ON EXISTENCE AND NONEXISTENCE IN THE LARGE FOR AN N-DIMENSIONAL SYSTEM OF HEAT EQUATIONS WITH NONTRIVIAL COUPLING AT THE BOUNDARY

Julio D. Rossi

The author studies the behavior of positive solutions of N heat equations, $(u_i)_t = \Delta u_i$, with boundary conditions $\frac{\partial u_i}{\partial \eta} = \ {}_{j=1}^{N}(u_j)^{p_{ij}}$. He finds necessary and sufficient conditions on the matrix $A = (p_{ij})$ (with $A -$Id nonsingular) that guarantee global existence of positive solutions.

I. Introduction.

In this paper we consider positive solutions of the following problem :

$$(1.1) \qquad (u_i)_t = \Delta u_i \qquad \text{in } \Omega \times (0,T)$$

with boundary conditions

$$(1.2) \qquad \frac{\partial u_i}{\partial \eta} = \prod_{j=1}^{N} (u_j)^{p_{ij}} \qquad \text{on } \partial\Omega \times (0,T)$$

and initial data

$$(1.3) \qquad u_i(x,0) = u_{i,0}(x) \qquad \text{in } \Omega$$

$$(i = 1, ..., N)$$

Here Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $\frac{\partial u_i}{\partial \eta}$ is the derivative of u_i in the direction of the outward normal, $A = (p_{ij})$ is a matrix with nonnegative entries such that $A - \mathrm{Id}$ is nonsingular, and the initial data $u_{i,0}$ are $C^2(\overline{\Omega})$ and positive.

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Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, (1428) Buenos Aires, Argentina. Supported by Universidad de Buenos Aires under grant EX117 and CONICET.

The system (1.1)-(1.3) is a simple example of a parabolic system with nontrivial coupling at the boundary. This type of system has been used to model different problems in biology, physics, etc. For example, in certain enzyme reaction diffusion problems we can find systems like (1.1)-(1.3), the boundary terms can represent the interaction at the interface between the enzimes (u_i are the concentrations of the different enzymes) (see [6]).

Under our hypothesis, local existence and uniqueness of a classical solution up to some time t_0 can be found in [1]. In that paper Amann proves that a necessary and sufficient condition for global existence is the boundedness of $\{u_i\}$ in L^∞-norm.

We are interested in global existence or blow-up (for us blow-up will mean finite time blow-up) of the solutions of (1.1)-(1.3). Observe that if the solution is nonglobal then, by the result of [1], there exists $T < +\infty$ such that

$$\limsup_{t \nearrow T} \sum_{i=1}^{N} \|u_i(x,t)\|_{L^\infty(\Omega)} = +\infty$$

Blow-up for nonlinear equations has received a great deal of interest. In particular for this kind of problems the blow-up phenomena is well known in the case of a single equation.

In [4] Levine and Payne prove that positive solutions of

$$(1.4) \qquad \begin{cases} u_t = \Delta u & \Omega \times (0,T) \\ \frac{\partial u}{\partial \eta} = f(u) & \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \Omega \end{cases}$$

blow up for every positive $u_0(x)$ if $f(u) = u^{1+\delta} h(u)$, with $\delta > 0$ and h nondecreasing. In [9] Walter proves that, for f convex, (1.4) has global solutions for every u_0 positive if and only if $\int^{+\infty} \frac{1}{ff'} = +\infty$. Lopez Gomez, Marquez and Wolanski ([5]) prove some blow-up vs. global existence results and localize the blow-up set of radial solutions of (1.4) on the boundary of a ball $B \subset R^n$.

In [7] we prove a blow-up result (without f convex) for f such that $\int^{+\infty} \frac{1}{f} < +\infty$ and localize the blow-up set on the boundary of a general Ω with the condition $\Delta u_0 > 0$. If $f(u) = u^p$ in [3] the blow-up set is localized on the boundary and the asymptotic behaviour near a blow-up point is obtained (see also [2] for the asymptotic behaviour).

In [8] we analyze the following system with nonlinear boundary conditions,

$$(1.5) \qquad \begin{cases} u_t = \Delta u + h(u,v) & v_t = \Delta v + r(u,v) \\ \frac{\partial u}{\partial \eta} = f(v) & \frac{\partial v}{\partial \eta} = g(u) \\ u(x,0) = u_0(x) & v(x,0) = v_0(x) \end{cases}$$

and prove blow-up or global existence results in terms of f and g (with some hypotheses on r and h).

For (1.1) no study has been done previously. In this paper we characterize the blow-up or global existence of a positive solution in terms of the matrix A.

We prove :

Theorem 1. *Let Ω be a bounded smooth domain.*

a) If for some $i = 1, ..., N$, $p_{ii} > 1$ then every positive continuous solution of (1.1)-(1.3) blows up.

b) Suppose $0 \leq p_{ii} \leq 1$. Let $(\alpha_1, ..., \alpha_N)$ be the solution of

$$(A - Id) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

then

b1) If $\min_{1 \leq i \leq N}(\alpha_i) > 0$ every positive continuous solution of (1.1)-(1.3) is global.

b2) If $\min_{1 \leq i \leq N}(\alpha_i) \leq 0$ every positive continuous solution blows up.

Remark 1. -For $N = 1$ we recover a well known result ([4], [9], [7])

-In the case $N = 2$ and $p_{11} = p_{22} = 0$ we have (1.5) with $f(v) = v^{p_{12}}$, $g(u) = u^{p_{21}}$ and $h = r = 0$. We obtain a result presented as an example in [8].

The paper is organized as follows, in Section II we prove a crucial result about a system of ordinary differential equations and in Section III we state a comparison result and we prove Theorem 1.

II. A result on a system of ordinary differential equations.

The following Theorem tells us about the behavior of the positive solutions of

$$\begin{cases} z_i'(s) = \prod_{j=1}^{N} (z_j)^{p_{ij}}(s) \\ z_i(0) = z_{i,0} > 0 \end{cases} \tag{2.1}$$

with all the $p_{ij} \geq 0$ and $0 \leq p_{ii} \leq 1$.

Theorem 2.1. *Let $\{z_i(s)\}$ be a positive solution of (2.1) with $A - Id$ nonsingular and $(\alpha_1, ..., \alpha_N)$ the solution of*

$$(A - Id) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

1) If $\min_{1 \leq i \leq N}(\alpha_i) > 0$ then (2.1) has a global supersolution of the form

$$z_i(s) = c_i(s + s_0)^{\alpha_i}$$

2) If $\min_{1 \leq i \leq N}(\alpha_i) \leq 0$ then every positive solution of (2.1) blows up.

Before the proof we state some Lemmas. First we prove a global existence result which gives the first part of Theorem 2.1.

Lemma 2.1. *If* $\min_{1\le i\le N}(\alpha_i) > 0$ *then (2.1) has a global supersolution of the form*

$$z_i(s) = c_i(s+s_0)^{\alpha_i}$$

Proof. We have to verify that $c_i(s+s_0)^{\alpha_i}$ is a solution of (2.1) with some appropriate positive constants c_i.

We want

$$c_i\alpha_i(s+s_0)^{\alpha_i-1} = \prod_{j=1}^{N}(c_j(s+s_0)^{\alpha_j})^{p_{ij}}$$

So we need

$$\alpha_i - 1 = \sum_{j=1}^{N}\alpha_j p_{ij} \tag{2.2}$$

and

$$\alpha_i c_i = \prod_{j=1}^{N}(c_j)^{p_{ij}} \tag{2.3}$$

(2.2) is a consequence of our choice of the α_i's as a solution of

$$(A - Id)\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_N\end{pmatrix} = \begin{pmatrix}-1\\ -1\\ \vdots\\ -1\end{pmatrix}$$

and (2.3) is equivalent to

$$(A - Id)\begin{pmatrix}\ln c_1\\ \ln c_2\\ \vdots\\ \ln c_N\end{pmatrix} = \begin{pmatrix}\ln(\alpha_1)\\ \ln(\alpha_2)\\ \vdots\\ \ln(\alpha_N)\end{pmatrix}$$

so that it is enough to take $c_i = e^{\beta_i}$ where β_i are the solutions of

$$(A - Id)\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_N\end{pmatrix} = \begin{pmatrix}\ln(\alpha_1)\\ \ln(\alpha_2)\\ \vdots\\ \ln(\alpha_N)\end{pmatrix}$$

(all the c_i's are positive). We choose s_0 large enough so

$$c_i(s_0)^{\alpha_i} > z_{i,0}$$

and hence we obtain the desired supersolution.

Lemma 2.2. *If* $\max_{1\le i\le N}(\alpha_i) < 0$ *then (2.1) have a positive subsolution of the form* $z_i(s) = c_i(s_0 - s)^{\alpha_i}$ *that blows up at an arbitrary time* s_0*. And hence every positive solution of (2.1) blows up.*

__Proof__. The verification is the same as in Lemma 2.1. If we take s_0 large enough we obtain $c_i s_0^{\alpha_i} < z_{i,0}$ and we can use a comparison argument to obtain the blow-up of the solution $\{z_i\}$.

We remark that we may consider subsystems of the form

$$\left\{\begin{array}{l} v'_{i_r}(s) = c\prod_{l=1}^{k}(v_{i_l})^{p_{i_l j_l}} \\ v_{i_r}(0) < z_{i_r,0} \end{array}\right. \tag{2.4}$$

where $\{i_1, ..., i_k\} \subset \{1, ..., N\}$ and $c > 0$.

In fact we may use a comparison argument, using the fact that

$$z'_{i_r}(s) \ge c\prod_{l=1}^{k}(z_{i_l})^{p_{i_l j_l}}$$

(all the z_i are positive and bounded below), to obtain $v_{i_l} < z_{i_l}$. Thus we have the following Lemma,

Lemma 2.3. *If every positive solution of (2.4) blows up the same is true for every positive solution of (2.1).*

We say that N is a submatrix of M if it is obtained by crossing out some of its rows and the corresponding columns. So that every submatrix of $Q = A - \mathrm{Id}$ has nonpositive entries in the diagonal and nonnegative ones off the diagonal.

__Notation__. $[M]_{i_1,....,i_r}$ is the submatrix of M obtained after deleting the rows and columns other than $i_1, ..., i_r$. When we write $[M]_{i_1,...(i_j)..,i_r}$ we mean that the row and column corresponding to the index in between parentheses has been deleted.

We prove the following algebraic Lemmas.

Lemma 2.4. *If for every* $r = 1, ..., k$ *any submatrix* $[Q]_{i_1,...,i_r}$ *of* Q *has*

$$sg(\det[Q]_{i_1,...,i_r}) = (-1)^r$$

then for any $r = 1, .., k$ *any matrix of the form*

$$D = \begin{pmatrix} +......+....+ & \\ + & \\ + & [Q]_{i_1,...,i_r} \\ + & \end{pmatrix} = \begin{pmatrix} d_{11} & d_{1j} \quad d_{1(r+1)} \\ d_{21} & \\ d_{j1} & [Q]_{i_1,...,i_r} \\ d_{(r+1)1} & \end{pmatrix} = (d_{ij})$$

has $sg(\det(D)) = (-1)^r$*. Here the* $+$ *means that the* d_{1j} *and* d_{j1} *are nonnegative.*

__Proof__. By induction in k. The case $k = 1$ is trivial. Now suppose that we have the result for every k less than k_0 and we want to obtain it for k_0. We take a submatrix $[Q]_{i_1,...,i_r}$ of Q and we observe that if $r < k_0$ then $r \le k_0 - 1$ and we conclude, by induction, that the statement is true. So we have to deal with the case $r = k_0$. In this case we compute $\det(D)$

$$\det(D) = \sum_{j=1}^{k_0+1} d_{1j}(-1)^{j+1}\det(D_{1j}) =$$

$$= d_{11}\det([Q]_{i_1,\ldots,i_{k_0}}) + \sum_{j=2}^{k_0+1}(-1)^{j+1}(-1)^{j-2}d_{1j}\det(\overline{D}_{1j})$$

where the D_{1j} are the minors of D. We observe that we can obtain a matrix $\overline{D}_{1j}$ from D_{1j} after $j-2$ transpositions of consecutive rows. Because the entries of $Q = A - ID$ are nonnegatives off the diagonal, $\overline{D}_{1j}$ has the following form,

$$\overline{D}_{1j} = \begin{pmatrix} +\ldots\ldots\ldots\ldots\ldots\ldots+ \\ + \\ +\ [Q]_{i_1,..(i_{j-1})..,i_{k_0}} \\ + \end{pmatrix}$$

With this we can write

$$\det(D) = d_{11}\det([Q]_{i_1,\ldots,i_{k_0}}) + \sum_{j=2}^{k_0+1}(-1)^{j+1}(-1)^{j-2}d_{1j}\det(\overline{D}_{1j})$$

By the inductive hypothesis we know that $\mathrm{sg}(\det(\overline{D}_{1j}) = (-1)^{k_0-1}$. Thus since $\mathrm{sg}(\det([Q]_{i_1,\ldots,i_{k_0}})) = (-1)^{k_0}$ and the d_{1j} are positive, we obtain $\mathrm{sg}(\det(D)) = (-1)^{k_0}$.

Lemma 2.5. *Let Q be a $k \times k$ matrix that has nonpositive entries in the diagonal and nonnegative ones out of the diagonal. If every submatrix $[Q]_{i_1,\ldots,i_r}$ ($1 \le r \le k$) of Q has $sg(\det([Q]_{i_1,\ldots,i_r})) = (-1)^r$ then the sign of the entries of Adj(Q) is $(-1)^{k-1}$ (if $k = 1$, that means that Q is a scalar, sg(Adj(Q))=sg(Q)).*

Proof. We use again induction on k, with $k \ge 2$), in the case $k = 1$ we observe that sg(Adj(Q))=sg(Q). The case $k = 2$ is again trivial. Next we compute $(\overline{q}_{ij}) = \mathrm{Adj}(Q)$

$$\overline{q}_{ij} = (-1)^{i+j}\det(Q_{ij})$$

If $i = j$ then we use the hypothesis and obtain

$$\overline{q}_{ii} = (-1)^{2i}\det(Q_{ii}) = \det([Q]_{1,..(i)..,k})$$

$$\mathrm{sg}(\overline{q}_{ii}) = (-1)^{k-1}$$

If $i > j$ then

$$\overline{q}_{ij} = (-1)^{i+j}\det(Q_{ij}) = (-1)^{(i+j+(j-1)+(i-2))}\det(Q_{ij}^{12})$$

where Q_{ij}^{12} is the matrix obtained from Q_{ij} after $j-1$ transpositions of consecutive rows and $i-2$ transpositions of consecutive columns. So it has the form

$$\begin{pmatrix} +..............+ \\ + \\ + \ [Q]_{1,..(i).(j)..k} \\ + \end{pmatrix}$$

By Lemma 2.4 we obtain

$$\mathrm{sg}(\overline{q}_{ij}) = (-1)\mathrm{sg}(\det(Q_{ij}^{12})) = (-1)^{k-1}$$

The analogous method works for $i < j$.

Proposition 2.1. *If the sign of* $\det(Q)$ *is* $(-1)^{N-1}$ *then every positive solution of (2.4) blows up (and then every positive solution of (2.1) blows up (Lemma 2.3)).*

Proof. By induction in N. The case $N = 1$ is easy, $(p_{11}-1)\alpha_1 = -1$, $\alpha_1 = \frac{-1}{(p_{11}-1)}$ and then $\alpha_1 < 0$ iff $p_{11} > 1$ (we can use Lemmas 2.1 and 2.2 here).

Next we observe that we may assume that the sign of the entries of the matrix $\mathrm{Adj}(Q)$ is $(-1)^{N-1}$. In fact if there exists a submatrix $[Q]_{i_1,\ldots,i_r}$ with the sign of $\det([Q]_{i_1,\ldots,i_r}) = (-1)^{r-1}$ the inductive hypothesis and Lemma 2.3 imply that every positive solution blows up and there is nothing else to prove. If every submatrix $[Q]_{i_1,\ldots,i_r}$ has sign of its determinanat $(-1)^r$ Lemma 2.5 implies that the entries of $\mathrm{Adj}(Q)$ have sign $(-1)^{N-1}$. So that,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = Q^{-1} \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} = \frac{\mathrm{Adj}(Q)^t}{\det(Q)} \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

satisfies $\alpha_i < 0$, $(i = 1, \ldots, N)$.

By Lemma 2.2 every positive solution blows up.

Proof of Theorem 2.1. We observe that if $\min_{1\le i\le N}(\alpha_i) = 0$ then (we assume $\alpha_1 = 0$),

$$(p_{11}-1)\alpha_1 + p_{12}\alpha_2 + \ldots. + p_{1N}\alpha_N = -1$$

is not possible (since all the p_{ij} , $i \neq j$ and α_j are nonnegative and the first term is zero).

So we have only to deal with

$$\min_{1\le i\le N} \alpha_i < 0 \qquad \text{and} \quad \max_{1\le i\le N} \alpha_i > 0$$

Without loss of generality we may assume that the first $\alpha_1, \ldots., \alpha_k$ are negative (< 0) and the rest $\alpha_{k+1}, \ldots., \alpha_N$ are nonnegative (≥ 0). By Lemma 2.3 we may consider the system

$$\text{(2.5)} \qquad \begin{cases} v_i'(s) = c\prod_{j=1}^k (v_j)^{p_{ij}} \\ v_i(0) < z_{i,0} \end{cases}$$

and prove that every positive solution to (2.5) blows up.

Let $Q = (p_{ij} - \mathrm{Id})_{1\le i,j\le k}$. We observe that,

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_k\end{pmatrix} = \frac{\mathrm{Adj}(Q)^t}{\det(Q)}\left\{\begin{pmatrix}-1\\ -1\\ \vdots\\ -1\end{pmatrix} - (p_{ij})\begin{pmatrix}\alpha_{k+1}\\ \alpha_{k+2}\\ \vdots\\ \alpha_N\end{pmatrix}\right\} \tag{2.6}$$

If there is a submatrix $[Q]_{i_1,\ldots,i_r}$ such that $\mathrm{sg}(\det([Q]_{i_1,\ldots,i_r})) = (-1)^{r-1}$ we may consider the system with matrix $[Q]_{i_1,\ldots,i_r}$ instead of Q as system (2.5). By Proposition 2.1 every positive solution blows up and by Lemma 2.3 this will be true for our original system.

If instead every submatrix has $\mathrm{sg}(\det([Q]_{i_1,\ldots,i_r})) = (-1)^r$, Lemma 2.5 implies that the entries of $\mathrm{Adj}(Q)$ have sign $(-1)^{k-1}$ and the relation (2.6) implies that $\mathrm{sg}(\det(Q)) = (-1)^{k-1}$. Again Proposition 2.1 gives the blow-up of positive solutions to (2.5)

III. Proof of Theorem 1.

We first prove a comparison result.

Lemma 3.1. *If $\underline{u}_i$, $\overline{u}_i$ are nonnegative solutions of*

$$(\underline{u}_i)_t \le \Delta \underline{u}_i \qquad (\overline{u}_i)_t \ge \Delta \overline{u}_i$$

$$\frac{\partial \underline{u}_i}{\partial \eta} \le \prod_{j=1}^{N} (\underline{u}_j)^{p_{ij}} \qquad \frac{\partial \overline{u}_i}{\partial \eta} \ge \prod_{j=1}^{N} (\overline{u}_j)^{p_{ij}}$$

with $\underline{u}_i(x,0) < \overline{u}_i(x,0)$, $\underline{u}_i, \overline{u}_i$ continuous up to $t = 0$, $(i = 1, \ldots., N)$.

Then $\underline{u}_i(x,t) < \overline{u}_i(x,t)$.

<u>*Note*</u>. We call $\underline{u}_i$ a subsolution of (1.1)-(1.3) and $\overline{u}_i$ a supersolution.

<u>*Proof*</u>. $w_i = (\overline{u}_i - \underline{u}_i)(x,t)$ satisfies

$$(w_i)_t \ge \Delta w_i$$

$$\frac{\partial w_i}{\partial \eta} \ge \prod_{j=1}^{N} (\overline{u}_j)^{p_{ij}} - \prod_{j=1}^{N} (\underline{u}_j)^{p_{ij}}$$

$$w_i(x,0) > \delta$$

for some $\delta > 0$.

Let T be the supremum of the set $\{t > 0; w_i(t) > \delta/2 \text{ for all } i\ \}$. $T > 0$ because of the initial inequality $w_i(x,0) > \delta$ and the continuity of w_i up to $t = 0$.

At T we have $w_{i_0}(x_0,T) = \delta/2$ for some $1 \le i_0 \le N$ and some $x_0 \in \overline{\Omega}$. Since w_{i_0} is a supercaloric function, its minimum in $Q_T = \{(x,t); x \in \overline{\Omega}, 0 \le t \le T\}$ is attained at the parabolic boundary, $\partial_p Q_T$, but it can not be at $t = 0$ ($w_{i_0} > \delta$) and on the lateral faces of $\partial_p Q_T$ we have

$$\frac{\partial w_{i_0}}{\partial \eta} \geq \prod_{j=1}^{N} (\overline{u}_j)^{p_{i_0 j}} - \prod_{j=1}^{N} (\underline{u}_j)^{p_{i_0 j}} > 0$$

which is a contradiction. Hence such T can not exist and it follows that

$$w_i > \delta/2$$

for every $i = 1, ..., N$, $x \in \overline{\Omega}$, $0 \leq t \leq t_0$ (if $\underline{u}_i, \overline{u}_i$ are all defined up to t_0).

Proof of theorem 1.

We begin by *a)*

Assume that $p_{11} > 1$ then u_1 satisfies

$$\left\{ \begin{array}{l} (u_1)_t = \Delta u_1 \\ \frac{\partial u_1}{\partial \eta} = (u_1)^{p_{11}} \prod_{j=2}^{N} (u_j)^{p_{ij}} \\ u_1(x,0) = u_{1,0}(x) > 0 \end{array} \right. \tag{3.1}$$

By an application of the minimum principle to $\{u_i\}, i = 2, .., N$ we may assume that $u_i \geq c > 0, i = 2, ..., N$ so u_1 is a solution of

$$\left\{ \begin{array}{l} w_t = \Delta w \\ \frac{\partial w}{\partial \eta} \geq c^{N-1} w^{p_{11}} \\ w(x,0) > 0 \end{array} \right. \tag{3.2}$$

And it is well known that every solution of (3.2) blows up in finite time if $p_{11} > 1$ (see for example [8] or [6]). Thus *a)* is proved.

In the rest of the paper we may assume that $0 \leq p_{ii} \leq 1$, (i=1,...,N). In order to prove *b)* the basic idea is to construct a subsolution (or supersolution) of (1.1)-(1.3) that blows up in finite time (or exists globally) and then use the comparison result (Lemma 3.1)

We propose

$$w_i(x,t) = \phi_i(a(x) + b(t)) \tag{3.3}$$

as the desired subsolution (supersolution). Then we compute

$$\left\{ \begin{array}{l} \nabla w_i(x,t) = \phi_i'(s) \nabla a(x) \\ \Delta w_i(x,t) = \phi_i'(s) \Delta a(x) + \phi_i''(s) |\nabla a(x)|^2 \\ (w_i)_t(x,t) = \phi_i'(s) b'(t) \end{array} \right. \tag{3.4}$$

$$s = a(x) + b(t)$$

We choose $\phi_i(s)$ a solution of the O.D.E. system

$$\left\{ \begin{array}{l} z_i'(s) = \prod_{j=1}^N (z_j(s))^{p_{ij}} \\ z_i(0) = z_{i,0} > 0 \end{array} \right. \tag{3.5}$$

(i.e. ϕ_i is a solution of (2.1)).

We observe that in the case *b1)* we can choose $\phi_i(s) = c_i(s+s_0)^{\alpha_i}$ and now we can construct a global supersolution of (1.1)-(1.3) that tends to infinity uniformly as t increases.

We need to satisfy, for $1 \le i \le N$

$$\phi_i'(s)b'(t) \ge \phi_i'(s)\Delta a(x) + \phi_i''(s)|\nabla a(x)|^2 \tag{3.6}$$

$$\phi_i'(s)\frac{\partial a(x)}{\partial \eta} \ge \prod_{j=1}^N (\phi_j(s))^{p_{ij}} \tag{3.7}$$

In order to satisfy (3.7) we take as $a(x)$ a $C^2(\overline{\Omega})$ function such that $\frac{\partial a}{\partial \eta} \ge 1$ at $\partial\Omega$ (for instance a smooth extension of a constant minus the distance to $\partial\Omega$). We can assume that $a(x) > 0$ in $\overline{\Omega}$ (just add a constant).

To satisfy (3.6) we have to choose $b(t)$ such that

$$b'(t) \ge L\left(1 + \frac{\phi_i''(s)}{\phi_i'(s)}\right)$$

where L is a bound of $|\Delta a|$ and $|\nabla a|^2$.

But we observe that

$$\frac{\phi_i''(s)}{\phi_i'(s)} \ge \frac{c}{s+s_0} \qquad , \; c > 0$$

where $c = \min\{\alpha_i\}$. So that

$$\frac{\phi_i''(a(x)+b(t))}{\phi_i'(a(x)+b(t))} \ge \frac{c}{a(x)+b(t)+s_0}$$

and we choose $b(t)$ as a solution of

$$b'(t) = L\left(1 + \frac{c}{b(t)+s_0}\right)$$

Such $b(t)$ is global and tends to infinity and hence $w_i(x,t)$ is a global supersolution of (1.1)-(1.3) that tends to infinity uniformly in $\overline{\Omega}$.

We apply Lemma 3.1 to obtain

$$u_i(x,t) < w_i(x,t+\tau)$$

(provided that $w_i(x,\tau) > u_{i,0}(x)$) and then we deduce that every continuous positive solution of (1.1)-(1.3) is global if *b1)* holds.

To prove *b2)* we observe that under our hypothesis we have that $\phi_i(s)$ blows up in finite time for every positive initial data. We have to choose $a(x)$, $b(t)$ in order to make $w_i(x,t)$ a subsolution of (1.1)-(1.3) with initial data $w_i(x,0) < u_{i0}(x)$. We have to satisfy

$$\phi_i'(s)b'(t) \le \phi_i'(s)\Delta a(x) + \phi_i''(s)|\nabla a|^2 \tag{3.8}$$

$$\phi_i'(s)\frac{\partial a}{\partial \eta} \le \prod_{j=1}^{N}(\phi_j(s))^{p_{ij}} \tag{3.9}$$

We take $b(t) = \varepsilon t$ and $a(x) = \delta \sum_{i=1}^{N}(x_i)^2 = \delta|x|^2$. Then, if δ is small enough (3.9) holds because of our choise of ϕ_i as solutions of (3.5).

Now we observe that $\frac{\phi_i''(s)}{\phi_i'(s)} \ge 0$ and therefore it is enough to choose ε such that

$$\varepsilon = b'(t) \le \Delta a(x) = 2\delta$$

(In particular (3.8) holds if $\varepsilon = 2\delta$)

Let us observe that if δ is small enough ϕ_i is defined at $s = a(x)$ for every $x \in \Omega$. Also if $\phi_i(0)$ is small we have, $w_i(x,0) < u_{i,0}(x)$ so that we can use Lemma 3.1 and obtain

$$w_i(x,t) < u_i(x,t)$$

To conclude with *b2)* of Theorem 1 we observe that since $b(t) = \varepsilon t$ and $\phi_i(s)$ blows up in finite time, $w_i(x,t)$ blows up and hence $u_i(x,t)$ cannot be global.

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