(p_n(x_n) - 2)|\phi(x_n)|^2}. \end{aligned}$$

If we take limit as $n \to \infty$ in the previous inequality, taking into account (1.4) we have that

$$\begin{aligned} &-\Delta_\infty\phi(x_0) - \log(|\nabla\phi(x_0)|)\langle\xi(x_0), \nabla\phi(x_0)\rangle \\ &\geq \lim_{n\to\infty}\left[\left(\frac{\Lambda_{p_n}^{1/n}\phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}}\right)^n \frac{|\nabla\phi(x_n)|^4\phi(x_n)}{(p_n(x_n) - 2)|\phi(x_n)|^2}\right]. \end{aligned} \tag{3.15}$$

For any $\phi$,

$$\lim_{n\to\infty} \frac{|\nabla\phi(x_n)|^4\phi(x_n)}{(p_n(x_n)-2)|\phi(x_n)|^2} = 0.$$

By (1.5) it also holds that

$$\lim_{n\to\infty} \frac{\Lambda_{p_n}^{1/n}\phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}} \to \frac{\Lambda_\infty\phi(x_0)^{q(x_0)}}{|\nabla\phi(x_0)|^{q(x_0)}}. \tag{3.16}$$

Now, we claim that the previous limit is smaller than one, namely,

$$|\nabla\phi(x_0)|^{q(x_0)} - \Lambda_\infty\phi(x_0)^{q(x_0)} \geq 0. \tag{3.17}$$

To prove this claim we argue by contradiction. Assume that

$$\frac{\Lambda_\infty\phi(x_0)^{q(x_0)}}{|\nabla\phi(x_0)|^{q(x_0)}} > 1.$$

Then, from (3.16) we conclude that there exists $\theta > 1$ such that

$$\frac{\Lambda_{p_n}^{1/n}\phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}} \geq \theta > 1.$$

for $n$ large. Therefore,

$$\lim_{n\to\infty}\left[\left(\frac{\Lambda_{p_n}^{1/n}\phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}}\right)^n \frac{|\nabla\phi(x_n)|^4\phi(x_n)}{(p_n(x_n)-2)|\phi(x_n)|^2}\right] \geq \lim_{n\to\infty}\frac{\theta^n}{n}\left[\frac{|\nabla\phi(x_n)|^4\phi(x_n)}{\frac{(p_n(x_n)-2)}{n}|\phi(x_n)|^2}\right] = \infty.$$

Hence the limit in (3.15) diverges, but the left hand side is bounded, so we reach a contradiction.

Now, if $u_\infty(x_0) = 0$ and $\nabla\phi(x_0) \neq 0$ we can use the same arguments to conclude that (3.17) holds, and if $\nabla\phi(x_0) = 0$, then (3.17) holds trivially.

On the other hand, it always holds that

$$-\Delta_\infty\phi(x_0) - \log(|\nabla\phi(x_0)|)\langle\xi(x_0), \nabla\phi(x_0)\rangle \geq 0. \tag{3.18}$$

Thus, we can combine the two equations (3.17) and (3.18) into the following

$$\min\{-\Delta_\infty\phi(x_0) - \log(|\nabla\phi(x_0)|)\langle\xi(x_0), \nabla\phi(x_0)\rangle, |\nabla\phi(x_0)|^{q(x_0)} - \Lambda_\infty\phi(x_0)^{q(x_0)}\} \geq 0. \tag{3.19}$$

To complete the proof it just remains to see that $u_\infty$ is a viscosity subsolution. Let us consider a point $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and $u_\infty(x) < \phi(x)$ for every $x$ in a neighbourhood of $x_0$. We want to show that

$$H_\infty(\phi(x_0), \nabla\phi(x_0), q(x_0), \xi(x_0), D^2\phi(x_0)) \leq 0.$$

We first observe that if $\nabla\phi(x_0) = 0$ the previous inequality trivially holds. Hence, let us assume that $\nabla\phi(x_0) \neq 0$. Now, we argue as follows: assuming that

$$|\nabla\phi(x_0)|^{q(x_0)} - \Lambda_\infty\phi(x_0)^{q(x_0)} > 0, \tag{3.20}$$

we will show that

$$-\Delta_\infty \phi(x_0) - \log(|\nabla\phi(x_0)|)\langle \xi(x_0), \nabla\phi(x_0)\rangle \le 0. \tag{3.21}$$

As before, we get a sequence of points $x_n \to x_0$ such that

$$\frac{-|\nabla\phi(x_n)|^2\Delta\phi(x_n) - \log(|\nabla\phi(x_n)|)\langle \nabla p_n(x_n), \nabla\phi(x_n)\rangle\Big)}{p_n(x_n)-2} - \langle \nabla\phi(x_n)D^2\phi(x_n), \nabla\phi(x_n)^t\rangle$$

$$\le \left(\frac{\Lambda_{p_n}^{1/n}\phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}}\right)^n \frac{|\nabla\phi(x_n)|^4\phi(x_n)}{(p_n(x_n)-2)|\phi(x_n)|^2}.$$

Taking limit as $n \to \infty$ in the above inequality we get an equation similar to (3.15), namely

$$-\Delta_\infty\phi(x_0) - \log(|\nabla\phi(x_0)|)\langle \xi(x_0), \nabla\phi(x_0)\rangle$$

$$\ge \lim_{n\to\infty}\left[\left(\frac{\Lambda_{p_n}^{1/n}\phi(x_n)^{\frac{p_n}{n}(x_n)}}{|\nabla\phi(x_n)|^{\frac{p_n}{n}(x_n)}}\right)^n \frac{|\nabla\phi(x_n)|^4\phi(x_n)}{(p_n(x_n)-2)|\phi(x_n)|^2}\right].$$

Now, we observe that the limit above is equal to zero, since we are assuming (3.20). Thus (3.21) holds and the proof is complete. □

**Acknowledgements.** The first author is supported by MTM2008-06326-C02-02. The second author supported by UBA X066 and CONICET (Argentina).

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