AN ANISOTROPIC INFINITY LAPLACIAN OBTAINED AS THE LIMIT OF THE ANISOTROPIC $(p, q)-$LAPLACIAN.

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Abstract. In this work we study the behaviour of the solutions to the following Dirichlet problem related to the anisotropic $(p, q)-$Laplacian operator

$$\begin{array}{ll} -\mathrm{div}_x(|\nabla_x u|^{p-2}\nabla_x u) - \mathrm{div}_y(|\nabla_y u|^{q-2}\nabla_y u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{array}$$

as $p, q \to \infty$. Here $\Omega \subset \mathbb{R}^N \times \mathbb{R}^K$ and $\nabla_x u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N})$ and $\nabla_y u = (\frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2}, \dots, \frac{\partial u}{\partial y_K})$ denote the gradient of u with respect to the first N variables (x variables) and with respect to the last K variables (y variables).

We consider a sequence of exponents (p_n, q_n) that goes to infinity with $p_n/q_n \to R$. We prove that u_n, the solution with $p = p_n$, $q = q_n$, verifies $u_n \to u_\infty$ uniformly in $\overline{\Omega}$, where u_∞ is the unique viscosity solution to

$$\begin{cases} -\Delta_{\infty,x} u_\infty = 0 & \text{for } |\nabla_y u_\infty|^R < |\nabla_x u_\infty|, \\ -R\Delta_{\infty,y} u_\infty = 0 & \text{for } |\nabla_y u_\infty|^R > |\nabla_x u_\infty|, \\ -\Delta_{\infty,x} u_\infty - R\Delta_{\infty,y} u_\infty = 0 & \text{for } |\nabla_y u_\infty|^R = |\nabla_x u_\infty|, \\ u_\infty = g & \text{on } \partial\Omega. \end{cases}$$

Here $\Delta_{\infty,x} u = \nabla_x u D_x^2 u (\nabla_x u)^t$ and $\Delta_{\infty,y} u = \nabla_y u D_y^2 u (\nabla_y u)^t$ are the infinity Laplacian in x variables and in y variables, respectively.

1. Introduction

In this work we analyze the behaviour of the solutions to the Dirichlet problem for the anisotropic $(p, q)-$Laplacian operator as $p, q \to \infty$. More precisely, we consider the following problem,

$$\begin{cases} -\mathrm{div}_x(|\nabla_x u|^{p-2}\nabla_x u) - \mathrm{div}_y(|\nabla_y u|^{q-2}\nabla_y u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $\Omega \subset \mathbb{R}^{N+K}$ being a bounded smooth domain. We have denoted by $\nabla_x u$ and $\nabla_y u$ the derivatives of u with respect to the first N variables and with respect to the last K variables, respectively, that is, $\nabla_x u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N})$ and $\nabla_y u = (\frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2}, \dots, \frac{\partial u}{\partial y_K})$. We assume that the boundary datum g is a Lipschitz function. Concerning the exponents p, q, we assume that we have a sequence (p_n, q_n) with $q_n > p_n$ and that there exists $1 \le R < \infty$ such that

$$\lim_{n\to\infty} \frac{q_n}{p_n} = R. \tag{1.2}$$

Weak solutions to the previous problem (1.1) can be easily obtained using variational arguments (see Section 2 for the details). In addition we show here that any continuous weak solution is also a viscosity solution.

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As we have mentioned, in this work we are interested in the behaviour of the solutions to (1.1) when we consider a sequence of exponents such that $p_n, q_n \to \infty$, as $n \to \infty$. Our main result reads as follows: we show that the sequence of solutions u_n to (1.1) with $p_n, q_n \to \infty$ verifying (1.2), converges uniformly to some function u_∞, that is a solution to a certain limit PDE. The precise statement is the following.

Theorem 1.1. *Let u_n be the sequence of solutions to* (1.1) *with $p_n, q_n \to \infty$ verifying* (1.2) *and a fixed boundary datum g Lipschitz. Then, up to a subsequence, $u_n \to u_\infty$ uniformly in Ω, and this function u_∞ verifies in the viscosity sense in Ω,*

$$\begin{cases} -\Delta_{\infty,x} u_\infty = 0 & \text{for } |\nabla_y u_\infty|^R < |\nabla_x u_\infty|, \\ -R\Delta_{\infty,y} u_\infty = 0 & \text{for } |\nabla_y u_\infty|^R > |\nabla_x u_\infty|, \\ -\Delta_{\infty,x} u_\infty - R\Delta_{\infty,y} u_\infty = 0 & \text{for } |\nabla_y u_\infty|^R = |\nabla_x u_\infty|, \end{cases} \tag{1.3}$$

and on $\partial\Omega$

$$u_\infty = g. \tag{1.4}$$

Here $\Delta_{\infty,x} u = \nabla_x u D_x^2 u (\nabla_x u)^t$ and $\Delta_{\infty,y} u = \nabla_y u D_y^2 u (\nabla_y u)^t$ are the infinity Laplacian in x variables and in y variables, respectively.

Moreover, the limit problem (1.3)-(1.4) *has a unique viscosity solution, and therefore the whole sequence $\{u_n\}$ converges uniformly to u_∞.*

We remark that, in our case, the limit problem depends strongly in the way that p and q go to infinity through the constant R that appears in the limit (1.2).

Concerning the optimal regularity of solutions to the limit PDE we have the following result.

Theorem 1.2. *Every continuous viscosity solution to* (1.3) *is locally Lipschitz. Moreover, this result is optimal for $N, K > 0$, since*

$$u(x, y) = x + \frac{1}{2}|y|, \tag{1.5}$$

is viscosity solution to (1.3)*, that has no further regularity than Lipschitz.*

Let us give some references and motivation for the analysis of this problem. The limit of $p-$harmonic functions (solutions to $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$) as $p \to \infty$ has been extensively studied in the literature (see [5] and the survey [1]) and leads naturally to the infinity Laplacian given by $\Delta_\infty u = \nabla u D^2 u (\nabla u)^t$. Infinity harmonic functions (solutions to $-\Delta_\infty u = 0$) are related to the optimal Lipschitz extension problem (see the survey [1]) and find applications in optimal transportation, image processing and tug-of-war games (see, e.g., [6], [10], [20], [21] and the references therein). Also limits of the eigenvalue problem related to the p-Laplacian have been exhaustively examined, see [14], [15], [22]. See also [17], [18], [19], [22], [23] and [24] for limits as $p(x) \to \infty$.

On the other hand, anisotropic problems like (1.1) have been analyzed for many years, specially concerning regularity of the solutions, see for example [9]. We refer to the survey [11] and references therein.

Concerning the limit as $p \to \infty$ of solutions to the anisotropic $p-$Laplacian, $-\tilde{\Delta}_p u = -\sum_i (|u_{x_i}|^{p-2} u_{x_i})_{x_i} = 0$ (note that here the exponent is the same for each term) we refer to [3], [12] and [25]. In this case the limit equation is known as the pseudo infinity Laplacian and is given by the expression $-\sum_{i \in I(\nabla u)} |u_{x_i}|^2 u_{x_i, x_i} = 0$, with $I(\xi) = \{i \ : \ |\xi_i| = \max_j |\xi_j|\}$. Note that this case corresponds to $p = q$ (hence

$R = 1$) and $N = K = 1$ in our setting. The optimal regularity for the pseudo infinity Laplacian was clarified in [25].

Therefore, it seems natural to look for the limit problem of the anisotropic $p, q-$Laplacian when different powers involved, and see how this fact affects the limit PDE. The simplest case to perform this study is problem (1.1). Let us point out that our analysis can be carried over for PDEs like

$$-\sum_i \operatorname{div}_{x_i}(|\nabla_{x_i} u|^{p_i-2}\nabla_{x_i} u) = 0,$$

but, for simplicity we focus on (1.1).

Regarding the ideas and methods used in the proofs we point out the following facts: the proof of the uniform convergence of u_n to u_∞ is based on a priori estimates (in some suitable anisotropic Sobolev spaces), that imply compactness of the sequence u_n; after that, one can verify the passage to the limit in the viscosity sense taking care of the different cases that appear (considering that the definition of viscosity solution has to take into account that the involved PDE is discontinuous with respect to ∇u), we use ideas from [12]; we follow [8] and [25] (using that solutions enjoy a comparison principle with appropriate cones) to show the optimal Lipschitz regularity for solutions to the limit PDE. Finally, uniqueness of the limit PDE is proved adapting ideas from [2].

The rest of the paper is organized as follows. In the next Section we show the existence of weak solutions to (1.1) using variational methods; in Section 3 we deal with viscosity solutions (we state the precise definition of viscosity solutions and prove that a weak solution to (1.1) is a viscosity solution) and next, in Section 4, we find the limit problem; in Section 5 we prove our Lipschitz regularity result, Theorem 1.2, and finally in Section 6 we obtain uniqueness of solutions to the limit PDE.

2. Weak solutions

First, we need to introduce an anisotropic Sobolev space. If $u : \Omega \to \mathbb{R}$ is a regular enough function, we will denote

$$\nabla_x u = \Big(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N}\Big) \quad \text{and} \quad \nabla_y u = \Big(\frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2}, \dots, \frac{\partial u}{\partial y_K}\Big).$$

Thus, the complete gradient of u reads as $\nabla u = (\nabla_x u, \nabla_y u)$. Throughout this paper we will denote by $W^{1,p,q}(\Omega)$, with $1 < p < q < \infty$, the anisotropic Sobolev space obtained as the completion of the space $C^\infty(\Omega)$ with respect to the norm $\|u\|_{p,q} = \|\nabla_x u\|_p + \|\nabla_y u\|_q + \|u\|_p$. These functions satisfy $\nabla_x u \in L^p(\Omega; \mathbb{R}^N)$ and $\nabla_y u \in L^q(\Omega; \mathbb{R}^K)$. Also, let $W_0^{1,p,q}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with the same norm used to define $W^{1,p,q}(\Omega)$.

Since $p < q$, it is immediate that $W^{1,p,q}(\Omega)$ is embedded in the usual Sobolev space $W^{1,p}(\Omega)$,

$$W^{1,p,q}(\Omega) \hookrightarrow W^{1,p}(\Omega).$$

Therefore there is a trace for functions in $W^{1,p,q}(\Omega)$ and it holds

$$W^{1,p,q}(\Omega) \hookrightarrow L^p(\partial\Omega).$$

Now we introduce the definition of a weak (or variational) solution to (1.1).

Definition 2.1. *We say that* $u \in W^{1,p,q}(\Omega)$ *is a weak solution to problem* (1.1) *if* $u - g \in W_0^{1,p,q}(\Omega)$ *and for every* $v \in C_0^1(\Omega)$ *there holds,*

$$\int_\Omega |\nabla_x u|^{p-2}\nabla_x u \nabla_x v + \int_\Omega |\nabla_y u|^{p-2}\nabla_y u \nabla_y v = 0.$$

Theorem 2.1. *There exists a unique weak solution to* (1.1)*.*

Proof. Solutions to the previous problem (1.1) can be easily obtained minimizing the functional

$$F(u) = \frac{1}{p}\int_\Omega |\nabla_x u|^p + \frac{1}{q}\int_\Omega |\nabla_x u|^q \tag{2.1}$$

in the space $S_{p,q} = \{u \in W^{1,p,q}(\Omega) \ : \ u - g \in W_0^{1,p,q}(\Omega)\}$. □

We have the following uniform estimates.

Theorem 2.2. *Given g Lipschitz, there exists a constant C independent of p, q such that u_n, the unique weak solution to* (1.1) *with p_n, q_n, verifies*

$$\left(\int_\Omega |\nabla_x u_n|^{p_n}\right)^{\frac{1}{p_n}}, \left(\int_\Omega |\nabla_y u_n|^{q_n}\right)^{\frac{1}{q_n}} \le C.$$

Proof. Let L denote the Lipschitz constant of g. Let v be a Lipschitz extension of g to Ω with the same Lipschitz constant L of g (note that we can choose v as the unique AMLE of g, see [1], and then the Lipschitz constant of v coincides with the corresponding one of g). To simplify the notation, along this proof we drop the subscript n. Since the solution u is obtained as a minimizer of (2.1), we have that

$$\int_\Omega \frac{|\nabla_x u|^p}{p} + \int_\Omega \frac{|\nabla_y u|^q}{q} \le \int_\Omega \frac{|\nabla_x v|^p}{p} + \int_\Omega \frac{|\nabla_y v|^q}{q} \le \frac{|\Omega| L^p}{p} + \frac{|\Omega| L^q}{q} \tag{2.2}$$

and hence, since $q > p$, assuming that $L \ge 1$,

$$\left(\int_\Omega |\nabla_x u|^p\right)^{\frac{1}{p}} \le (2|\Omega|p)^{\frac{1}{p}} L^{\frac{q}{p}} \qquad \text{and} \qquad \left(\int_\Omega |\nabla_y u|^q\right)^{\frac{1}{q}} \le (2|\Omega|q)^{\frac{1}{q}} L.$$

Recalling assumption (1.2), we have that $q/p \le C$. Taking into account this fact in the previous inequalities, we obtain that there exists C such that

$$\left(\int_\Omega |\nabla_x u|^p\right)^{\frac{1}{p}}, \left(\int_\Omega |\nabla_y u|^q\right)^{\frac{1}{q}} \le C. \tag{2.3}$$

Now, if $L < 1$ from (2.2) we obtain

$$\left(\int_\Omega |\nabla_x u|^p\right)^{\frac{1}{p}} \le (2|\Omega|p)^{\frac{1}{p}} \qquad \text{and} \qquad \left(\int_\Omega |\nabla_y u|^q\right)^{\frac{1}{q}} \le (2|\Omega|q)^{\frac{1}{q}}.$$

From these estimates (2.3) immediately follows (note that in this case we don't need to use that $q/p \le C$). □

Corollary 2.3. *Given g Lipschitz there exists a subsequence (named again u_n) and $u \in W^{1,\infty}(\Omega)$ such that*

$$u_n \rightharpoonup u \qquad \textit{weackly in } W^{1,r}(\Omega),$$

for every $1 < r < \infty$ and

$$u_n \to u$$

uniformly in $\overline{\Omega}$.

Proof. From

$$\left(\int_\Omega |\nabla_x u_n|^{p_n}\right)^{\frac{1}{p_n}}, \left(\int_\Omega |\nabla_y u_n|^{q_n}\right)^{\frac{1}{q_n}} \leq C.$$

we can obtain for any $1 < r < p < q$ fixed,

$$\begin{aligned}&\|\nabla_x u_n\|_{L^r(\Omega)} + \|\nabla_y u_n\|_{L^r(\Omega)} \\ &\qquad \leq \|\nabla_x u_n\|_{L^{p_n}(\Omega)}|\Omega|^{\frac{p_n-r}{p_n r}} + \|\nabla_y u_n\|_{L^{q_n}(\Omega)}|\Omega|^{\frac{q_n-r}{q_n r}} \leq C.\end{aligned}$$

Hence we have that u_n is bounded in $W^{1,r}(\Omega)$ and then we can extract a subsequence such that $u_n \rightharpoonup u$ weackly in $W^{1,r}(\Omega)$. Moreover, we can also take $N+K < r < p < q$ and by the above estimates we obtain for a subsequence, $u_n \to u$ uniformly in $\overline{\Omega}$. We conclude the proof using a diagonal procedure. □

3. Viscosity solutions.

In this section we introduce the concept of viscosity solutions to problem (1.1). Assuming that u_n are smooth enough to differentiate (1.1), we get

$$\begin{aligned}&-|\nabla_x u|^{p-2}\Delta_x u - (p-2)|\nabla_x u|^{p-4}\nabla_x u D_x^2 u \nabla_x u \\ &\qquad -|\nabla_y u|^{q-2}\Delta_y u - (q-2)|\nabla_y u|^{p-4}\nabla_y u D_y^2 u \nabla_y u = 0.\end{aligned} \tag{3.1}$$

This equation is nonlinear but elliptic (degenerate), thus it makes sense to consider viscosity subsolutions and supersolutions of it.

Now, let us recall the definition of viscosity sub and supersolution to a nonlinear PDE problem of the form

$$\begin{cases} H(Du, D^2u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

In general the function H can be discontinuous. Then we denote by H^* and H_* the upper and lower semicontinuous envelopes of H, respectively, defined as

$$H^*(z,S) = \lim_{\varepsilon\to 0} \sup\{H(z',S') \ : \ |z-z'| + |S-S'| < \varepsilon\}$$

for $z \in \mathbb{R}^{N+K}$ and $S \in \mathbb{S}^{N+K}$ (we denote by $\mathbb{S}^L$ the set of symmetric matrices in $\mathbb{R}^{L\times L}$) and

$$H_*(z,S) = -(-H)^*(z,S).$$

Definition 3.1. *An upper semicontinuous function u defined in Ω is a viscosity subsolution of (3.2) if, $u|_{\partial\Omega} \leq g$ and, whenever $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ are such that $u - \psi$ has a maximum at x_0, then*

$$H_*(\nabla\psi(x_0), D^2\psi(x_0)) \leq 0.$$

Definition 3.2. *A lower semicontinuous function u defined in Ω is a viscosity supersolution of (3.2) if, $u|_{\partial\Omega} \geq g$ and, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $u - \phi$ has a minimum at x_0, then*

$$H^*(\nabla\phi(x_0), D^2\phi(x_0)) \geq 0.$$

In what follows we will keep the notation used in the above definitions. That is, by ϕ we will denote the test functions such that $u-\phi$ has a minimum in Ω and by ψ the test functions such that $u - \psi$ has a maximum somewhere in Ω.

We refer to [7] for more details about general theory of viscosity solutions, and [13], [16] for viscosity solutions related to the $\infty-$Laplacian and the $p-$Laplacian operators.

Now, let $z \in \mathbb{R}^{N+K}$, and $S \in \mathbb{S}^{N+K}$. To simplify the notation we will call

$$w_1 = (z_1, ..., z_N), \qquad \text{and} \qquad w_2 = (z_{N+1}, ..., z_{N+K}),$$

so w_1 stands for the first N components of z and w_2 for the last K components. Also we will call

$$S_1 = (s_{ij})_{1\le i,j\le N}$$

the first $N \times N$ minor of the matrix S and

$$S_2 = (s_{ij})_{N+1\le i,j\le N+K}$$

the last $K \times K$ minor of S.

Let consider the following continuous function

$$\begin{aligned} H_n(z,S) = -|w_1|^{p-2}(\text{trace}(S_1)) - (p-2)|w_1|^{p-4}\langle S_1 \cdot w_1, w_1\rangle \\ +|w_2|^{q-2}(\text{trace}(S_2) - (q-2)|w_2|^{q-4}\langle S_2 \cdot w_2, w_2\rangle. \end{aligned} \tag{3.3}$$

Solutions to (3.1) are to be considered as solutions to

$$\begin{cases} H_n(\nabla u_n, D^2 u_n) = 0, & \text{in } \Omega, \\ u_n = g & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

in the sense of Definitions 3.1 and 3.2. We remark here that, since H_n is continuous, we have that $H_n = (H_n)_* = (H_n)^*$.

The following result can be shown as in [15]. We include the proof for convenience of the reader.

Lemma 3.1. *A continuous weak solution to equation* (1.1) *is a viscosity solution to* (3.4)*.*

Proof. Let $x_0 \in \Omega$ and a let ϕ be a test function such that $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict minimum at x_0 (we may assume that the minimum is strict, see [7]). We want to show that

$$\begin{aligned} -|\nabla_x \phi|^{p-2}\Delta_x\phi(x_0) - (p-2)|\nabla_x\phi|^{p-4}\nabla_x\phi D_x^2\phi\nabla_x\phi(x_0) \\ -|\nabla_y \phi|^{q-2}\Delta_y\phi(x_0) - (q-2)|\nabla_y\phi|^{q-4}\nabla_y\phi D_y^2\phi\nabla_y\phi(x_0) \ge 0 \end{aligned}$$

Assume, *ad contrarium*, that this is not the case; then there exists a radius $r > 0$ such that $B(x_0, r) \subset \Omega$ and

$$\begin{aligned} -|\nabla_x \phi|^{p-2}\Delta_x\phi(x) - (p-2)|\nabla_x\phi|^{p-4}\nabla_x\phi D_x^2\phi\nabla_x\phi(x) \\ -|\nabla_y \phi|^{q-2}\Delta_y\phi(x) - (q-2)|\nabla_y\phi|^{q-4}\nabla_y\phi D_y^2\phi\nabla_y\phi(x) < 0 \end{aligned}$$

for every $x \in B(x_0, r)$. Set

$$m = \inf_{|x-x_0|=r} (u - \phi)(x)$$

and let $\Phi(x) = \phi(x) + m/2$. This function Φ verifies $\Phi(x_0) > u(x_0)$ and

$$-\Delta_{p,x}\Phi(x) - \Delta_{q,y}\Phi(x) < 0 \qquad \text{in } B(x_0, r). \tag{3.5}$$

Multiplying (3.5) by $(\Phi - u)^+$, which vanishes on the boundary of $B(x_0, r)$, we get

$$\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla_x \Phi|^{p-2} \nabla_x \Phi \cdot \nabla_x (\Phi - u) \\ + \int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla_y \Phi|^{q-2} \nabla_y \Phi \cdot \nabla_y (\Phi - u) < 0.$$

On the other hand, taking $(\Phi - u)^+$, extended by zero outside $B(x_0, r)$, as test function in the weak formulation of (1.1), we obtain

$$\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla_x u|^{p-2} \nabla_x u \cdot \nabla_x (\Phi - u) \\ + \int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla_y u|^{q-2} \nabla_y u \cdot \nabla_y (\Phi - u) = 0.$$

Upon subtraction and using a well know inequality, we conclude

$$\begin{aligned} 0 \quad > \quad & \int_{B(x_0,r)\cap\{\Phi>u\}} \big(|\nabla_x \Phi|^{p-2} \nabla_x \Phi - |\nabla_x u|^{p-2} \nabla_x u\big) \cdot \nabla_x (\Phi - u) \\ & + \int_{B(x_0,r)\cap\{\Phi>u\}} \big(|\nabla_y \Phi|^{q-2} \nabla_y \Phi - |\nabla_y u|^{q-2} \nabla_y u\big) \cdot \nabla_y (\Phi - u) \\ \geq \quad & c \int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla_x \Phi - \nabla_x u|^p + c \int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla_y \Phi - \nabla_y u|^q, \end{aligned}$$

a contradiction.

This proves that u is a viscosity supersolution. The proof that u is a viscosity subsolution runs as above and we omit the details. □

4. The limit problem as $(p, q) \to \infty$.

In a previous section (Section 2) we have proved that if we take limit in (1.1) as $p, q \to \infty$ the solutions converge uniformly in $C(\overline{\Omega})$ to some function u_∞. Let us determine the equation satisfied by u_∞. We have developed in Section 3 all the ingredients to compute the limit of the equation $H_n(\nabla u_n, D^2 u_n) = 0$ with H_n given by (3.3), as $p_n, q_n \to \infty$ in the viscosity sense, that is to identify the limit equation verified by any uniform limit u_∞.

We define for $z \in \mathbb{R}^{N+K}$ and $S \in \mathbb{S}^{N+K}$ a symmetric real matrix,

$$H_\infty(z, S) = \begin{cases} -\langle S_1 \cdot w_1, w_1 \rangle & \text{for } |w_2|^R < |w_1|, \\ -R\langle S_2 \cdot w_2, w_2 \rangle & \text{for } |w_2|^R > |w_1|, \\ -\langle S_1 \cdot w_1, w_1 \rangle - R\langle S_2 \cdot w_2, w_2 \rangle & \text{for } |w_2|^R = |w_1|. \end{cases} \tag{4.1}$$

As this function H_∞ is discontinuous, our first step is to characterize its upper and lower semicontinuous envelopes, $(H_\infty)^*$ and $(H_\infty)_*$.

Lemma 4.1. *The upper semicontinuous envelope of H_∞ is given by*

$$(H_\infty)^*(z, S) = \begin{cases} -\langle S_1 \cdot w_1, w_1 \rangle & \textit{for } |w_2|^R < |w_1|, \\ -R\langle S_2 \cdot w_2, w_2 \rangle & \textit{for } |w_2|^R > |w_1|, \\ \max\Big\{ - \langle S_1 \cdot w_1, w_1 \rangle - R\langle S_2 \cdot w_2, w_2 \rangle; & \\ \qquad -\langle S_1 \cdot w_1, w_1 \rangle; -R\langle S_2 \cdot w_2, w_2 \rangle \Big\} & \textit{for } |w_2|^R = |w_1|. \end{cases}$$

The lower semicontinuous envelope has the same expression except for the last case in which the max *is replaced by*

$$\min\Big\{ -\langle S_1\cdot w_1, w_1\rangle - R\langle S_2\cdot w_2, w_2\rangle ; -\langle S_1\cdot w_1, w_1\rangle ; -R\langle S_2\cdot w_2, w_2\rangle \Big\}.$$

Proof. If $z=0$ the statement is trivial, hence we may assume that $z\neq 0$.

First, suppose that $|w_2|^R<|w_1|$. Then, for ε small enough we also have $|w_2'|^R<|w_1'|$, for every z' such that $|z'-z|<\varepsilon$. Therefore,

$$H_\infty(z',S') = -\langle S_1'\cdot w_1', w_1'\rangle$$

for every $|z'-z|<\varepsilon$ and we conclude that

$$\begin{aligned}
(H_\infty)^*(z,S) &= \lim_{\varepsilon\to 0}\sup\{H_\infty(z',S') \ : \ |z-z'|+|S-S'|<\varepsilon\}\\
&= \lim_{\varepsilon\to 0}\sup\{-\langle S_1'\cdot w_1', w_1'\rangle \ : \ |z-z'|+|S-S'|<\varepsilon\}\\
&= -\langle S_1\cdot w_1, w_1\rangle.
\end{aligned}$$

The case $|w_2|^R>|w_1|$ can be handled analogously.

Thus, we are left with the case $|w_2|^R=|w_1|$. First, let us show that

$$(H_\infty)^*(z,S)\geq -\langle S_1\cdot w_1, w_1\rangle.$$

In fact, let $z_k=(z_1,..,z_N,k(z_{N-1},...,z_{N+K}))$ with $k<1$ and let $k\nearrow 1$ to obtain

$$\begin{aligned}
(H_\infty)^*(z,S) &= \lim_{\varepsilon\to 0}\sup\{H_\infty(z',S') \ : \ |z-z'|+|S-S'|<\varepsilon\}\\
&\geq \lim_{k\nearrow 1}\sup_{z_k}\{-\langle S_1\cdot w_1, w_1\rangle\} = -\langle S_1\cdot w_1, w_1\rangle.
\end{aligned}$$

Analogously one can see that

$$(H_\infty)^*(z,S)\geq -R\langle S_2\cdot w_2, w_2\rangle.$$

Finally,

$$\begin{aligned}
(H_\infty)^*(z,S) &= \lim_{\varepsilon\to 0}\sup\{H_\infty(z',S') \ : \ |z-z'|+|S-S'|<\varepsilon\}\\
&\geq H_\infty(z,S) = -\langle S_1\cdot w_1, w_1\rangle - R\langle S_2\cdot w_2, w_2\rangle.
\end{aligned}$$

Now, since the possible limit of $H_\infty(z',S')$ as $z'\to z$ and $S'\to S$ is given by

$$-\langle S_1\cdot w_1, w_1\rangle - R\langle S_2\cdot w_2, w_2\rangle, \qquad -\langle S_1\cdot w_1, w_1\rangle, \qquad \text{or} \qquad -R\langle S_2\cdot w_2, w_2\rangle,$$

the result follows.

The analogous result for the lower envelope $(H_\infty)_*$ can be proved in the same way and thus we omit the details. □

In the sequel we assume that we have a subsequence $p_{n_i}, q_{n_i}\to\infty$ with the assumptions stated in the introduction such that

$$\lim_{i\to\infty} u_{n_i} = u_\infty$$

uniformly in Ω.

Theorem 4.1. *A function u_∞ obtained as a uniform limit of a subsequence of $\{u_n\}$ verifies $u_\infty=g$ on $\partial\Omega$ and the following PDE in the viscosity sense*

$$\begin{cases}
-\Delta_{\infty,x}u_\infty = 0 & \text{for } |\nabla_y u_\infty|^R < |\nabla_x u_\infty|,\\
-R\Delta_{\infty,y}u_\infty = 0 & \text{for } |\nabla_y u_\infty|^R > |\nabla_x u_\infty|,\\
-\Delta_{\infty,x}u_\infty - R\Delta_{\infty,y}u_\infty = 0 & \text{for } |\nabla_y u_\infty|^R = |\nabla_x u_\infty|.
\end{cases} \tag{4.2}$$

Proof. Our task is to show that u_∞ is a viscosity solution to (4.2). Since every u_n takes the boundary datum g, we get $u_\infty = g$ on $\partial\Omega$. Hence, it just remains to see that u_∞ verifies the equation, that is, u_∞ is a solution to

$$H_\infty(\nabla u, D^2 u) = 0$$

in the sense of Definitions 3.1 and 3.2.

To prove that u_∞ is a viscosity supersolution of (4.2), let ϕ be such that $u - \phi$ has a strict local minimum at $x_0 \in \Omega$, with $\phi(x_0) = u(x_0)$. We want to prove that

$$(H_\infty)^*(\nabla\phi(x_0), D^2\phi(x_0)) \geq 0. \tag{4.3}$$

Since $u_n \to u$ uniformly, there is a sequence $(x_n)_n$ such that $x_n \to x_0$ and $u_n - \phi$ has a local minimum at x_n. As u_n is a viscosity solution of (1.1) (cf. Lemma 3.1), we have

$$\begin{aligned} -|\nabla_x\phi|^{p-2}\Delta_x\phi(x_n) - (p-2)|\nabla_x\phi|^{p-4}\nabla_x\phi D^2_x\phi\nabla_x\phi(x_n) \\ -|\nabla_y\phi|^{q-2}\Delta_y\phi(x_n) - (q-2)|\nabla_y\phi|^{q-4}\nabla_y\phi D^2_y\phi\nabla_y\phi(x_n) \geq 0. \end{aligned}$$

Hence, assuming that $\nabla_x\phi(x_0) \neq 0$ we obtain,

$$\begin{aligned} &\left(\frac{|\nabla_x\phi|^2\Delta_x\phi}{p-2} + \Delta_{\infty,x}\phi\right)(x_n) \\ &\qquad \leq -\left(\frac{|\nabla_y\phi|^{q-4}}{|\nabla_x\phi|^{p-4}}\right)\left(\frac{|\nabla_y\phi|^2\Delta_y\phi}{p-2} + \frac{(q-2)}{(p-2)}\Delta_{\infty,y}\phi\right)(x_n). \end{aligned} \tag{4.4}$$

Now, we observe that, as $n \to \infty$,

$$\left(\frac{|\nabla_x\phi|^2\Delta_x\phi}{p-2} + \Delta_{\infty,x}\phi\right)(x_n) \to \Delta_{\infty,x}\phi(x_0),$$

$$\left(\frac{|\nabla_y\phi|^{\frac{q-4}{p-4}}}{|\nabla_x\phi|}\right)(x_n) \to \left(\frac{|\nabla_y\phi|^R}{|\nabla_x\phi|}\right)(x_0)$$

and

$$\left(\frac{|\nabla_y\phi|^2\Delta_y\phi}{p-2} + \frac{(q-2)}{(p-2)}\Delta_{\infty,y}\phi\right)(x_n) \to R\Delta_{\infty,y}\phi(x_0).$$

From those limits we deduce that, if

$$\left(\frac{|\nabla_y\phi|^R}{|\nabla_x\phi|}\right)(x_0) < 1, \text{ then } \Delta_{\infty,x}\phi(x_0) \leq 0,$$

and, if

$$\left(\frac{|\nabla_y\phi|^R}{|\nabla_x\phi|}\right)(x_0) > 1, \text{ then } \Delta_{\infty,y}\phi(x_0) \leq 0.$$

In the case

$$\left(\frac{|\nabla_y\phi|^R}{|\nabla_x\phi|}\right)(x_0) = 1$$

we argue by contradiction, assuming that

$$-\Delta_{\infty,x}\phi(x_0) < 0, \qquad \text{and} \qquad -R\Delta_{\infty,y}\phi(x_0) < 0. \tag{4.5}$$

Note that from these inequalities we have that

$$-\Delta_{\infty,x}\phi(x_0) - R\Delta_{\infty,y}\phi(x_0) < 0.$$

Also note that (4.5) implies that $\nabla_x\phi(x_0) \neq 0$ and $\nabla_y\phi(x_0) \neq 0$.

Suppose first that
$$\left(\frac{|\nabla_y \phi|^{q-4}}{|\nabla_x \phi|^{p-4}}\right)(x_0) \geq 1.$$
Going back to equation (4.4) and rearranging it as follows, along a subsequence $n_i \to \infty$, we obtain
$$\begin{aligned}\left(\frac{|\nabla_x \phi|^2 \Delta_x \phi}{p-2}\right)(x_{n_i}) &\leq -\Delta_{\infty,x}\phi(x_{n_i}) \\ &- \left(\frac{|\nabla_y \phi|^{q-4}}{|\nabla_x \phi|^{p-4}}\right)\left(\frac{|\nabla_y \phi|^2 \Delta_y \phi}{p-2} + \frac{(q-2)}{(p-2)}\Delta_{\infty,y}\phi\right)(x_{n_i}) < 0,\end{aligned}$$
for n_i large enough. Taking limit as $n_i \to \infty$, we get a contradiction.

Now, we treat the case
$$\left(\frac{|\nabla_y \phi|^{q-4}}{|\nabla_x \phi|^{p-4}}\right)(x_0) < 1.$$
Then, we argue as before with
$$\begin{aligned} 0 \quad \geq & \left(\frac{|\nabla_x \phi|^{p-4}}{|\nabla_y \phi|^{q-4}}\right)\left(\frac{|\nabla_x \phi|^2 \Delta_x \phi}{p-2} + \Delta_{\infty,x}\phi\right)(x_n) \\ & + \left(\frac{|\nabla_y \phi|^2 \Delta_y \phi}{p-2} + \frac{(q-2)}{(p-2)}\Delta_{\infty,y}\phi\right)(x_n),\end{aligned}$$
using that
$$\left(\frac{|\nabla_x \phi|^{p-4}}{|\nabla_y \phi|^{q-4}}\right)(x_0) > 1.$$
The fact that u_∞ is a viscosity subsolution of (4.2) can be proved analogously. □

5. Lipschitz regularity

First, we prove that viscosity solutions enjoy comparison with some special cones in (x, y), that take into account the anisotropy. These cones are defined by
$$C^b_{x_0,y_0}(x, y) = A + b^R|x - x_0| + b|y - y_0|.$$
We denote the corresponding ball by
$$B^b_r(x_0, y_0) = \left\{(x, y) \ : \ b^R|x - x_0| + b|y - y_0| \leq r\right\}. \tag{5.6}$$
It holds comparison with cones from below.

Lemma 5.1. *Let u be a viscosity supersolution to* (1.3). *If $u(x, y) \geq C^b_{x_0,y_0}(x, y)$ for $(x, y) \in \partial(B^b_r(x_0, y_0)\setminus\{(x_0, y_0)\})$, then $u(x, y) \geq C^b_{x_0,y_0}(x, y)$ for $(x, y) \in B^b_r(x_0, y_0)$.*

Proof. We follow [8] and [25] and argue by contradiction. Suppose that $u(x, y) < C^b_{x_0,y_0}(x, y)$ for some $(x, y) \in B^b_r(x_0, y_0) \setminus (x_0, y_0)$ and consider the perturbation of the cone,
$$w(x, y) = \tilde{C}^b_{x_0,y_0}(x, y) - \varepsilon\left(L^2 - |x - x_0|^2 - |y - y_0|^2\right),$$
where $\tilde{C}^b_{x_0,y_0}$ is a smooth approximation of $C^b_{x_0,y_0}$, one can choose
$$\tilde{C}^b_{x_0,y_0}(x, y) = A + b^R|x - x_0|^a + b|y - y_0|^a$$
with $a > 1$ close to 1. If L is large enough and ε is small enough we obtain $w(x, y) \leq u(x, y)$ for $(x, y) \in \partial(B^b_r(x_0, y_0) \setminus \{(x_0, y_0)\})$ and $\min(u - w) = u(z) - w(z) < 0$. A direct computation shows $-\Delta_{\infty,x}w(z) < 0$ and $-\Delta_{\infty,y}w(z) < 0$ for ε small, which contradicts the fact that u is a viscosity supersolution. □

Similarly we have comparison with cones from above.

Lemma 5.2. *Let u be a viscosity subsolution to* (1.3). *If* $u(x,y) \le C^b_{x_0,y_0}(x,y)$ *for* $(x,y) \in \partial(B^b_r(x_0,y_0)\backslash\{(x_0,y_0)\})$, *then* $u(x,y) \le C^b_{x_0,y_0}(x,y)$ *for* $(x,y) \in B^b_r(x_0,y_0)$.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. First we show, following [8], that every viscosity solution is locally Lipschitz. Consider the cones centered at (x_0,y_0),

$$C^b_+(x,y) = u(x_0,y_0) + b^R|x-x_0| + b|y-y_0|$$

and

$$C^b_-(x,y) = u(x_0,y_0) - b^R|x-x_0| - b|y-y_0|.$$

Let us choose $\delta > 0$ such that

$$B^b_\delta(x_0,y_0) = \left\{(x,y) \ : \ b^{R-1}|x-x_0| + |y-y_0| \le \frac{\delta}{b}\right\} \subset \Omega$$

for every $b \ge 1$. Now, since u is locally bounded we can choose b sufficiently large (it is enough to take $b = \max\{2\sup_{\overline{B^1_\delta}}|u|; 1\}$) such that

$$C^b_+(x,y) \ge u(x,y) \ge C^b_-(x,y),$$

on $\partial B^b_\delta(x_0,y_0)$. As

$$C^b_+(x_0,y_0) = u(x_0,y_0) = C^b_-(x_0,y_0)$$

using the previous lemmas (comparison with cones) we obtain that there exists $b > 0$ such that

$$b^R|x-x_0| + b|y-y_0| \ge u(x,y) - u(x_0,y_0) \ge -b^R|x-x_0| - b|y-y_0|,$$

for every (x,y) in a neigbourhood of (x_0,y_0). This proves that u is locally Lipschitz.

To finish the proof of the theorem we have to show that

$$u(x,y) = x + \frac{1}{2}|y| \tag{5.7}$$

is a viscosity solution. First, assume that $u-\psi$ has a maximum at (x_0,y_0) with $y_0 \ne 0$. Then, since $u-\psi$ is smooth, satisfies

$$(u-\psi)_x(x_0,y_0) = 0, \qquad (u-\psi)_y(x_0,y_0) = 0$$

and

$$(u-\psi)_{xx}(x_0,y_0) \le 0.$$

As $u_x(x_0,y_0) = 1$, $|u_y(x_0,y_0)| = 1/2$ and $u_{xx}(x_0,y_0) \le 0$ we get

$$\psi_x(x_0,y_0) = 1 > |\psi_y(x_0,y_0)|^R = (1/2)^R \qquad \text{and} \qquad -\psi_{xx}(x_0,y_0) \le 0.$$

Hence, we obtain

$$-\Delta_{\infty,x}\psi(x_0,y_0) = -(\psi_x)^2\psi_{xx}(x_0,y_0) \le 0.$$

Analogously, if $u-\phi$ has a minimum at (x_0,y_0) with $y_0 \ne 0$ we have

$$-\Delta_{\infty,x}\phi(x_0,y_0) \ge 0.$$

Now, if $y_0 = 0$ and ψ is smooth, $u-\psi$ cannot have a maximum at (x_0,y_0). On the other hand, if $u-\phi$ has a minimum at this point we obtain

$$\phi_x(x_0,y_0) = 1, \ |\phi_y(x_0,y_0)| \le 1/2 \text{ and } -\phi_{xx}(x_0,y_0) \ge 0.$$

Therefore, $|\phi_x(x_0,y_0)| > |\phi_y(x_0,y_0)|^R$ and

$$-\Delta_{\infty,x}\phi(x_0,y_0) = -(\phi_x)^2\phi_{xx}(x_0,y_0) \geq 0.$$

This finishes the proof of Theorem 1.2. □

Remark 5.1. The cones $C^b_{x_0,y_0}$ play the same role as the one played by the l^2-cones in the theory for the usual infinity Laplacian, Δ_∞, see [1]. Note that these cones $C^b_{x_0,y_0}$ are not differentiable along the planes $x = x_0$, $y = y_0$. This can explain the fact that solutions to our limit problem are locally Lipschitz but not locally C^1.

One may think that the lack of C^1 regularity comes from the fact that the operator that appears in the limit PDE is discontinuous. However, this is not always the case, as in our example $x + \frac{1}{2}|y|$, where $|u_x| > |u_y|^R$ at every point.

As trivial examples of solutions we can consider bilinear functions, that is, $u(x,y) = a_1xy + a_2x + a_3y + a_4$. With these examples it is easy to find classical solutions in which $|u_x| > |u_y|^R$ or $|u_x| < |u_y|^R$ or $|u_x| = |u_y|^R$, depending on the point.

6. Uniqueness of solutions to the limit PDE

We will follow closely the arguments in [2], see also [25], where similar ideas where used to show uniqueness for the pseudo infinity Laplacian. The main point of the proof is to obtain an equivalent result to Lemma 3.2 in [2], that is, to prove a version of Hopf's Lemma. To this end we will use trick of comparison with cones, that we have also applied to prove Lipschitz regularity for the solutions in the previous section. Note that the results of [2] are not directly applicable, since our limit PDE does not verify the structural assumption (F3) in that reference.

Lemma 6.1 (Hopf's Lemma)**.** *Assume that w is a viscosity supersolution to equation* (1.3) *with a local minimum at y_0. Then w is constant in a neighborhood of y_0.*

Proof. Let w_β be the inf-convolution of w, that is

$$w_\beta(y) = \inf_z \left(w(z) + \frac{|y-z|^2}{\beta^2} \right).$$

We choose β small enough such that w_β is well defined and semi-convex in a neighborhood of y_0, $w_\beta(y_0) = w(y_0)$. Moreover, it is possible to show that w_β is also a viscosity supersolution with a minimum at y_0. See [7].

We will prove the result by contradiction. Since w_β is a semi-convex function, by translating it, we can assume that there is a ball B^b_r of the form (5.6) $w_\beta > 0$ on ∂B^b_r and $w_\beta(z_0) = 0$ for some $z_0 \in B^b_{2r} \setminus \overline{B^b_r}$.

We consider a multiple of the cone $C^b_{x_0,y_0}(x,y)$, that is,

$$\chi_\alpha(x,y) = \alpha 2r + \alpha b^R|x - x_0| + \alpha b|y - y_0|,$$

and observe that, if we choose α small enough, we get that $\chi_\alpha < w_\beta$ on ∂B^b_r, $\chi_\alpha \leq w_\beta$ on ∂B^b_{2r} and $\chi_\alpha(z_0) > w_\beta(z_0) = 0$.

Now, consider the perturbation of the function χ_α,

$$\Theta(x,y) = \tilde{\chi}_\alpha(x,y) - \varepsilon\left(L^2 - |x - x_0|^2 - |y - y_0|^2\right),$$

where $\tilde{\chi}_\alpha$ is a smooth approximation of χ_α. If L is large enough and ε is small enough we obtain $\Theta(x,y) \leq w_\beta(x,y)$ for $(x,y) \in \partial B^b_r$ and for $(x,y) \in \partial B^b_{2r}$. Moreover, we have $\min(w_\beta - \Theta) = w_\beta(z) - \Theta(z) < 0$. A direct computation shows

$-\Delta_{\infty,x}\Theta(z) < 0$ and $-\Delta_{\infty,y}\Theta(z) < 0$ for ε small, which contradicts the fact that w_β is a viscosity supersolution. □

By using the sup-convolution we can prove a similar statement to Lemma 6.1 when a viscosity subsolution has a local maximum.

Lemma 6.2 (Hopf's Lemma)**.** *Assume that w is a viscosity subsolution to equation* (1.3) *with a local maximum at y_0. Then w is constant in a neighborhood of y_0.*

Once these Lemmas are established, the rest of the proof of uniqueness is contained in the comparison principle proved in [2] (see also [25]). For completeness, we briefly sketch the proof here.

Theorem 6.1. *Let u be a bounded continuous subsolution and v a bounded continuous supersolution to* (1.3) *such that $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.*

Proof. By regularizing u and v by sup and inf convolution and taking $u-\eta$ instead of u, we can assume u is semi-convex, v is semi-concave and $u < v$ on $\partial\Omega$. We need to show $u \le v$ in Ω. We argue by contradiction and suppose that

$$\max_{\overline{\Omega}}(u-v) > 0. \tag{6.8}$$

Define for h small

$$M(h) = \max_{x\in\overline{\Omega}_h}(u(x+h)-v(x)),$$

where $\Omega_h = \{x \in \Omega : d(x,\partial\Omega) > h\}$. Notice that (6.8) implies that $M(0) > 0$. Hence, if (6.8) holds, we must have for h small enough that $M(h) > 0$.

Before finding the contradiction we will show that necessarily one of the following holds:

(1) There is a sequence $h_n \to 0$ such that at any maximum point x_{h_n} of $u(\cdot + h_n) - v(\cdot)$ it holds $Du(x_{h_n}+h_n) = Dv(x_{h_n}) \neq 0$ for every n. or
(2) There is a neighborhood of 0 such that for every h in this neighborhood $M(h) = M(0)$.

If (1) occurs we are going to reach the contradiction by proving that necessarily for n large enough $M(h_n) \le 0$, which contradicts (6.8). On the other hand, if (2) holds, we show that Lemma 6.1 implies that the set where $M(0)$ is achieved is open and closed, hence equal to $\overline{\Omega}$, contradicting that $u < v$ on $\partial\Omega$ (when $M(0) > 0$).

Let us start by showing that either (1) or (2) occurs. Suppose that (1) does not take place. Notice that M is a maximum of semi-convex functions, which implies that M is semi-convex in a neighborhood of 0. Let $x_h \in \Omega$ be a maximum point of $u(\cdot + h) - v(\cdot)$. By general properties of semi-convex functions (see (DMP) in [2]), $u(\cdot + h)$ and $v(\cdot)$ are differentiable at x_h and $Du(x_h + h) = Dv(x_h)$.

The semi-convexity of u implies that, if $Du(x_h + h) = Dv(x_h) = 0$ for some h, then there exists a constant r, small enough, such that for every $h' \in B_r(h)$

$$M(h') \ge u(x_h + h') - v(x_h) \ge u(x_h + h) - v(x_h) - C|h-h'|^2.$$

It follows that $M(h') \ge M(h) - C|h-h'|^2$, which means that $0 \in \partial M(h)$. Let us recall here that the subdifferential of f at $\hat{x}$ is given by

$$\partial f(\hat{x}) := \{y \,:\, f(x) \ge f(\hat{x}) + \langle y, x-\hat{x}\rangle - c\|x-\hat{x}\|^2, \text{ for some } c > 0\}$$

and that for a convex function we can simple write

$$\partial f(\hat{x}) := \{y \,:\, f(x) \ge f(\hat{x}) + \langle y, x-\hat{x}\rangle\,\}.$$

Since (1) does not take place, necessarily for any h in a neighborhood of 0 we must have that $0 \in \partial M(h)$, or equivalently $M(h) = M(0)$ for h in some neighborhood of 0. That is (2) occurs.

Now we are left to show that any of the two alternatives lead us to a contradiction.

(a) If (2) holds: then for every h in a neighborhood of 0

$$u(x_0) - v(x_0) = M(0) = M(h) \geq u(x_0 + h) - v(x_0).$$

That is, x_0 is a local maximum for u. Lemma 6.1 implies that u is constant in a neighborhood of x_0. Since $u - v$ attains a local maximum at x_0 and u is constant, v must attain a local minimum at x_0. Using once more Hopf's Lemma, we conclude that v must also be constant in a neighborhood of x_0. It follows that the set where $M(0)$ is attained is open. By continuity of u and v it must be also closed, hence it must equal $\overline{\Omega}$, which contradicts that $u < v$ on $\partial\Omega$.

(b) If (1) holds: consider φ_ε given by

$$\varphi'_\epsilon(t) = \exp\left(\int_0^t \exp\left(-\varepsilon^{-1}(s + \varepsilon^{-1})\right) ds\right) \tag{6.9}$$

and denote $\psi_\varepsilon = \phi_\varepsilon^{-1}$. Let

$$U_\varepsilon = \psi_\varepsilon(u), \qquad V_\varepsilon = \psi_\varepsilon(v).$$

Notice that $U_\varepsilon \to u$ and $V_\varepsilon \to v$ as $\varepsilon \to 0$. For a smooth function ϕ, letting $\Phi_\varepsilon = \psi_\varepsilon(\phi)$ we have

$$\nabla_x \Phi_\varepsilon = \psi'_\varepsilon(\phi)\nabla_x \phi \qquad \text{and} \qquad D_x^2 \Phi_\varepsilon = \psi'_\varepsilon(\phi) D_x^2 \phi + \psi''_\varepsilon(\phi)(\nabla_x \phi \otimes \nabla_x \phi),$$

and analogous expressions for the derivatives with respect to the y variables. Hence we have that U_ε is a viscosity subsolution of

$$G_\varepsilon\left(U_\varepsilon, \frac{\partial U_\varepsilon}{\partial x_i}, D^2 U_\varepsilon\right) = 0$$

and analogously V_ε is a supersolution. Here G_ε is given by

$$G_\varepsilon(a, z, S) = \begin{cases} -(\varphi'_\varepsilon(a))^2 \langle (\varphi'_\varepsilon(a) S_1 + \varphi''_\varepsilon(a)(w_1 \otimes w_1)) \cdot w_1, w_1 \rangle \\ -R(\varphi'_\varepsilon(a))^2 \langle (\varphi'_\varepsilon(a) S_2 + \varphi''_\varepsilon(a)(w_2 \otimes w_2)) \cdot w_2, w_2 \rangle \\ -(\varphi'_\varepsilon(a))^2 \langle (\varphi'_\varepsilon(a) S_1 + \varphi''_\varepsilon(a)(w_1 \otimes w_1)) \cdot w_1, w_1 \rangle \\ \qquad -R(\varphi'_\varepsilon(a))^2 \langle (\varphi'_\varepsilon(a) S_2 + \varphi''_\varepsilon(a)(w_2 \otimes w_2)) \cdot w_2, w_2 \rangle, \end{cases}$$

according to the sign of $(\varphi'_\varepsilon(a))^{R-1}|w_2|^R - |w_1|$.

By definition of φ_ε we have that U_ε and V_ε are, respectively, semi-convex and semi-concave. It holds that $U_\varepsilon(\cdot + h_n) - V_\varepsilon(\cdot)$ attains its maximum at some interior point $x_\varepsilon \in \Omega$. As U_ε and V_ε are semi-convex and semi-concave (property (DMP) in [2]), both of them are differentiable at x_ε and $|DU_\varepsilon(x_\varepsilon + h_n)| = |DV_\varepsilon(x_\varepsilon)|$. Since $U_\varepsilon \to u$ and $V_\varepsilon \to v$ as $\varepsilon \to 0$, we can find a sequence $\varepsilon_k \to 0$ such that $x_{\varepsilon_k} \to \overline{x}$, where $\overline{x}$ is a maximum of $u(\cdot + h_n) - v(\cdot)$. In the sequel, let us denote such sequences by ε and x_ε. Since (1) holds, we have $|Du(\overline{x} + h_n)| = |Dv(\overline{x})| \geq \delta(n) > 0$. By general properties of semi-convex and semi-concave functions (property (PGC) in [2]) and the definition of φ_ε, we have for ε small enough $|DU_\varepsilon(x_\varepsilon + h_n)| = |DV_\varepsilon(x_\varepsilon)| \geq \frac{\delta(n)}{2}$.

Note that one can construct a sequence of points $p_m \to 0$ and a sequence functions $f_m(x) = U_\varepsilon(x + h_n) - V_\varepsilon(x) - \langle p_m, x\rangle$, such that f_m has a strict maxima at

x_ε^m and $x_\varepsilon^m \to x_\varepsilon$ as $m \to \infty$. Lemma A.3 in [7] shows that if $r > 0$ is small enough, there is a $\overline{\rho} > 0$ such that the set of maximum points in $B_r(x_\varepsilon^m)$ of

$$g_m(x) = U_\varepsilon(x + h_n) - V_\varepsilon(x) - \langle p_m, x\rangle - \langle q, x\rangle$$

with $q \in B_\rho(0)$ ($\rho \leq \overline{\rho}$), contains a set of positive Lebesgue measure. By Alexansandrov's result, $U_\varepsilon(\cdot + h_n)$ and $V_\varepsilon(\cdot)$ are twice differentiable a.e. Therefore, for r small and $\rho \leq \overline{\rho}$, there exists a $z \in B_r(x_\varepsilon^m)$ and $q \in B_\rho(0)$ such that z is a maximum of g_m and U_ε and V_ε are twice differentiable at z. Since z is a maximum it holds $DU_\varepsilon(z + h_n) = DV_\varepsilon(z) + p_m + q$. As before, for q, ρ small and m large

$$|DU_\varepsilon(z + h_n)| = |DV_\varepsilon(z) + p_m + q| \geq \frac{\delta(n)}{4}.$$

Moreover, since U_ε is semi-convex and V_ε semi-concave, it holds

$$-C \cdot Id \leq D^2U_\varepsilon(z + h_n) \leq D^2V_\varepsilon(z) \leq C \cdot Id,$$

for some constant $C > 0$ independent of ρ, r and m. Evaluating at z we have by the definition of G

$$G_\varepsilon(U_\varepsilon(z + h_n), DV_\varepsilon(z) + p_m + q, D^2V_\varepsilon(z)) \leq 0 \leq G_\varepsilon(V_\varepsilon(z), DV_\varepsilon(z), D^2V_\varepsilon(z)).$$

Since $DV_\varepsilon(z)$ and $D^2V_\varepsilon(z)$ are bounded, by taking a subsequence when $\rho, r \to 0$ and $m \to \infty$, we can find $\overline{P} \geq \frac{\delta(n)}{4}$ and $\overline{X}$ such that $DV_\varepsilon(z) \to \overline{P}$ and $D^2V_\varepsilon(z) \to \overline{X}$.

Taking limits we obtain

$$G_\varepsilon(U_\varepsilon(z + h_n), \overline{P}, \overline{X}) \leq 0 \leq G_\varepsilon(V_\varepsilon(z), \overline{P}, \overline{X}). \tag{6.10}$$

On the other hand, it is easy to see by the definition of G_ε and φ_ε that, for any $\delta > 0$, ε small enough, $|a| \leq \delta^{-1}$, $\delta \leq |z| \leq \delta^{-1}$ and $|S| \leq \delta^{-1}$ it holds

$$\frac{\partial G_\varepsilon(a, z, S)}{\partial a} > 0,$$

which contradicts (6.10), finishing the proof of uniqueness. □

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