THE LIMIT AS $p(x) \to \infty$ OF SOLUTIONS TO THE INHOMOGENEOUS DIRICHLET PROBLEM OF THE $p(x)-$LAPLACIAN.

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Abstract. In this work we study the behaviour of the solutions to the following Dirichlet problem related to the $p(x)-$Laplacian operator

$$\begin{array}{ll} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{array}$$

as $p(x) \to \infty$, for some suitable functions f. We consider a sequence of functions $p_n(x)$ that goes to infinity uniformly in $\overline{\Omega}$. Under adequate hypotheses on the sequence p_n, basically, that the following two limits exist,

$$\lim_{n\to\infty} \nabla \ln p_n(x) = \xi(x), \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{\max_{x\in\overline{\Omega}} p_n}{\min_{x\in\overline{\Omega}} p_n} \le k, \quad \text{for some } \ k>0,$$

we prove that $u_{p_n} \to u_\infty$ uniformly in $\overline{\Omega}$. In addition, we find that u_∞ solves a certain PDE problem (that depends on f) in viscosity sense. In particular, when $f \equiv 1$ in Ω we get $u_\infty(x) = \mathrm{dist}(x, \partial\Omega)$ and it turns out that the limit equation is $|\nabla u| = 1$.

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1. Introduction

In this work we analyze the behaviour of the solutions to the inhomogeneous Dirichlet problem involving the $p(x)-$Laplacian operator as $p(x) \to \infty$. More precisely, we consider

$$(1.1) \qquad \left\{ \begin{array}{ll} -\mathrm{div}(|\nabla u|^{p_n(x)-2}\nabla u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{array} \right.$$

with $\Omega \subset \mathbb{R}^N$ being a bounded smooth domain, and a sequence of functions $p_n : \overline{\Omega} \to \mathbb{R}$ such that $p_n \in C(\overline{\Omega})$ and $p_n(x) > 1$, for every $n \ge 1$ and every $x \in \overline{\Omega}$. For n fixed, existence of solutions to the previous problem (1.1) is analyzed in [11]. In this work we are interested in the behaviour of the solutions to (1.1) when we consider a sequence of functions such that $p_n(x) \to \infty$ for every $x \in \overline{\Omega}$, as $n \to \infty$. As right hand side we will take a fixed function, f.

Let us give some motivation for this study. When p is constant in Ω and $f \equiv 0$ (in this case we need to consider a nontrivial boundary datum $u|_{\partial\Omega} = g$ in order to obtain nontrivial solutions) it is known that solutions to the $p-$Laplacian converge to a solution to $\Delta_\infty u_\infty = 0$, where the infinity Laplacian, Δ_∞, is given by, $\Delta_\infty u := \big(D^2 u\, \nabla u\big) \cdot \nabla u = \sum_{i,j=1}^N \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i x_j}$. This operator appears naturally when one considers absolutely minimizing Lipschitz extensions of a boundary function g; see the survey [2]. A fundamental result contained in [15] establishes that the Dirichlet problem for Δ_∞ is well posed in the viscosity sense. Solutions to $-\Delta_\infty u = 0$ (known as infinity harmonic functions) are also used in several applications, for instance, in optimal transportation and image processing (see, e.g., [10, 13, 4, 24, 25] and the references therein). Also limits of the eigenvalue problem related to the p-laplacian have been exhaustively studied, see [6, 16, 17, 18]. When $f > 0$ in Ω the limit of (1.1) as $p \to \infty$ has been analyzed in [3], and as limit of the solutions u_p it gives the function $u_\infty(x) = \operatorname{dist}(x, \partial\Omega)$ that is a solution to the Eikonal equation $|\nabla u_\infty| = 1$.

On the other hand, problems related to PDEs involving variable exponents have deserved a great deal of attention in recent years. Its interest is widely justified with many physical examples, such as elasticity and electrorheological fluids. Consequently, there has been an extensive development of the functional analytical tools, needed for the analysis of such problems. See [9, 20] and also the recent survey [14] and references therein.

Although a natural extension of the theory, the problem addressed here is a natural continuation of recent papers. In [22], the authors treat the case of a variable exponent that equals infinity in a subdomain of Ω and in [21], [23], the limit of $p(x)$-harmonic functions is studied, that is, the limit as $p(x) \to \infty$ of solutions to $\Delta_{p(x)} u = 0$ with $u = g$ on $\partial\Omega$. In addition, in [26] the authors deal with the limit of the eigenvalue problem.

Now, let us state our assumptions on the sequence p_n. We will assume that $p_n(x)$ is a sequence of C^1 functions in $\overline{\Omega}$ such that

$$p_n(x) \to +\infty, \quad \text{uniformly in } \Omega; \tag{1.2}$$

$$\limsup_{n\to\infty} \frac{p_n^+}{p_n^-} \le k, ; \tag{1.3}$$

where

$$p_n^- = \min_{x\in\overline{\Omega}} p_n(x), \qquad p_n^+ = \max_{x\in\overline{\Omega}} p_n(x), \tag{1.4}$$

and

$$\nabla \ln p_n(x) \to \xi(x), \quad \text{uniformly in } \Omega, \tag{1.5}$$

where $\xi \in C(\Omega : \mathbb{R}^N)$.

Let us now present some examples of possible sequences $p_n(x)$. We are specially interested in understanding the hypothesis (1.3) and (1.5).

(1) $p_n(x) = n$; we have $\xi = 0$ and $k = 1$.
(2) $p_n(x) = p(x) + n$; we get $\xi = 0$ and $k = 1$.
(3) $p_n(x) = np(x)$; now we get a nontrivial vector field $\xi(x) = \nabla(\ln(p(x)))$ and $k = \frac{\max_{x\in\overline{\Omega}} p}{\min_{x\in\overline{\Omega}} p}$.
(4) $p_n(x) = n^a p(x/n)$ [scaling in x]; in this case, we have

$$\nabla(\ln p_n(x)) = \frac{\nabla p}{p}(x/n)\frac{1}{n} \to 0$$

and so $\xi = 0$. Moreover, we have also $k = 1$. The conclusion also holds for $p_n(x) = n^a + p(x/n)$, we have $\xi = 0$ and $k = 1$.
(5) $p_n(x) = n^a p(nx)$; we get

$$\nabla(\ln p_n(x)) = \frac{n\nabla p}{p}(nx),$$

which does not have a limit as $n \to \infty$. The same happens with $p_n(x) = n + p(nx)$, for which

$$\nabla(\ln p_n(x)) = \frac{n\nabla p(nx)}{n + p(nx)},$$

that does not have a uniform limit (although it is bounded).

(6) We can modify the previous example to get a nontrivial limit. Assume that $r = r(\theta)$ is a function of the angular variable and that $0 \notin \Omega$; then consider $p_n(x) = n + r(nx)$ to obtain

$$\nabla(\ln p_n(x)) = \frac{n\nabla r(nx)}{n + r(nx)} \to \nabla r(\theta).$$

In this case we get $k = 1$.

(7) Finally, we can combine examples (3) and (6). Let $p_n(x) = np(x) + r(nx)$, with Ω as in (6). We get

$$\nabla(\ln p_n(x)) = \frac{n\nabla p(x) + n\nabla r(nx)}{np(x) + r(nx)} \to \frac{\nabla p(x) + \nabla r(\theta)}{p(x)}.$$

In this case $k = \frac{\max_{x\in\overline{\Omega}} p}{\min_{x\in\overline{\Omega}} p}$.

Our first result reads as follows:

Theorem 1.1. *There exists a subsequence $\{u_{p_{n_i}}\}$ of solutions that converge to some nontrivial function u_∞ in $C^\beta(\Omega)$, for some $0 < \beta < 1$. Moreover, the limit u_∞ verifies*

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \le 1, \tag{1.6}$$

and is a maximizer of the following problem

$$\max_K \int_\Omega fv\,dx, \qquad K = \{v \in W_0^{1,\infty}(\Omega), |\nabla v| \le 1\}. \tag{1.7}$$

Concerning the equation verified by the limit, we have the following result. For $z, \theta \in \mathbb{R}^N$ and S a symmetric real matrix let

$$H_\infty(z, \theta, S) = -\langle S \cdot z, z\rangle - |z|^2 \log(|z|)\langle \theta, z\rangle. \tag{1.8}$$

Theorem 1.2. *A function u_∞ obtained as a uniform limit of a subsequence of $\{u_{p_n}\}$ verifies in the viscosity sense*

$$|\nabla u_\infty| \le 1, \qquad -|\nabla u_\infty| \ge -1. \tag{1.9}$$

Moreover, it verifies the following

$$\begin{aligned}
&H_\infty(\nabla u_\infty, \xi, D^2 u_\infty) = 0, && \text{in } \Omega \setminus \mathit{supp}\, f,\\
&|\nabla u_\infty| = 1, && \text{in } \{f > 0\}^\circ,\\
&-|\nabla u_\infty| = -1, && \text{in } \{f < 0\}^\circ,\\
&H_\infty(\nabla u_\infty, \xi, D^2 u_\infty) \ge 0, && \text{in } \Omega \cap \partial\{f > 0\} \setminus \partial\{f < 0\},\\
&H_\infty(\nabla u_\infty, \xi, D^2 u_\infty) \le 0, && \text{in } \Omega \cap \partial\{f < 0\} \setminus \partial\{f > 0\},
\end{aligned}$$

with H_∞ defined in (1.8)*, and ξ given by* (1.5)*.*

The rest of the paper is organized as follows. In the next Section we introduce some notation and preliminary results. Section 3 is devoted to the study of Problem (1.1), we find the convergence as $p_n(x) \to \infty$ to some function and determine the equation satisfied by this limit. We also give some explicit examples in special cases.

2. Preliminaries

First of all, let us give some brief introduction to variable exponent Sobolev and Lebesgue spaces, and some of their main properties, that we will use in the sequel. See [8], [9], [12], [20] and the survey [14] for more details. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as follows

$$L^{p(x)}(\Omega) = \left\{ u \ \text{ such that } u : \Omega \to \mathbb{R} \text{ is measureable and } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \right\},$$

and is endowed with the norm

$$|u|_{p(x)} = \inf \left\{ \tau > 0 \ \text{ such that } \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \le 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is given by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \ \text{ such that } |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with the norm

$$\|u\| = \inf \left\{ \tau > 0 \ \text{ such that } \int_\Omega \left| \frac{\nabla u(x)}{\tau} \right|^{p(x)} + \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \le 1 \right\}.$$

Let us denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The following result holds.

Proposition 2.1.

i) *The spaces* $\left(L^{p(x)}(\Omega), |\cdot|_{p(x)}\right)$, $\left(W^{1,p(x)}(\Omega), \ \|\cdot\|\right)$ *and* $\left(W_0^{1,p(x)}(\Omega), \ \|\cdot\|\right)$ *are separable, reflexive and uniformly convex Banach spaces.*

ii) *Hölder's inequality holds, namely*

$$\int_\Omega |uv|\, dx \le 2|u|_{p(x)}|v|_{q(x)}, \quad \forall u \in L^{p(x)}(\Omega), \forall v \in L^{q(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

iii) *If* $q \in C(\overline{\Omega})$ *and* $0 < q(x) < p^*(x)$ *for every* $x \in \overline{\Omega}$, *then the imbedding from* $W^{1,p(x)}(\Omega)$ *to* $L^{q(x)}(\Omega)$ *is compact and continuous, where* $p^*(x)$ *is given by*

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)}, & p(x) < N, \\ \infty, & p(x) > N. \end{cases}$$

iv) *There exists a constant* $C > 0$ *such that*

$$|u|_{p(x)} \le C|\nabla u|_{p(x)}, \qquad \textit{for every } u \in W_0^{1,p(x)}(\Omega).$$

Therefore, $|\nabla u|_{p(x)}$ *and* $\|u\|$ *are equivalent norms on* $W_0^{1,p(x)}(\Omega)$.

Let us introduce now some results concerning to problem (1.1) for fixed n, see [11] for details. Namely, we consider the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Definition 2.1. *We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to problem* (2.1) *if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx = \int_\Omega f v\, dx, \quad \forall v \in W_0^{1,p(x)}(\Omega).$$

Let us denote $X = W_0^{1,p(x)}(\Omega)$ and define the operator

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx.$$

It is well known that $J \in C^1(X, \mathbb{R})$ and that the $p(x)-$Laplacian operator is the derivative in the weak sense of J, see [5]. Let us denote $J' := L : X \to X^*$ that is given by

$$(L(u), v) = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx.$$

In [11] it is shown that L is a continuous, bounded and strictly monotone operator. Moreover, L is a homeomorphism. With those properties the authors of [11] obtain the following existence result.

Theorem 2.1. *If $f \in L^{\alpha(x)}(\Omega)$, where $\alpha \in C(\overline{\Omega})$, $\alpha > 1$ in $\overline{\Omega}$, satisfies $\frac{1}{p(x)} + \frac{1}{\alpha(x)} = 1$, then* (2.1) *has a unique weak solution. Moreover, the solution minimizes the following functional*

$$\inf_{v\in X, v\neq 0} \int_\Omega \frac{1}{p(x)} |\nabla v|^{p(x)} dx - \int_\Omega f v\, dx. \tag{2.2}$$

As we mentioned in the introduction, our purpose is to show that if we take limit in (1.1) as the function $p_n(x)$ goes to infinity the solutions converge uniformly in $C(\overline{\Omega})$ to some function u_∞. Thus, in order to identify the limit problem satisfied by any cluster point u_∞ we introduce the concept of viscosity solutions to problem (1.1). Formally, if we expand the derivatives of u_{p_n} in (1.1), we get

$$\begin{aligned} &-|\nabla u_{p_n}|^{p_n(x)-2}\left(\Delta u_{p_n} + \log(|\nabla u_{p_n}|)\sum_{i=1}^N \frac{\partial u_{p_n}}{\partial x_i}\frac{\partial p_n(x)}{\partial x_i}\right) \\ &\qquad -(p_n(x)-2)|\nabla u_{p_n}|^{p_n(x)-4}\sum_{i,j=1}^N \frac{\partial u_{p_n}}{\partial x_i}\frac{\partial u_{p_n}}{\partial x_j}\frac{\partial^2 u_{p_n}}{\partial x_i \partial x_j} = f(x). \end{aligned} \tag{2.3}$$

Equation (2.3) is nonlinear but elliptic (degenerate), thus it makes sense to consider viscosity subsolutions and supersolutions of it. Let $y \in \mathbb{R}$, z, $\theta \in \mathbb{R}^N$, and S a real symmetric matrix. Let us consider the following continuous function

$$\begin{aligned} H_{p_n(x)}(y, z, \theta, S) = \quad & -|z|^{p_n(x)-2}\Big(\mathrm{trace}(S) + \log(|z|)\langle z, \theta\rangle\Big) \\ & -(p_n(x)-2)|z|^{p_n(x)-4}\langle S\cdot z, z\rangle - f(x). \end{aligned} \tag{2.4}$$

Let us state now the definition of viscosity super and subsolutions of this partial differential equation. We refer to [7] for the notion of viscosity solutions.

$$\begin{cases} H_{p_n(x)}(u_{p_n}, \nabla u_{p_n}, \nabla p_n, D^2 u_{p_n}) = 0, & \text{in } \Omega, \\ u_{p_n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Definition 2.2. *An upper semicontinuous function u defined in Ω is a viscosity subsolution of* (2.5) *if, $u|_{\partial\Omega} \le 0$ and, whenever $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ are such that $u(x_0) = \psi(x_0)$ and $u(x) < \psi(x)$, if $x \neq x_0$, then*

$$H_{p_n(x)}(\phi(x_0), \nabla\psi(x_0), \nabla p_n(x_0), D^2\psi(x_0)) \le 0.$$

Definition 2.3. *A lower semicontinuous function u defined in Ω is a viscosity supersolution of* (2.5) *if, $u|_{\partial\Omega} \ge 0$ and, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $u(x_0) = \phi(x_0)$ and $u(x) > \phi(x)$, if $x \neq x_0$, then*

$$H_{p_n(x)}(\phi(x_0), \nabla\phi(x_0), \nabla p_n(x_0), D^2\phi(x_0)) \ge 0.$$

In the sequel we will keep the notation used in the above definitions. That is, by ϕ we will denote the test functions touching from bellow the graph of u_∞ and by ψ the test functions touching the graph of u_∞ from above.

Note that in both of the above definitions the strict inequality can be relaxed, since the second condition is required just in a neigbourhood of x_0. We refer to [7] for more details about general theory of viscosity solutions, and [16], [19] for viscosity solutions related to the $\infty-$Laplacian and the $p-$Laplacian operators. The following result can be shown as in [18], see also [22], but we include the proof for completeness.

Lemma 2.1. *A continuous weak solution to* (1.1) *is a viscosity solution to* (2.5)*.*

Proof. We omit the subscript n along this proof. Let us show that if u is a continuous weak supersolution then, it is a viscosity supersolution. Let $x_0 \in \Omega$ and a let ϕ be a test function such that $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict minimum at x_0. We want to show that

$$\begin{aligned} -\Delta_{p(x_0)}\phi(x_0) &= -|\nabla\phi(x_0)|^{p(x_0)-2}\Delta\phi(x_0) - (p(x_0)-2)|\nabla\phi(x_0)|^{p(x_0)-4}\Delta_\infty\phi(x_0) \\ &\quad -|\nabla\phi(x_0)|^{p(x_0)-2}\ln(|\nabla\phi|)(x_0)\,\langle\nabla\phi(x_0), \nabla p(x_0)\rangle \\ &\ge f(x_0). \end{aligned}$$

Assume, *ad contrarium*, that this is not the case; then there exists a radius $r > 0$ such that $B(x_0, r) \subset \Omega$ and

$$\begin{aligned} -\Delta_{p(x)}\phi(x) &= -|\nabla\phi(x)|^{p(x)-2}\Delta\phi(x) - (p(x)-2)|\nabla\phi(x)|^{p(x)-4}\Delta_\infty\phi(x) \\ &\quad -|\nabla\phi(x)|^{p(x)-2}\ln(|\nabla\phi|)(x)\langle\nabla\phi(x), \nabla p(x)\rangle \\ &< f(x), \end{aligned}$$

for every $x \in B(x_0, r)$. Set

$$m = \inf_{|x-x_0|=r} (u - \phi)(x)$$

and let $\Phi(x) = \phi(x) + m/2$. This function Φ verifies $\Phi(x_0) > u(x_0)$, $\Phi < u$ on $\partial B(x_0, r)$ and

$$-\Delta_{p(x)}\Phi = -\mathrm{div}(|\nabla\Phi|^{p(x)-2}\nabla\Phi) < f(x), \qquad \text{in } B(x_0, r). \tag{2.6}$$

Multiplying (2.6) by $(\Phi - u)^+$, which vanishes on the boundary of $B(x_0, r)$, we get

$$\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla(\Phi - u)\,dx < \int_{B(x_0,r)\cap\{\Phi>u\}} f(x)(\Phi - u)\,dx.$$

On the other hand, taking $(\Phi - u)^+$, extended by zero outside $B(x_0, r)$, as test function in the weak formulation of the problem, we obtain

$$\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla(\Phi - u)\,dx = \int_{B(x_0,r)\cap\{\Phi>u\}} f(x)(\Phi - u)\,dx.$$

Upon subtraction and using a well known inequality we conclude

$$\begin{aligned} 0 &> \int_{B(x_0,r)\cap\{\Phi>u\}} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\,dx \\ &\geq c\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla\Phi-\nabla u|^{p(x)}\,dx, \end{aligned}$$

a contradiction.

This proves that u is a viscosity supersolution. The proof that u is a viscosity subsolution runs as above and we omit the details. □

3. The limit problem as $p_n(x)\to\infty$.

Our purpose in this section is to analyze the behaviour of the solutions to problem (1.1) as the function $p_n(x)$ goes to infinity as $n\to\infty$. First we show that there exists a nontrivial limit that maximizes (1.7).

Proof of Theorem 1.1. If we consider the trivial function in (2.2) we get

$$\int_\Omega \frac{1}{p_n(x)}|\nabla u_{p_n}|^{p_n(x)}dx - \int_\Omega f u_{p_n}\,dx \leq 0.$$

Then,

$$\int_\Omega \frac{1}{p_n(x)}|\nabla u_{p_n}|^{p_n(x)}dx \leq \int_\Omega f u_{p_n}\,dx \leq \|f\|_{L^{q'}(\Omega)}\|u_{p_n}\|_{L^q(\Omega)} \leq C(\Omega,f,q)\|\nabla u_{p_n}\|_{L^q(\Omega)},$$

where $\frac{1}{q}+\frac{1}{q'}=1$ to apply Hölder inequality. Now we claim that

$$\|\nabla u_{p_n}\|_{L^q(\Omega)} \leq C(\Omega,q)|\nabla u_{p_n}|_{p_n(x)}. \tag{3.1}$$

Indeed, if we apply Hölder inequality for variable exponent Sobolev spaces, see Proposition 2.1, we get

$$\|\nabla u_{p_n}\|^q_{L^q(\Omega)} \leq 2|1|_{a_n'(x)}||\nabla u_{p_n}|^q|_{a_n(x)} \leq 2\max\{1,\mu(\Omega)\}|\nabla u_{p_n}|^q_{p_n(x)},$$

where $qa_n(x)=p_n(x)$ and $\frac{1}{a_n(x)}+\frac{1}{a_n'(x)}=1$. Hence, from the above estimate (3.1) straight follows. Thus summing up we have shown that

$$\int_\Omega \frac{1}{p_n(x)}|\nabla u_{p_n}|^{p_n(x)}dx \leq C(\Omega,f,q)|\nabla u_{p_n}|_{p_n(x)}. \tag{3.2}$$

Next, we take τ_0 such that

$$\frac{1}{2} \leq \int_\Omega \left|\frac{\nabla u_{p_n}}{\tau_0}\right|^{p_n(x)} dx \leq 1. \tag{3.3}$$

Taking into account (3.2) and (3.3) we deduce that

$$\frac{\min\{\tau_0^{p_n^+},\tau_0^{p_n^-}\}}{2p_n^+} \leq \int_\Omega \frac{1}{p_n(x)}|\nabla u_{p_n}|^{p_n(x)}dx \leq C(f,\Omega,q)\tau_0, \tag{3.4}$$

with p_n^+, p_n^- defined in (1.4). Now we claim that

$$|\nabla u_{p_n}|_{p_n(x)} \leq C(n), \quad \text{with } C(n)\to 1, \quad \text{as } n\to\infty. \tag{3.5}$$

If $|\nabla u_{p_n}|_{p_n(x)} \leq 1$, then (3.5) easily follows. Then let us assume that $|\nabla u_{p_n}|_{p_n(x)} > 1$ and let $\tau_0 > 1$ such that (3.3) holds. Note that, taking into account (1.3) it holds that

$$\limsup_{n\to\infty} \frac{\log(p_n^+)}{p_n^- - 1} = 0. \tag{3.6}$$

Therefore, by (3.4) and (3.6) we obtain that

$$\tau_0 \leq (C(f,\Omega,q)p_n^+)^{\frac{1}{p_n^- -1}} \to 1, \quad \text{as } n \to \infty,$$

which proves (3.5). By $iii)$ in Proposition 2.1 it follows that u_{p_n} is uniformly bounded in $W^{1,p_n}(\Omega)$. We can assume that $p_n > q > N$ for every $x \in \Omega$ assuring that $W^{1,q}(\Omega)$ embeds compactly into $C^\beta(\Omega)$, for some $0 < \beta < 1$. Then, from (3.5) we get for a subsequence $\{u_{p_{n_i}(x)}\}$ that

$$u_{p_{n_i}(x)} \rightharpoonup u_\infty, \text{ weakly in } W^{1,q}(\Omega) \quad \text{and} \quad u_{p_{n_i}(x)} \to u_\infty, \text{ strongly in } C^\beta(\Omega). \tag{3.7}$$

Moreover, by the convergence in (3.7) and the lower semicontinuity of the norm, we have that

$$\begin{aligned} |\nabla u_\infty|_{L^q(\Omega)} &\leq \liminf_{n\to\infty} |\nabla u_{p_n}|_{L^q(\Omega)} \leq \liminf_{n\to\infty} \Big(2\max\{1,\mu(\Omega)\}\Big)^{\frac{1}{q}} |\nabla u_{p_n}|_{p_n(x)} \\ &\leq \liminf_{n\to\infty} \Big(2\max\{1,\mu(\Omega)\}\Big)^{\frac{1}{q}} C(n) = \Big(2\max\{1,\mu(\Omega)\}\Big)^{\frac{1}{q}}, \end{aligned}$$

Passing to the limit as $q \to \infty$ in the previous estimate we get

$$|\nabla u_\infty|_{L^\infty(\Omega)} \leq 1,$$

that is, (1.6).

It remains to see that u_∞ maximizes (1.7), (thus u_∞ is nontrivial). Note that for n fixed, by (2.2) we have that

$$\int_\Omega \frac{1}{p_n(x)} |\nabla u_{p_n}|^{p_n(x)} dx - \int_\Omega f u_{p_n}\, dx \leq \int_\Omega \frac{1}{p_n(x)}\, dx - \int_\Omega f v\, dx,$$

for any $v \in K$. Neglecting the first positive term on the left hand side and rearranging we obtain

$$\int_\Omega f v\, dx \leq \int_\Omega f u_{p_n}\, dx + \int_\Omega \frac{1}{p_n(x)}\, dx.$$

Now, passing to the limit as $n \to \infty$ in the previous expression, taking into account (1.2) and (3.7), we get that

$$\int_\Omega f v\, dx \leq \int_\Omega f u_\infty\, dx,$$

for any function $v \in K$, thus (1.7) holds. □

We will see that (1.7) will allow us to find some explicit examples of limits u_∞. But before, let us determine the equation satisfied by u_∞. We give now some results that will be useful for such task. The next Lemma can be found in [3].

Lemma 3.1. *Assume that $\|\nabla u_\infty\|_{L^\infty(\Omega)} \leq 1$. If $\psi \in C^2(\Omega)$ is such that $u_\infty - \psi$ attains its maximum at $x_0 \in \Omega$, then $|\nabla\psi(x_0)| \leq 1$. Analogously, if x_0 is a local minimum for $u_\infty - \phi$, then $|\nabla\phi(x_0)| \leq 1$.*

We have all the ingredients to compute the limit of the equation given in (2.4),

$$H_{p_n(x)}(u_{p_n}, \nabla u_{p_n}, \nabla p_n, D^2 u_{p_n}) = 0,$$

as $p_n(x) \to \infty$ in the viscosity sense, that is, to identify the limit equation verified by any limit u_∞ as in (3.7). In the sequel we assume that we have a subsequence $p_{n_i}(x) \to \infty$ with the assumptions stated in the introduction such that $u_{p_{n_i}} \to u_\infty$ as $i \to \infty$ uniformly in $\overline{\Omega}$. We will still denote the solution as u_{p_n} for readable reasons (understanding that we are considering only a convergent sequence).

Proof of Theorem 1.2. We observe that (1.9) follows directly from (1.6) and Lemma 3.1. Also it is clear that $u_\infty = 0$ on $\partial\Omega$.

Now, as usual, we consider a point x_0, where $u_\infty - \phi$ attains a minimum or $u_\infty - \psi$ attains a maximum. Depending on the sign of the function f at that point we have a different limit of the equation (2.5). Let us consider separately each case.

1. Let x_0 be in $\Omega \setminus \operatorname{supp} f$: Since u_{p_n} converge uniformly to u_∞ (see Theorem 1.1), there exists a sequence of points $x_{p_n} \in \Omega \setminus \operatorname{supp} f$, where $u_{p_n} - \phi$ attains a minimum, with ϕ satisfying equation (2.3) with the inequality '$\geq$'. Let us suppose that $|\nabla\phi(x_0)| \neq 0$. Then, $|\nabla\phi(x_{p_n})| \neq 0$ for n large and we can multiply equation (2.3) by $(p_n(x_n) - 2)^{-1} |\nabla\phi(x_n)|^{-(p_n(x_n)-4)}$ to obtain

$$\frac{-|\nabla\phi(x_n)|^2 \Big(\Delta\phi(x_n) + \log(|\nabla\phi(x_n)|)\langle \nabla p_n(x_n), \nabla\phi(x_n)\rangle\Big)}{p_n(x_n) - 2} - \langle \nabla\phi(x_n) D^2\phi(x_n), \nabla\phi(x_n)^t\rangle \geq 0.$$

If we pass to the limit as $n \to \infty$ in this expression taking into account (1.2) and (1.5) we get

$$H_\infty(\nabla\phi, \xi, D^2\phi)(x_0) \geq 0. \tag{3.8}$$

On the other hand, if $|\nabla\phi(x_0)| = 0$ then (3.8) trivially holds. Arguing analogously for a test function ψ we deduce that

$$H_\infty(\nabla\psi, \xi, D^2\psi)(x_0) \leq 0, \tag{3.9}$$

and we have shown the first statement in Theorem 1.2.

2. Let x_0 be in $\{f > 0\}^\circ$. In this case there exists $x_{p_n} \to x_0$ such that $f(x_{p_n}) > 0$ where $u_{p_n} - \phi$ attains a minimum. It implies that $|\nabla\phi(x_{p_n})| \neq 0$ for n large, so we can multiply (2.3) again by $(p_n(x_n) - 2)^{-1} |\nabla\phi(x_n)|^{-(p_n(x_n)-4)}$ getting

$$\begin{aligned} &\frac{-|\nabla\phi(x_n)|^2 \Big(\Delta\phi(x_n) + \log(|\nabla\phi(x_n)|)\langle \nabla p_n(x_n), \nabla\phi(x_n)\rangle\Big)}{p_n(x_n) - 2} - \langle \nabla\phi(x_n) D^2\phi(x_n), \nabla\phi(x_n)^t\rangle \\ &\geq \frac{f(x_{p_n})}{(p_n(x_{p_n}) - 2)|\nabla\phi(x_{p_n})|^{p_n(x_{p_n})-4}}. \end{aligned} \tag{3.10}$$

Now we observe that the left hand side of this inequality is bounded for every n. Therefore, the limit as $n \to \infty$ on the right hand side cannot diverge. Hence, $|\nabla\phi(x_0)| \geq 1$. By Lemma 3.1 we know that $|\nabla\psi(x_0)| \leq 1$. Thus, it holds that

$$|\nabla u_\infty| = 1,$$

in the viscosity sense and this case is complete.

3. Let x_0 be in $\{f < 0\}^\circ$. This case is analogous to the previous one, but we sketch the proof for completeness. We begin by considering first the viscosity subsolutions. Arguing as above we get that (3.10) holds for ψ with the reverse inequality, where $f(x_{p_n}) < 0$. It follows that

$|\nabla\psi(x_0)| \geq 1$ for this limit to be finite. For the viscosity supersolutions we get that $|\nabla\phi(x_0)| \leq 1$ by Lemma 3.1 and then

$$-|\nabla u_\infty| = -1.$$

4. Let x_0 be in $\Omega \cap \partial\{f > 0\} \setminus \partial\{f < 0\}$. In other words, we have that $f(x_0) = 0$ and we can approximate x_0 by points in $\{f > 0\}$ or in $\{f = 0\}$. Let us consider the sequence x_{p_n}, where $u_{p_n} - \phi$ attains a minimum. At least for a subsequence we have that $f(x_{p_n}) > 0$ or $f(x_{p_n}) = 0$. In case that $f(x_{p_n}) = 0$ we can argue as in the first step to obtain (3.8). If $f(x_{p_n}) > 0$ we can proceed as in step 2 and conclude that for the viscosity supersolutions (3.8) holds. For the subsolutions, if $f(x_{p_n}) = 0$ we can argue as in step 1 to deduce (3.9). On the other hand, if $f(x_{p_n}) > 0$ we get $|\nabla u_\infty| \leq 1$. Summing up, we conclude for this case that u_∞ verifies

$$H_\infty(\nabla u_\infty, \xi, D^2 u_\infty)(x_0) \geq 0,$$

together with the general viscosity estimates for the gradient in the whole domain, (1.9).

5. Let x_0 be in $\Omega \cap \partial\{f < 0\} \setminus \partial\{f > 0\}$. In an analogous way, we can show that in this case u_∞ satisfies

$$H_\infty(\nabla u_\infty, \xi, D^2 u_\infty)(x_0) \leq 0,$$

together with the general viscosity estimates for the gradient in the whole domain, (1.9).

6. Finally, let us consider x_0 in $\Omega \cap \partial\{f > 0\} \cap \partial\{f < 0\}$. That is, x_0 is such that $f(x_0) = 0$, and it can be reached by sequences contained either in $\{f > 0\}$, either in $\{f = 0\}$, either in $\{f < 0\}$. Using the same arguments as in the previous cases we get for this set that u_∞ satisfies (1.9). □

Explicit examples. We finish this article giving some examples in which the limits u_∞ can be computed.

Example 1. If $f(x) > 0$ in Ω, then $u_\infty(x) = \text{dist}(x, \partial\Omega)$. To prove this fact, we will see that $\text{dist}(\cdot, \partial\Omega)$ is the unique maximizer to (1.7). Any $v \in K$ verifies that $v(x) \leq \text{dist}(x, \partial\Omega)$. Thus, since $f > 0$ we have that

$$\int_\Omega vf\,dx \leq \int_\Omega \text{dist}(\cdot, \partial\Omega) f\,dx,$$

and the inequality is strict, unless $\text{dist}(\cdot, \partial\Omega) \equiv v$ in $\overline{\Omega}$. Therefore, $u_\infty(x) = \text{dist}(x, \partial\Omega)$.

Example 2. If $f(x) < 0$ in Ω, then $u_\infty(x) = -\text{dist}(x, \partial\Omega)$. The proof of this fact runs as before.

Example 3. Let us consider $f \geq 0$ in Ω with $f > 0$ in $D \subset \Omega$. By the same arguments used in the first example we know that $u_\infty(x) = \text{dist}(x, \partial\Omega)$ in $\text{supp}f$. To find an explicit solution in the whole domain, let us restrict to the case $\Omega = [0, L]$ with $L > 2$ and $\text{supp}f = [0, 1]$. Then, u_∞ is the straight line with slope 1 in $[0, 1]$, whereas in $[1, L]$, according to Theorem 1.2, verifies

$$-u''_\infty (u'_\infty)^2 - (u'_\infty)^2 \log(|u'_\infty|)\xi u'_\infty = 0.$$

After some computations, where we have assumed that $u' < 0$ in $[1, L]$, we obtain that

$$u_\infty(x) - u_\infty(L) = \int_x^L \exp\left(K \exp\left(-\int_1^t \xi(s)\,ds\right)\right) dt,$$

where K is a constant to be determined. Now, since $u_\infty(L) = 0$ we have to choose the free constant K such that

$$1 = u(1) = \int_1^L \exp\left(K \exp\left(-\int_1^t \xi(s)\,ds\right)\right) dt.$$

Note that the previous expression is continuous and decreasing in K. If $K = 0$ the right hand side is $L - 1 > 1$, while if $K \to -\infty$, then the right hand side tends to zero. Therefore, there exists some $K_0 < 0$ such that $u_\infty(1) = 1$. This value K_0 is unique.

When $L \le 2$ the analysis performed before shows that $u_\infty(x) = \mathrm{dist}(x, \partial\Omega)$, that is

$$u_\infty(x) = \begin{cases} x, & 0 \le x < L/2; \\ -x + L, & L/2 \le x < L. \end{cases}$$

Example 4. Finally let us consider a function f that changes sign, that is, $f > 0$ in $D_1 \subset \Omega$ and $f < 0$ in $D_2 \subset \Omega$. To find an explicit limit solution, we again consider this case in one dimension. Let us take $\Omega = [-1, 1]$ and assume that f is an odd function which is strictly positive in $[-1, 0)$. In addition we assume that the exponents $p_n(x)$ are even for every n. In this case the limit u_∞ is also odd, thus $u_\infty(0) = 0$. Since we showed that u_∞ maximizes (1.7), then we have an unique possible choice for u_∞, which is

$$u_\infty(x) = \begin{cases} x + 1, & -1 \le x < -1/2; \\ -x, & -1/2 \le x < 1/2, \\ x - 1, & 1/2 \le x \le 1. \end{cases}$$

Note that, according to Theorem 1.2 in $[-1, 1] \setminus \{0\}$ it holds that $|u'_\infty| = 1$ in $(-1, 0)$ and $-|u'_\infty| = -1$ in $(0, 1)$.

Remark 3.1. *We wish to stress that, in case f does not change sign and does not vanish, that is, examples 1 and 2, the limit solution u_∞ does not depend on the sequence $p_n(x)$ and coincides with the limit of the problems $\Delta_p u = f$ as $p \to \infty$, with p constant, that is $u_\infty(x) \equiv dist(x, \partial\Omega)$.*

Moreover, even in the case that $f \ge 0$ and $f > 0$ in some set $D \subset \Omega$ containing the all the points in Ω at which the function $dist(\cdot, \Omega)$ is not differentiable, then we can deduce also that $u_\infty(x) = dist(x, \Omega)$ in Ω. Note that it is not the case of example 3 with $L > 2$.

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