THE HEAT CONTENT FOR NONLOCAL DIFFUSION WITH NON–SINGULAR KERNELS

J. M. MAZON, J. D. ROSSI, J. TOLEDO

Abstract. We study the behaviour of the heat content for a nonlocal evolution problem. We obtain an asymptotic expansion for the heat content of a set D, defined as $\mathbb{H}_D^J(t) := \int_D u(x,t)dx$ being u the solution to $u_t = J * u - u$ with initial condition $u_0 = \chi_D$. This expansion is given in terms of geometric values of D. As a consequence, we obtain that $\mathbb{H}_D^J(t) = |D| - P_J(D)t + o(t)$ as $t \downarrow 0$. We also recover the usual heat content for the heat equation when we rescale the kernel J in an appropriate way. Finally, we also find an asymptotic expansion for the nonlocal analogous to the spectral heat content that is defined as before but considering $u(x,t)$ a solution to the equation $u_t = J * u - u$ inside D with $u = 0$ in $\mathbb{R}^N \setminus D$ and initial condition $u_0 = \chi_D$.

To Ireneo Peral, a great mathematician and friend.

1. Introduction

Let us start with a brief description of the classical heat content for the heat equation. In [5] (see also [6]) it is defined the *heat content* of a Borel measurable set $D \subset \mathbb{R}^N$ at time t as

$$\mathbb{H}_D(t) := \int_D T(t)\chi_D(x)dx,$$

being $(T(t))_{t\geq 0}$ the heat semigroup in $L^2(\mathbb{R}^N)$. Therefore, $\mathbb{H}_D(t)$ represents the amount of heat in D at time t if in D the initial temperature is 1 and in $\mathbb{R}^N \setminus D$ the initial temperature is 0. As a consequence of the following characterization for the perimeter of a set $D \subset \mathbb{R}^N$ with finite perimeter, given in [13, Theorem 3.3],

$$\lim_{t\to 0^+} \sqrt{\frac{\pi}{t}} \int_{\mathbb{R}^N\setminus D} T(t)\chi_D(x)dx = P(D), \tag{1.1}$$

Key words and phrases. Heat content, nonlocal evolution problems, non-singular kernels.

2010 *Mathematics Subject Classification.* 45M05, 35K08.

The auhtors have been partially supported by the Spanish MEC and FEDER, project MTM2015-70227-P.

the following result was presented in [5] for the heat content: for D an open subset in $\mathbb{R}^N$ with finite Lebesgue measure and finite perimeter,

$$\mathbb{H}_D(t) = |D| - \sqrt{\frac{t}{\pi}} P(D) + o(\sqrt{t}), \quad t \downarrow 0. \tag{1.2}$$

In [1] and [2] the concept of heat content was extended to more general diffusion processes. More precisely, for $0 < \alpha \leq 2$, let $p_t^{(\alpha)} : \mathbb{R}^N \to [0, \infty)$ be the probability density such that its Fourier transform verifies

$$\widehat{p_t^{(\alpha)}}(x) = e^{-t|x|^\alpha}.$$

If one considers

$$T_t^{(\alpha)}(f)(x) := \int_{\mathbb{R}^N} f(y) p_t^{(\alpha)}(x-y) dy,$$

then $u(x,t) = T_t^{(\alpha)}(f)(x)$ is the unique weak solution of the initial valued problem

$$\begin{cases} u_t(x,t) = -(-\Delta)^{\frac{\alpha}{2}} u(x,t), & (x,t) \in \mathbb{R}^N \times [0,\infty), \\ u(x,0) = f(x) & x \in \mathbb{R}^N. \end{cases}$$

And, in this context, the *heat content* of a Borel measurable set $D \subset \mathbb{R}^N$ at time t is defined as

$$\mathbb{H}_D^{(\alpha)}(t) := \int_D T_t^{(\alpha)} \chi_D(x) dx.$$

Note that for $\alpha = 2$, $p_t^{(2)}$ is the Gaussian kernel

$$p_t^{(2)} = (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right),$$

and consequently, $\mathbb{H}_D^{(2)}(t) = \mathbb{H}_D(t)$. While, for $\alpha = 1$ and $k_N = \frac{\Gamma\left(\frac{N+1}{2}\right)}{\pi^{\frac{N+1}{2}}}$,

$$p_t^{(1)}(x) = \frac{k_N t}{(t + |x|)^{\frac{N+1}{2}}}$$

is the Poisson heat kernel.

In recent years nonlocal diffusion problems with non singular kernels, and related problems, have also been studied, see for example [3] and the references therein. More concretely, for $J : \mathbb{R}^N \to [0, +\infty)$ a measurable, nonnegative and radially symmetric function verifying $\int_{\mathbb{R}^N} J(z)dz = 1$, in [3] the following nonlocal Cauchy problem has been studied:

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy, & (x,t) \in \mathbb{R}^N \times [0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

This equation appears naturally from the following considerations: If $u(x,t)$ is thought of as a density at a point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} J(y-x)u(y,t)\,dy = (J*u)(x,t)$ is the rate at which individuals are arriving at position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (1.3).

Recall from [3] that a solution of problem (1.3) in the time interval $[0,T]$ is a function $u \in W^{1,1}(0,T;L^2(\mathbb{R}^N))$ which satisfies $u(x,0) = u_0$ and

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))\,dy \qquad \text{a.e. in } (x,t) \in \mathbb{R}^N \times (0,T).$$

A simple integration of the equation in (1.3) in space gives that the total mass is preserved, that is,

$$\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx \qquad \forall\, t \geq 0. \tag{1.4}$$

Our aim is to study the heat content associated with the above nonlocal diffusion processes and to relate it with the (local) heat content.

Let us fix the hypothesis on the kernel J that we will assume in this paper:

$$(\mathbf{H_J}) \qquad \begin{array}{c} J \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}) \text{ is non–negative, radially decreasing} \\ \text{and compactly supported with } \displaystyle\int_{\mathbb{R}^N} J(x)\,dx = 1. \end{array}$$

Now we give the following definition.

Definition 1.1. Given a Lebesgue measurable set $D \subset \mathbb{R}^N$ with finite measure, we define the *J-heat content* of D in $\mathbb{R}^N$ at time t by

$$\mathbb{H}_D^J(t) := \int_D u(x,t)dx,$$

being u the solution of (1.3) with datum $u_0 = \chi_D$.

Note that, from (1.4), we have

$$\mathbb{H}_D^J(0) = |D|$$

and that no further regularity is required for D besides having finite measure.

We have the following interpretation of the J-heat content, $\mathbb{H}_D^J(t)$: if $u(x,t)$ represents the density of a population at a point $x \in \mathbb{R}^N$ at time t with initial condition $u(x,0) = \chi_D(x)$, then the J-heat content of D at time t represents the size of the population that remained inside D at that time when in D the initial density of the population is 1 and in $\mathbb{R}^N \setminus D$ the initial density of population is 0.

Our first result gives an asymptotic expansion of the J-heat content form which a similar result to (1.2) follows. This expansion, obtained from a classical Taylor's expansion, is given for any time and for any order and with the main fact that all the terms involved in the expansion can be expressed using nonlocal perimeters of the set D with different kernels. We will use the following notation:

$$J = (J*)^1,$$

$$J * J = (J*)^2,$$

and for the convolution of n kernels:

$$J * J * \ldots * J = (J*)^n.$$

Observe that, for all n, $(J*)^n$ is nonnegative, radially decreasing and compactly supported with

$$\int_{\mathbb{R}^N} (J*)^n(x) dx = 1.$$

One of our main results is the following.

Theorem 1.2. *For $\mathbb{H}_D^J(t)$ the J-heat content of D given in Definition 1.1 it holds that*

$$\mathbb{H}_D^J(t) = |D| - \sum_{n=1}^{+\infty} \left(\sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} P_{(J*)^k}(D) \right) \frac{t^n}{n!} \qquad \forall t > 0, \tag{1.5}$$

where $P_{(J*)^k}(D)$ denotes the $(J*)^k$–nonlocal perimeter given by

$$P_{(J*)^k}(D) := \int_D \int_{\mathbb{R}^N \setminus D} (J*)^k(x-y)\, dy\, dx.$$

Moreover, for $L \geq 1$,

$$\left| \mathbb{H}_D^J(t) - |D| + \sum_{n=1}^{L} \left(\sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} P_{(J*)^k}(D) \right) \frac{t^n}{n!} \right| \leq \frac{|D|\, 2^L}{(L+1)!}\, t^{L+1} \qquad \forall t > 0. \tag{1.6}$$

We also have the following expression:

$$\mathbb{H}_D^J(t) = \sum_{n=0}^{+\infty} \left(\int_D \int_D (J*)^n (x-y) dy dx \right) \frac{e^{-t} t^n}{n!} \qquad \forall t > 0, \tag{1.7}$$

where $\int_D \int_D (J*)^0 (x-y) dy dx = |D|$.

The J–nonlocal perimeter of a measurable set was studied in [12] for non–singular kernels in connection with other geometrical properties. Here we obtain, in particular, the following nonlocal version of (1.1) and (1.2): let u be the solution of (1.3) for datum $u_0 = \chi_D$, D a finite Lebesgue measurable set in $\mathbb{R}^N$, then

$$\lim_{t \to 0^+} \frac{1}{t} \int_{\mathbb{R}^N \setminus D} u(x,t) dx = -(\mathbb{H}_D^J)'(0) = P_J(D), \tag{1.8}$$

or equivalently

$$\mathbb{H}_D^J(t) = |D| - P_J(D)t + o(t) \qquad t \downarrow 0. \tag{1.9}$$

Remark 1.3.

1. There is a remarkable point concerning (1.9) and (1.2): from the small asymptotic expansion of $\mathbb{H}_D^J(t)$ and $\mathbb{H}_D(t)$, it is possible to recover a geometric feature of the set D, in addition to its volume, namely, we can obtain the J-perimeter and the classical perimeter respectively.

2. The result given in (1.8) is similar to the one obtained in [2] for $\mathbb{H}_D^{(\alpha)}(t)$, $0 < \alpha < 1$, which says that

$$\left(\mathbb{H}_D^{(\alpha)}\right)'(0) = -\mathcal{A}_{\alpha,N} \mathcal{P}_\alpha(D),$$

where $\mathcal{A}_{\alpha,N}$ is a positive constant and $\mathcal{P}_\alpha(D)$ is the α–perimenter

$$\mathcal{P}_\alpha(D) := \int_D \int_{\mathbb{R}^N \setminus D} \frac{dx\, dy}{|x-y|^{N+\alpha}}.$$

We also show that we can recover the classical heat content from the J-heat content rescaling the kernel. Let

$$J_\epsilon(x) := \frac{1}{\epsilon^N} J\left(\frac{x}{\epsilon}\right).$$

Associated with the rescaled kernel J_ϵ we can consider the J_ϵ-heat content just by taking u as the solution to (1.3) with $J = J_\epsilon$ in Definition 1.1.

Theorem 1.4. *For D a subset in $\mathbb{R}^N$ with finite Lebesgue measure, we have*
$$\lim_{\epsilon\to 0^+} \mathbb{H}_D^{J_\epsilon}\left(\frac{C_J}{\epsilon^2}t\right) = \mathbb{H}_D(t) \quad \textit{for all } t>0.$$
Here C_J is the normalizing constant given by
$$C_J = \frac{2}{\displaystyle\int_{\mathbb{R}^N} J(x)|x_N|^2 dx}.$$

Let us now consider the *spectral heat content* given by
$$\mathbb{Q}_D(t) := \int_D u(x,t)dx$$
for the solution u of the Dirichlet problem
$$\begin{cases} u_t(x,t) = \Delta u(x,t), & (x,t)\in D\times[0,\infty), \\ u(x,t)=0, & (x,t)\in \partial D\times(0,\infty), \\ u(x,0) = \chi_D(x) & x\in D. \end{cases}$$
The following result was given in [7] for smooth bounded domains:
$$\mathbb{Q}_D(t) = |D| - \frac{2}{\sqrt{\pi}}P(D)\sqrt{t} + \frac{1}{2}(N-1)\int_{\partial D}\mathcal{H}_{\partial D}(x)d\mathcal{H}^{N-1}(x)\,t + O(t^{3/2}), \quad t\downarrow 0,$$
where $\mathcal{H}_{\partial D}$ is the mean curvature of ∂D.

In [3] the following nonlocal Dirichlet problem has also been studied
$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy, & (x,t)\in D\times[0,\infty), \\ u(x,t)=0, & (x,t)\in(\mathbb{R}^N\setminus D)\times[0,\infty), \\ u(x,0)=u_0(x) & x\in\mathbb{R}^N. \end{cases} \tag{1.10}$$
Therefore, we can also define
$$\mathbb{Q}_D^J(t) := \int_D u(x,t)dx,$$
being u the solution of (1.10) for datum $u_0=\chi_D$. Observe that also
$$\mathbb{Q}_D^J(0) = |D|.$$

We prove the following result.

Theorem 1.5. *For $\mathbb{Q}_D^J(t)$ the J-spectral heat content of D it holds that*

$$\mathbb{Q}_D^J(t) = |D| - P_J(D)\,t + \frac{1}{2}\int_D \mathcal{H}_{\partial D}^J(x)dx\,t^2 + \frac{1}{2}\int_D\int_D\int_D J(x-y)J(y-z)dzdydx\,t^2 + O(t^3) \qquad \text{as } t \downarrow 0,$$

where $\mathcal{H}_{\partial D}^J(x)$ is the J-mean curvature at x defined by

$$\mathcal{H}_{\partial D}^J(x) := \int_{\mathbb{R}^N} J(x-y)(\chi_{\mathbb{R}^N\setminus D}(y) - \chi_D(y))dy.$$

2. The J-heat content. Proofs of the results

We will denote by Δ_J the nonlocal operator of the equation (1.3), i.e., we let

$$\Delta_J u(x) := \int_{\mathbb{R}^N} J(x-y)(u(y)-u(x))dy.$$

In [3], it is showed that the operator B in $L^2(\Omega)$ with domain $D(B) = L^2(\Omega)$ defined by

$$B(u) = v \iff v(x) = -\Delta_J u(x) \quad \forall\, x \in \Omega$$

is m-completely accretive in $L^2(\Omega)$. Then, this operator B generates a C_0-semigroup $(T_J(t))_{t\geq 0}$ in $L^2(\Omega)$ which solves the Cauchy problem (1.3). With this operator we can describe the J-heat content of a bounded subset $D \subset \mathbb{R}^N$ as

$$\mathbb{H}_D^J(t) = \int_D T_J(t)\chi_D(x)dx.$$

Proposition 2.1. *Given a Lebesgue measurable set $D \subset \mathbb{R}^N$ with finite measure, we have*

$$\mathbb{H}_D^J(t) = \left\| T_J\left(\frac{t}{2}\right)\chi_D \right\|_{L^2}^2.$$

Proof. It is enough to prove that the operators $T_J(t)$ are selfadjoint, since then

$$\left\| T_J\left(\frac{t}{2}\right)\chi_D \right\|_{L^2}^2 = \left\langle T_J\left(\frac{t}{2}\right)\chi_D, T_J\left(\frac{t}{2}\right)\chi_D \right\rangle = \langle T_J(t)\chi_D, \chi_D\rangle = \mathbb{H}_D^J(t).$$

So, let us see that $T_J(t)$ is selfadjoint. Let $\mathcal{F} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ the Fourier-Plancherel transform. We write $\hat{f} := \mathcal{F}(f)$. If $u(t)(x) = u(x,t) := T_J(t)f(x)$, since $u_t(t) = J * u(t) - u(t)$, applying the Fourier-Plancherel transform, we have

$$\hat{u}_t(\xi,t) = (\hat{J}-1)\hat{u}(\xi,t), \tag{2.1}$$

from where it follows that

$$\hat{u}(\xi,t) = e^{(\hat{J}(\xi)-1)t}\widehat{f}(\xi).$$

Therefore, given $f, g \in L^2(\mathbb{R}^N)$, we have

$$\langle T_J(t)f, g\rangle = \langle \mathcal{F}(T_J(t)f), \mathcal{F}(g)\rangle = \langle e^{(\hat{J}(\xi)-1)t}\widehat{f}, \widehat{g}\rangle = \langle \widehat{f}, e^{(\hat{J}(\xi)-1)t}\widehat{g}\rangle = \langle f, T_J(t)g\rangle,$$

as we wanted to show. $\square$

Associated with the non–singular kernel J, there is a nonlocal version of the usual perimeter of a set and a nonlocal concept of mean curvature (see [9], [12]): let $E \subset \mathbb{R}^N$ be a measurable set, the *nonlocal J-perimeter of E* is given by

$$P_J(E) := \int_E \left(\int_{\mathbb{R}^N \setminus E} J(x-y)dy \right) dx,$$

and the *J-mean curvature* at a point x is defined by

$$\mathcal{H}^J_{\partial E}(x) := \int_{\mathbb{R}^N} J(x-y)(\chi_{\mathbb{R}^N\setminus E}(y) - \chi_E(y))dy.$$

Observe that, since $\int_{\mathbb{R}^N} J(x)dx = 1$,

$$|E| = \int_{\mathbb{R}^N}\int_E J(x-y)dydx = \int_E\int_E J(x-y)dydx + \int_{\mathbb{R}^N\setminus E}\int_E J(x-y)dydx,$$

that is,

$$|E| = \int_E\int_E J(x-y)dydx + P_J(E), \tag{2.2}$$

and also

$$\begin{aligned}\int_E \mathcal{H}^J_{\partial E}(x)\, dx &= \int_E \left(1 - 2\int_E J(x-y)dy\right) dx \\ &= |E| - 2\int_E\int_E J(x-y)\, dy\, dx,\end{aligned}$$

from where, on account of (2.2),

$$\int_E \mathcal{H}^J_{\partial E}(x)dx = 2P_J(E) - |E|. \tag{2.3}$$

2.1. **The asymptotic expansion of the J–heat content.**

Proof of Theorem 1.2. From (2.1), we have that, for $u(\cdot,t)=T_J(t)\chi_D$, the Fourier transform verifies the evolution problem

$$\hat{u}_t(\xi,t)=(\hat{J}(\xi)-1)\hat{u}(\xi,t),$$

with the initial condition $\hat{u}(\xi,0)=\widehat{\chi_D}(\xi)$. Hence,

$$\hat{u}(\xi,t)=e^{(\hat{J}(\xi)-1)t}\widehat{\chi_D}(\xi), \tag{2.4}$$

and, using Taylor expansion of the exponential,

$$\begin{aligned}
\hat{u}(\xi,t) &= \sum_{n=0}^{L}(\hat{J}(\xi)-1)^n\widehat{\chi_D}(\xi)\frac{t^n}{n!}+(\hat{J}(\xi)-1)^{L+1}e^{(\hat{J}(\xi)-1)s}\widehat{\chi_D}(\xi)\frac{t^{L+1}}{(L+1)!}\\
&= \sum_{n=0}^{L}\left(\widehat{\chi_D}(\xi)+\sum_{k=1}^{n}\binom{n}{k}(-1)^{n-k}\hat{J}(\xi)^k\widehat{\chi_D}(\xi)\right)\frac{t^n}{n!}\\
&\qquad +(\hat{J}(\xi)-1)^{L+1}e^{(\hat{J}(\xi)-1)s}\widehat{\chi_D}(\xi)\frac{t^{L+1}}{(L+1)!}\\
&= \sum_{n=0}^{L}\left(\widehat{\chi_D}(\xi)+\sum_{k=1}^{n}\binom{n}{k}(-1)^{n-k}\hat{J}(\xi)^k\widehat{\chi_D}(\xi)\right)\frac{t^n}{n!}\\
&\qquad +(\hat{J}(\xi)-1)^{L+1}\hat{u}(\xi,s)\frac{t^{L+1}}{(L+1)!},
\end{aligned}$$

for $0<s<t$. Taking now the inverse Fourier-Plancheral transform and integrating over D we get, for $n\geq 1$,

$$\begin{aligned}
\mathbb{H}_D^J(t) &= \sum_{n=0}^{L}\left((-1)^n|D|+\sum_{k=1}^{n}\binom{n}{k}(-1)^{n-k}\int_D\int_D (J*)^k(x-y)\,dydx\right)\frac{t^n}{n!}\\
&\qquad +\int_D \mathcal{F}^{-1}\left((\hat{J}(\xi)-1)^{L+1}\hat{u}(\xi,s)\right)(x)\,dx\frac{t^{L+1}}{(L+1)!},
\end{aligned}$$

Now, for $n \geq 1$, using (2.2),

$$
\begin{aligned}
(-1)^n|D| + \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \int_D \int_D (J*)^k(x-y)dydx \\
&= (-1)^n|D| + \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \big(|D| - P_{(J*)^k}(D)\big) \\
&= -\sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} P_{(J*)^k}(D).
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\int_D \mathcal{F}^{-1}\left((\hat{J}(\xi)-1)^{L+1}\hat{u}(\xi,s)\right)(x)dx \\
&\qquad = (-1)^{L+1}|D| + \sum_{k=1}^{L+1} \binom{L+1}{k} (-1)^{L+1-k} \int_D (J*)^k * u(x,s)dx.
\end{aligned}
\tag{2.5}
$$

Using the fact that for this problem we have

$$
0 \leq u \leq 1,
$$

this follows from the maximum principle, we get

$$
0 \leq \int_D \big((J*)^k * u\big)(x,t)dx \leq |D| \qquad \forall k \in \mathbb{N}.
$$

Hence, from (2.5) we have that

$$
\left|\int_D \mathcal{F}^{-1}\left((\hat{J}(\xi)-1)^{L+1}\hat{u}(\xi,s)\right)(x)dx\right| \leq \frac{1}{2}\sum_{k=0}^{L+1}\binom{L+1}{k}|D| = 2^L|D|.
$$

From where (1.5) and (1.6) follow.

From (2.4) we also have

$$
\hat{u}(\xi,t) = e^{-t}\widehat{\chi_D}(\xi)e^{t\hat{J}(\xi)} = e^{-t}\sum_{n=0}^{\infty}(\hat{J}(\xi))^n\widehat{\chi_D}(\xi)\frac{t^n}{n!}.
$$

Taking here the inverse Fourier transform we get that

$$
u(x,t) = e^{-t}\sum_{n=0}^{\infty}\mathcal{F}^{-1}\left((\hat{J}(\xi))^n\widehat{\chi_D}(\xi)\right)(x,t)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\int_D (J*)^n(x-y)dy\frac{e^{-t}t^n}{n!},
\tag{2.6}
$$

where $\displaystyle\int_D (J*)^0(x-y)dy = \chi_D(x)$; then, integrating over D we get

$$\mathbb{H}_D^J(t) = \sum_{n=0}^{\infty} \int_D \int_D (J*)^n(x-y)dydx \frac{e^{-t}t^n}{n!},$$

and then (1.7) is proved. □

Remark 2.2.
1. We can also proceed in the following way: Integrating in (1.3) over D, we get

$$\begin{aligned}(\mathbb{H}_D^J)'(t) &= \int_D \int_{\mathbb{R}^N} J(x-y)u(y,t)dydx - \int_D u(x,t)dx \\ &= \int_D \int_{\mathbb{R}^N} J(x-y)u(y,t)dydx - \mathbb{H}_D^J(t),\end{aligned}$$

and hence,

$$\mathbb{H}_D^J(t) + (\mathbb{H}_D^J)'(t) = \int_D \int_{\mathbb{R}^N} J(x-y)u(y,t)dydx. \tag{2.7}$$

Taking the time derivative in (2.7) and using again (2.7), we get

$$\begin{aligned}(\mathbb{H}_D^J)'(t) + (\mathbb{H}_D^J)''(t) &= \int_D \int_{\mathbb{R}^N} J(x-y)u_t(y,t)dydx \\ &= \int_D \int_{\mathbb{R}^N} J(x-y) \left(\int_{\mathbb{R}^N} (J(y-z)u(z,t) - u(y,t))dx \right) dydx \\ &= \int_D \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)J(y-z)u(z,t)dzdydx \\ &\quad - \int_D \int_{\mathbb{R}^N} J(x-y)u(y,t)dydx \\ &= \int_D \big((J*)^2 * u\big)(x,t)dx - \mathbb{H}_D^J(t) - (\mathbb{H}_D^J)'(t),\end{aligned}$$

hence

$$\mathbb{H}_D^J(t) + 2(\mathbb{H}_D^J)'(t) + (\mathbb{H}_D^J)''(t) = \int_D \big((J*)^2 * u\big)(x,t)dx.$$

By induction, it is easy to see that, for any $n \geq 1$,

$$\sum_{k=0}^{n} \binom{n}{k} (\mathbb{H}_D^J)^{(k)}(t) = \int_D \big((J*)^n * u\big)(x,t)dx,$$

which is equivalent to what was obtained before.

2. In particular, we have

$$\mathbb{H}_D^J(0) + 2(\mathbb{H}_D^J)'(0) + (\mathbb{H}_D^J)''(0) = \int_D \int_{\mathbb{R}^N} \int_D J(x-y)J(y-z)dzdydx. \tag{2.8}$$

Then, having in mind (2.3), we get

$$(\mathbb{H}_D^J)''(0) = \int_D \int_{\mathbb{R}^N} \int_D J(x-y)J(y-z)dzdydx + \int_D \mathcal{H}_{\partial D}^J(x)dx. \tag{2.9}$$

Therefore, applying Talor's expansion we obtain

$$\begin{aligned}\mathbb{H}_D^J(t) = |D| - P_J(D)\,t + \frac{1}{2}\int_D \mathcal{H}_{\partial D}^J(x)dx\, t^2 \\ + \frac{1}{2}\int_D \int_{\mathbb{R}^N} \int_D J(x-y)J(y-z)dzdydx\, t^2 + O(t^3) \qquad \text{as } t \downarrow 0.\end{aligned}$$

3. For $n \geq 2$, we can express the coeficients $(\mathbb{H}_D^J)^{(n)}(0)$ using the nonlocal curvature as follows:

$$\begin{aligned}(\mathbb{H}_D^J)^{(n)}(0) = (-1)^n P_J(D) + \\ + \frac{1}{2}\sum_{k=1}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} \left(\int_D \mathcal{H}_{\partial D}^{(J*)^k}(x)dx - \int_D \mathcal{H}_{\partial D}^{(J*)^{(k+1)}}(x)dx \right).\end{aligned} \tag{2.10}$$

Indeed,

$$\begin{aligned}(\mathbb{H}_D^J)^{(n)}(0) &= -\sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} P_{(J*)^k}(D) \\ &= \sum_{k=1}^{n-1} (-1)^{n-1-k} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) P_{(J*)^k}(D) - P_{(J*)^n}(D) \\ &= (-1)^n P_J(D) + \sum_{k=1}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} \Big(P_{(J*)^k}(D) - P_{(J*)^{(k+1)}}(D) \Big),\end{aligned}$$

and hence, using (2.3), we can write

$$P_{(J*)^k}(D) - P_{(J*)^{(k+1)}}(D) = \frac{1}{2}\left(\int_D \mathcal{H}_{\partial D}^{(J*)^k}(x)dx - \int_D \mathcal{H}_{\partial D}^{(J*)^{(k+1)}}(x)dx \right)$$

to get (2.10).

Remark 2.3. Now, let us make a comment on the relation of the semigroup $T_J(t)$ and the operator $\Delta_J \chi_D$.

Observe that for $\phi(x,t) := T_J(t)\chi_D(x) - \chi_D(x)$, by (1.4),

$$\int_{\mathbb{R}^N} \phi(x,t)dx = 0 \quad \forall\, t \geq 0,$$

hence

$$\int_{\mathbb{R}^N} \phi^+(x,t)dx = \int_{\mathbb{R}^N} \phi^-(x,t)dx \quad \forall\, t \geq 0.$$

Therefore, since

$$\phi^+(x,t) = (T_J(t)\chi_D(x) - \chi_D(x))\chi_{\mathbb{R}^N\setminus D}(x),$$

we have

$$\|T_J(t)\chi_D - \chi_D\|_{L^1} = 2\int_{\mathbb{R}^N} \phi^+(x,t)dx = 2\int_{\mathbb{R}^N\setminus D} T_J(t)\chi_D(x)dx.$$

Then, by (1.8), we get:

$$\lim_{t\to 0^+} \frac{1}{2t}\|T_J(t)\chi_D - \chi_D\|_{L^1} = P_J(D), \tag{2.11}$$

note that this formula is similar to the one that holds for the calssical heat content, see [13].

Now, since

$$\begin{aligned} P_J(D) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\Big|\chi_D(y) - \chi_D(x)\Big|dydx \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N} J(x-y)(\chi_D(y) - \chi_D(x))dy\right|dx = \frac{1}{2}\|\Delta_J\chi_D\|_{L^1} \end{aligned}$$

we can write (2.11) as

$$\lim_{t\to 0^+} \frac{1}{t}\|T_J(t)\chi_D - \chi_D\|_{L^1} = \|\Delta_J\chi_D\|_{L^1}. \tag{2.12}$$

Moreover, since the operator $B = \Delta_J$ is the infinitesimal generator of the C_0-semigroup $(T_J(t))_{t\geq 0}$ in $L^2(\mathbb{R}^N)$, we have

$$\Delta_J\chi_D = \lim_{t\to 0^+} \frac{1}{t}\left(T_J(t)\chi_D - \chi_D\right) \quad \text{in} \quad L^2(\mathbb{R}^N). \tag{2.13}$$

Hence, (2.12) and (2.13) imply that also:

$$\Delta_J\chi_D = \lim_{t\to 0^+} \frac{1}{t}\left(T_J(t)\chi_D - \chi_D\right) \quad \text{in} \quad L^1(\mathbb{R}^N).$$

2.2. **A probabilistic interpretation.** Let us see that the formula given in (1.7), that is,

$$\mathbb{H}_D^J(t) = \sum_{k=0}^{+\infty} \left(\int_D \int_D (J*)^k (x-y) dy dx \right) \frac{e^{-t} t^k}{k!},$$

has a probabilistic interpretation.

As mentioned in the Introduction, $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then

$$\int_{\mathbb{R}^N} J(x-s)J(s-y)ds$$

is the probability of jumping from location y to location x in two jumps (passing through a point s the probability $J(x-s)J(s-y)$ and we integrate for $s \in \mathbb{R}^N$);

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-s)J(s-w)J(w-y)dsdw$$

is the probability of jumping from location y to location x in three jumps.

Then, from $J(x-y)$ we obtain the probability of a transition from x to D in k steps as

$$F^{(k)}(x, D) = \int_D (J*)^k (x-y) dy,$$

and we can also set

$$F^{(0)}(x, D) = \chi_D(x).$$

Observe that $F^{(k)}(x, D)$ also determines how may matter of D goes to x after k jumps, even for $k = 0$. Also, if we define

$$f(0) := \int_D \int_D (J*)^0 (x-y) dy dx = |D|,$$

and

$$f(k) := \int_D \int_D (J*)^k (x-y) dy dx, \qquad k = 1, 2, \dots,$$

we have that $f(k)$ is the amount of matter of D remaining in D after k jumps, for any $k \geq 0$. Let us call this jumps as J-jumps.

From (2.6) we have that $u(x, t)$ is the expected value of the amount of matter of D that goes to x when this matter moves by J-jumps and the number of J-jumps up to time t, N_t, follows a Poisson distribution with rate t:

$$u(x, t) = \sum_{k=0}^{+\infty} F^{(k)}(x, D) \frac{e^{-t} t^k}{k!}.$$

It is well known that this function is the transition probability of a pseudo-Poisson process of intensity 1 (see [10, Ch. X]).

Moreover, from (1.7), we have

$$\mathbb{H}_D^J(t) = \sum_{k=0}^{+\infty} f(k) \frac{e^{-t}t^k}{k!} = \mathbb{E}(f(N_t))$$

is the expected value of the amount of matter of D that remains in D when this matter moves by J-jumps and the number of J-jumps up to time t follows a Poisson distribution with rate t.

2.3. **The J-heat loss of D.** Using (2.2), from (1.7) we can get the following expansion for the nonlocal J-heat loss of D in $\mathbb{R}^N$ at t (see [6]):

$$|D| - \mathbb{H}_D^J(t) = \sum_{k=0}^{+\infty} P_{(J*)^k}(D) \frac{e^{-t}t^k}{k!},$$

setting $P_{(J*)^0}(D) = 0$. Using the above notation,

$$|D| - \mathbb{H}_D^J(t) = \mathbb{E}(g(N_t)),$$

where $g(k) := P_{(J*)^k}(D)$.

2.4. **The J–heat content and a non-local isoperimetric inequality.** The following isoperimetric inequality was given in [12]:

$$P_J(B_r) \leq P_J(D) \quad \text{for } B_r \text{ a ball of radius } r > 0 \text{ such that } |B_r| = |D|. \tag{2.14}$$

The proof of this fact uses the Riesz Rearrangement Inequality, and as a consequence of [8, Theorem 1], in the case of radially nonincreasing J having compact support $B_\delta(0)$, the ball of radius $\delta > 0$ centered at 0, the equality in (2.14) holds if and only if D is a ball of radius r, when $r > \frac{\delta}{2}$.

From (2.14) we have the following result.

Corollary 2.4. *Let J be radiallly nonincreasing and having compact support $B_\delta(0)$. For any bounded subset $D \subset \mathbb{R}^N$ with $|D| > \frac{\delta}{2}$ we have*

$$\mathbb{H}_D^J(t) \leq \mathbb{H}_{B_r}^J(t) \quad \textit{for small } t > 0,$$

where B_r is a ball such that $|B_r| = |D|$.

Proof. The result is true when D is a ball of radius r, so let us suppose that this is not the case. Then, on account of the comment after (2.4), we have

$$\mathbb{H}^J_{B_r}(0) = |D| = \mathbb{H}^J_D(0),$$

and

$$(\mathbb{H}^J_{B_r})'(0) = -P_J(B_r) > -P_J(D) = (\mathbb{H}^J_D)'(0).$$

Then, by (1.9), we get

$$\mathbb{H}^J_D(t) < \mathbb{H}^J_{B_r}(t) \quad \text{for small } t > 0,$$

and the result follows. □

As consequence of Corollary 2.4 and Proposition 2.1 we have the following characterization.

Corollary 2.5. *Let J be radially nonincreasing and having compact support $B_\delta(0)$. The Isoperimetric Inequality* (2.14) *is equivalent to the inequality*

$$\|T_J(t)\chi_D\|_{L^2} \le \|T_J(t)\chi_{B_r}\|_{L^2} \quad \textit{for small } t > 0,$$

being B_r a ball such that $|B_r| = |D|$, when $r > \frac{\delta}{2}$.

A similar result for the local case was proved in [14] (see also [4] and [11]).

2.5. **Convergence to the heat content when rescaling the kernel.** For D a subset in $\mathbb{R}^N$ with finite Lebesgue measure, we will call J-heat content of α-intensity of D to

$$H^{J,\alpha}_D(t) = \int_D u(x,t),$$

where u is the solution of

$$\begin{cases} u_t(x,t) = \alpha \displaystyle\int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy, & (x,t) \in \mathbb{R}^N \times [0,\infty), \\ u(x,0) = \chi_D(x), & x \in \mathbb{R}^N. \end{cases}$$

Observe that $H^J_D(t)$ is the J-heat content of 1-intensity of D; also, $H^{J,\alpha}_D(t) = H^J_D(\alpha t)$.

Let us consider the rescaled kernel for $\epsilon > 0$:

$$J_\epsilon(x) = \frac{1}{\epsilon^N} J\left(\frac{x}{\epsilon}\right);$$

and let

$$C_J = \frac{2}{\displaystyle\int_{\mathbb{R}^N} J(x)|x_N|^2 dx}.$$

Let v_ϵ the solution of

$$\begin{cases} (v_\epsilon)_t(x,t) = \dfrac{1}{\epsilon^2}\left[J_\epsilon * v_\epsilon(x,t) - v_\epsilon(x,t)\right], & x \in \mathbb{R}^N,\ t \in [0,\infty), \\ v_\epsilon(x,0) = \chi_D(x), & x \in \mathbb{R}^N. \end{cases}$$

By [3, Theorem 1.30], we have that for every $T > 0$,

$$\lim_{\epsilon \to 0^+} \|v_\epsilon - v\|_{L^\infty(\mathbb{R}^N \times (0,T))} = 0,$$

being v the solution of the heat equation

$$\begin{cases} v_t(x,t) = \dfrac{1}{C_J}\Delta v(x,t), & x \in \mathbb{R}^N,\ t \in [0,\infty), \\ v(x,0) = \chi_D(x), & x \in \mathbb{R}^N. \end{cases}$$

Set now $u(x,t) = v(x, C_J t)$. Then u verifies

$$\begin{cases} u_t(x,t) = \Delta u(x,t), & x \in \mathbb{R}^N,\ t \in [0,\infty), \\ u(x,0) = \chi_D(x), & x \in \mathbb{R}^N. \end{cases}$$

Hence for u_ϵ the solution of problem

$$\begin{cases} (u_\epsilon)_t(x,t) = \frac{C_J}{\epsilon^2}\left(J_\epsilon * u_\epsilon(x,t) - u_\epsilon(x,t)\right), & x \in \mathbb{R}^N,\ t \in [0,\infty), \\ u_\epsilon(x,0) = \chi_D(x), & x \in \mathbb{R}^N, \end{cases}$$

we have that

$$\mathbb{H}_D^{J_\epsilon, \frac{C_J}{\epsilon^2}}(t) = \int_D u_\epsilon(x,t)\,dx = \int_D v_\epsilon(x, C_J t)\,dx,$$

and

$$\lim_{\epsilon \to 0} \int_D v_\epsilon(x, C_J t)\,dx = \int_D v(x, C_J t)\,dx = \int_D u(x,t)\,dx = \mathbb{H}_D(t).$$

Consequently, we have proved the following result (which is a restatement of Theorem 1.4 with the notation introduced in this section):

Theorem 2.6. *For D a subset in $\mathbb{R}^N$ with finite Lebesgue measure, we have*

$$\lim_{\epsilon \to 0^+} \mathbb{H}_D^{J_\epsilon, \frac{C_J}{\epsilon^2}}(t) = \mathbb{H}_D(t) \quad \textit{for all } t > 0.$$

That is, if the jumps are rescaled to occur in a ball of radius ϵ and the intensity of the Poisson process that controls the intensity of the jumps is rescaled to the size $\frac{C_J}{\epsilon^2}$ then we are approaching, for ϵ small, the Gaussian heat content.

Remark 2.7. In [9] (see also [12]) it is shown that, for $E \subset \mathbb{R}^N$ a bounded set of finite perimeter,

$$\lim_{\epsilon \downarrow 0} \frac{C_{J,1}}{\epsilon} P_{J_\epsilon}(E) = P(E), \tag{2.15}$$

where

$$C_{J,1} = \frac{2}{\displaystyle\int_{\mathbb{R}^N} J(x)|x_N|dx}.$$

Observe that

$$(J_\epsilon *)^k = [(J*)^k]_\epsilon.$$

Then, by Theorem 1.2, for $n \geq 1$,

$$(\mathbb{H}_D^{J_\epsilon})^{(n)}(0) = -\sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} P_{[(J*)^k]_\epsilon}(D),$$

and consequently, by (2.15),

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} (\mathbb{H}_D^{J_\epsilon})^{(n)}(0) = c_n\, P(D),$$

where

$$\sum_{k=1}^{n} \binom{n}{k} c_k = \frac{1}{2} \int_{\mathbb{R}^N} (J*)^n(x)|x_N|dx\,,$$

that is,

$$c_n = -\frac{1}{2} \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \int_{\mathbb{R}^N} (J*)^k(x)|x_N|dx\,.$$

3. The spectral heat content

Recall from the introduction that in [3] the following nonlocal Dirichlet problem has been studied

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy, & (x,t) \in D \times [0,\infty), \\ u(x,t) = 0, & (x,t) \in (\mathbb{R}^N \setminus D) \times [0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

and hence we defined

$$\mathbb{Q}_D^J(t) := \int_D u(x,t)dx,$$

being u the solution of (3.1) for datum $u_0 = \chi_D$.

Now, our task is to obtain the asymptotic expansion of $\mathbb{Q}_D^J(t)$.

First, we observe that

$$\mathbb{Q}_D^J(0) = |D|.$$

For the second term in the expansion, is is easy to see that again

$$\mathbb{Q}_D^J(t) + (\mathbb{Q}_D^J)'(t) = \int_D \int_D J(x-y)u(y,t)dydx,$$

$$\mathbb{Q}_D^J(0) + (\mathbb{Q}_D^J)'(0) = \int_D \int_D J(x-y)dydx,$$

and hence

$$(\mathbb{Q}_D^J)'(0) = -P_J(D).$$

Note that the first two terms in the expansion of $\mathbb{Q}_D^J(t)$ and of $\mathbb{H}_D^J(t)$ coincide. Now the expression for the next terms differs from that of the J–heat content. For example, now, instead of (2.8) we have

$$\mathbb{Q}_D^J(0) + 2(\mathbb{Q}_D^J)'(0) + (\mathbb{Q}_D^J)''(0) = \int_D \int_D \int_D J(x-y)J(y-z)dzdydx,$$

Hence,

$$(\mathbb{Q}_D^J)''(0) = \int_D \int_D \int_D J(x-y)J(y-z)dzdydx + 2P_J(D) - |D|,$$

that can be written as

$$(\mathbb{Q}_D^J)''(0) = \int_D \int_D \int_D J(x-y)J(y-z)dzdydx + \int_D \mathcal{H}_{\partial D}^J(x)dx,$$

which different from (2.9) in the term with three integrals.

Gathering this information we now have

$$\begin{aligned}\mathbb{Q}_D^J(t) = |D| - P_J(D)\,t + \frac{1}{2}\int_D \mathcal{H}_{\partial D}^J(x)dx\,t^2 \\ + \frac{1}{2}\int_D \int_D \int_D J(x-y)J(y-z)dzdydx\,t^2 + O(t^3) \qquad \text{as } t \downarrow 0.\end{aligned}$$

This proves Theorem 1.5.

References

[1] L. Acuña Valverde, *Heat content estimates over sets of finite perimeter*, J.Math. Anal. Appl. **441** (2016), 104-120.

[2] L. Acuña Valverde, *Heat content for Stable Processes in Domains of* $\mathbb{R}^d$r , J.Geom Anal. DOI 10.1007/s12220-016-9688-9.

[3] F. Andreu, J. M. Mazón, J. Rossi and J. Toledo, *Nonlocal Diffusion Problems.* Mathematical Surveys and Monographs, vol. 165, AMS, Providence, 2010.

[4] A. Baernstein, *Integral means, univalent functions and circular symmetrization,* Acta Math. **133** (1974), 139–169.

[5] M. van der Berg , *Heat Flow and Perimeter in* $\mathbb{R}^m$. Potential Anal. **39** (2013), 369–387.

[6] M. van der Berg and K. Gitting, *Uniform bounds for the heat content of open ses in Euclidean spaces*. Diff. Geom. Appl. **40** (2015), 67–85.

[7] M. van der Berg and J.F. Le Gall, *Mean curvature and the heat equation*. Math. Z. **215** (1994), 437–464.

[8] A. Burchard, *Cases of Equality in the Riesz Rearrangement Inequality*, Annals of Mathematics, Second Series, Vol. 143, No. 3, 499–527.

[9] H. Brezis, *How to recognize constant functions*. Uspekhi Mat. Nauk 57 (2002) (in Russian). English translation in Russian Math. Surveys 57 (2002), p. 693-708.

[10] W. Feller. *Introduction to probability theory and its applications*, vol. ii pod. 1974.

[11] M. Ledoux, *Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space*, Bull. Sci. Math. **118** (1994), 485–510.

[12] J. M. Mazón, J. D. Rossi and J. Toledo, *Nonlocal Perimeter, Curvature and Minimal Surfaces for measurable sets.* Preprint.

[13] M. Jr., Miranda, D. Pallara, F. Paronetto and M. Preunkert, *Short-time heat flow and functions of bounded variation in* $\mathbb{R}^N$. Ann. Fac. Sci. Toulouse **16**, (2007), 125–145.

[14] M. Preunkert, *A Semigroup version of the isoperimetric inequality.* Semigroup Forum 68, 233245 (2004)

J. M. Mazón: Departament d'Anàlisi Matemàtica, Universitat de València, Dr. Moliner 50, 46100 Burjassot, Spain. mazon@uv.es

J. D. Rossi: Dpto. de Matemáticas, FCEyN, Universidad de Buenos Aires, 1428, Buenos Aires, Argentina. jrossi@dm.uba.ar

J. Toledo: Departament d'Anàlisi Matemàtica, Universitat de València, Dr. Moliner 50, 46100 Burjassot, Spain. toledojj@uv.es