BLOW-UP VS. GLOBAL EXISTENCE FOR QUASILINEAR PARABOLIC SYSTEMS WITH A NONLINEAR BOUNDARY CONDITION

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We study the behaviour of positive solutions of the system

$$u_t = \mathrm{div}(a(u)\nabla u) + f(u,v) \qquad v_t = \mathrm{div}(b(v)\nabla v) + g(u,v)$$

in Ω a bounded domain with the boundary conditions $\frac{\partial u}{\partial \eta} = r(u,v)$, $\frac{\partial v}{\partial \eta} = s(u,v)$ on $\partial\Omega$ and the initial data (u_0, v_0). We find conditions on the functions a, b, f, g, r, s that guarantee the global existence (or finite time blow-up) of positive solutions for every (u_0, v_0).

I. Introduction.

Let Ω be a bounded domain in ^{n} with smooth boundary $\partial\Omega$. In this paper we consider positive solutions of the following system :

$$(1.1) \qquad \begin{cases} u_t = \mathrm{div}(a(u)\nabla u) + f(u,v) & \text{in } \Omega \times (0,T) \\ v_t = \mathrm{div}(b(v)\nabla v) + g(u,v) \end{cases}$$

where $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are positive C^2 functions nondecreasing in each variable and $a(\cdot), b(\cdot)$ are positive $(a \geq c > 0, b \geq c > 0)$, nondecreasing and C^2.

With boundary conditions

$$(1.2) \qquad \begin{cases} \frac{\partial u}{\partial \eta} = r(u,v) & \text{on } \partial\Omega \times (0,T) \\ \frac{\partial v}{\partial \eta} = s(u,v) \end{cases}$$

where $r(\cdot,\cdot), s(\cdot,\cdot)$ are positive , nondecreasing in each variable and C^2 .

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And initial data

$$
(1.3) \qquad \begin{cases} u(x,0) = u_0(x) & \text{in } \Omega \\ v(x,0) = v_0(x) \end{cases}
$$

u_0, v_0 are $C^2(\overline{\Omega})$, positive, real functions.

Problem (1.1)-(1.3) have been formulated from physical models arising in various fields of the applied sciences. For example it can be viewed as a heat conduction problem with nonlinear diffusivity, source and a nonlinear radiation law coupling on the boundary of the material body.

Local in time existence and uniqueness of positive classical solutions was proved by Amann ([1]) , who gives a necessary and sufficient condition for global existence, mainly the boundedness of (u, v) in L^∞-norm in domains of the form $\overline{\Omega} \times [0, T)$.

We are interested in global existence of the solutions of (1.1)-(1.3). Observe that if the solution is nonglobal then, by the result of ([1]), there exist $T < +\infty$ such that

$$
\limsup_{t \nearrow T} (\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)}) = +\infty
$$

In this case we say that the solution has finite time blow-up.

Blow-up for nonlinear equations has deserved a great deal of interest. In particular for this kind of problems the blow-up phenomena is well known in the case of a single equation.

In [7] Levine and Payne prove that a positive solution of

$$
(1.4) \qquad \begin{cases} u_t = \Delta u & \Omega \times (0,T) \\ \frac{\partial u}{\partial \eta} = f(u) & \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \Omega \end{cases}
$$

blows-up for every positive $u_0(x)$ if $f(u) = u^{1+\delta} h(u)$, with $\delta > 0$ and h nondecreasing. In [13] Walter proves that, for f convex, (1.4) has global solutions for every u_0 positive if and only if $\int^{+\infty} \frac{1}{ff'} = +\infty$. Lopez Gomez, Marquez and Wolanski [8] prove some blow-up vs. global existence results and localize the blow-up set of radial solutions of (1.4) on the boundary of a ball $B \subset {}^n$. In [10] Rial and Rossi proves a blow-up result for f such that $\int^{+\infty} \frac{1}{f} < +\infty$ (without the convexity hypotheses) and in the case that f is convex they localize the blow-up set on the boundary of a general Ω if $\Delta u_0 > 0$. If $f(u) = u^p$ $(p > 1)$ in [2] the blow-up set is also proved to be localized on the boundary and the asymptotic behavior near a blow-up point is obtained (see also [6] for the asymptotic behavior).

For results about blow-up vs. global existence of solutions of parabolic systems we refer among others to [3] , [4] , [5] , [11] and [12].

To our knowledge , no general study was available for (1.1) prior to this work .

The main Theorem presented here shows, as was seen before (see [11] and [12]), that the blow-up of solutions of (1.1)-(1.3) is related to the behavior of positive solutions of the ordinary system

$$
(1.5) \qquad \begin{cases} \varphi'(\sigma) = r(\varphi(\sigma), \psi(\sigma)) \\ \psi'(\sigma) = s(\varphi(\sigma), \psi(\sigma)) \\ \varphi(0) = \varphi_0 \\ \psi(0) = \psi_0 \end{cases}
$$

In fact we prove :

Theorem 1. *a) If every positive solution of (1.5) blows-up then every positive solution of (1.1)-(1.3) blows-up.*
b)Suppose that (1.5) has global positive solutions.

Also suppose that $\frac{F(\sigma)}{\varphi'(\sigma)}, \frac{G(\sigma)}{\psi'(\sigma)}$ *are monotone increasing or decreasing simultaneously, where the functions* F *and* G *are given by*

$$\{b(\psi(\sigma))\psi'(\sigma) + (b(\psi(\sigma))\psi'(\sigma))' + g(\varphi(\sigma), \psi(\sigma))\} = G(\sigma)$$

$$\{a(\varphi(\sigma))\varphi'(\sigma) + (a(\varphi(\sigma))\varphi'(\sigma))' + f(\varphi(\sigma), \psi(\sigma))\} = F(\sigma)$$

Then it holds,

b1) If

$$\int^{+\infty} \frac{1}{\min\left\{\frac{F(\sigma)}{\varphi'(\sigma)}, \frac{G(\sigma)}{\psi'(\sigma)}\right\}}\, d\sigma < +\infty$$

then every positive solution of (1.1)-(1.3) blows-up.

b2) If

$$\int^{+\infty} \frac{1}{\max\left\{\frac{F(\sigma)}{\varphi'(\sigma)}, \frac{G(\sigma)}{\psi'(\sigma)}\right\}}\, d\sigma = +\infty$$

then every positive solution of (1.1)-(1.3) is global.

Note 1. The monotonicity required in b) can be replaced by

b'1) If F *and* G *are increasing and for some* $k > 0$

$$\int^{+\infty} \frac{1}{\min\left\{\frac{F(\sigma)}{\varphi'(\sigma+k)}, \frac{G(\sigma)}{\psi'(\sigma+k)}\right\}}\, d\sigma < +\infty$$

then the conclusion of b1) holds.

b'2) If F *and* G *are increasing and for every* $d > 0$

$$\int^{+\infty} \frac{1}{\max\left\{\frac{F(\sigma+d)}{\varphi'(\sigma)}, \frac{G(\sigma+d)}{\psi'(\sigma)}\right\}}\, d\sigma = +\infty$$

then the conclusion of b2) holds.

The proof of this Note 1 follows easily by using arguments similar to the ones in the proof of Theorem 1 and is left to the reader.

This Theorem applies in several examples, giving conditions for global existence (or finite time blow-up) that are easy to check.

In [11] one of the authors examines the existence of global positive solutions of

$$(1.6) \qquad \begin{cases} (u_i)_t = \Delta u_i \\ \frac{\partial u_i}{\partial \eta} = \prod_{j=1}^{N} (u_j)^{p_{ij}} \end{cases}$$

and obtains blow-up vs. global existence results in terms of the matrix $P = (p_{ij})$.

The natural extension of (1.6) presented here as a particular case of (1.1)-(1.3) is the problem of global existence of positive solutions of the following system:

$$(1.7) \qquad \begin{cases} u_t = \operatorname{div}((u)^{(n-1)}\nabla u) + u^{q_{11}}v^{q_{12}} & \text{in } \Omega \times (0,T) \\ v_t = \operatorname{div}((v)^{(m-1)}\nabla v) + u^{q_{21}}v^{q_{22}} \end{cases}$$

$$(1.8) \qquad \begin{cases} \frac{\partial u}{\partial \eta} = u^{p_{11}}v^{p_{12}} & \text{on } \partial\Omega \times (0,T) \\ \frac{\partial v}{\partial \eta} = u^{p_{21}}v^{p_{22}} \end{cases}$$

$$(1.9) \qquad \begin{cases} u(x,0) = u_0(x) & \text{in } \Omega \\ v(x,0) = v_0(x) \end{cases}$$

where p_{ij} and q_{ij} are nonnegative and $n, m \geq 1$. We are interested in the coupled case, so we also suppose that the matrix $P = (p_{ij})$ is strictly cooperative (i.e. $p_{21} \neq 0$ and $p_{12} \neq 0$).

To use Theorem 1 we need a result that tells us about the behavior of the positive solutions of the system of ordinary differential equations (1.5)

$$(1.10) \qquad \begin{cases} z_1' = (z_1)^{p_{11}}(z_2)^{p_{12}} \\ z_2' = (z_1)^{p_{21}}(z_2)^{p_{22}} \\ z_1(0) = z_{1,0} > 0 \qquad z_2(0) = z_{2,0} > 0 \end{cases}$$

with all the $p_{ij} \geq 0$ and $p_{12} > 0$, $p_{21} > 0$. More precisely we prove:

Theorem 2. *Let $\{z_i(s)\}$ be a positive solution of (1.10)*

1) If $p_{ii} > 1$ for some $1 \leq i \leq 2$ then every positive solution of (1.10) blows-up.

2) Assume that all the $p_{ii} \leq 1$, and that $P - Id$ is nonsingular. Let (α_1, α_2) be the solution of

$$(P - Id)\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

(observe that α_1 and α_2 have the same sign).

2.1) If $(\alpha_i) < 0$ then every positive solution of (1.10) blows-up.

2.2) If $(\alpha_i) > 0$ then (1.10) has a global solution of the form

$$z_i(s) = c_i(s + s_0)^{\alpha_i}$$

3) Assume that all the $p_{ii} \leq 1$, and that $P - Id$ is singular and strictly cooperative (this is $p_{21} \neq 0$ and $p_{12} \neq 0$). Then there exists a vector (β_1, β_2), $\beta_i > 0$, wich is a solution of

$$(P - Id)\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and (1.10) has global solutions of the form

$$z_i(s) = c_i e^{\beta_i s}$$

Then we can obtain as an easy corollary of Theorem 1 the following result:

Theorem 3.
a)If the positive solutions of (1.10) blows-up (parts 1) and 2.1) of Theorem 2) then also every positive solution of (1.7)-(1.9) blows-up.

b)Suppose that we are under the hypothesis of part 2.2 of Theorem 2 (the solutions of (1.10) are global). Let

$$M_1 = \max\{\alpha_1(n-1), (\alpha_1(q_{11}-1)+\alpha_2 q_{12})+1\}$$
$$M_2 = \max\{\alpha_2(m-1), (\alpha_1 q_{21}+\alpha_2(q_{22}-1))+1\}$$

b1) If $M_1 > 1$ and $M_2 > 1$ then every solution of (1.7)-(1.9) blows-up.
b2) If $M_1 \le 1$ and $M_2 \le 1$ then every solution of (1.7)-(1.9) is global.

c)Suppose that the hypothesis of part 3) of Theorem 2 is true. Let

$$K_1 = \max\{\beta_1(n-1), (\beta_1(q_{11}-1)+\beta_2 q_{12})\}$$
$$K_2 = \max\{\beta_2(m-1), (\beta_1 q_{21}+\beta_2(q_{22}-1))\}$$

c1) If $K_1 > 0$ and $K_2 > 0$ then every solution of (1.7)-(1.9) blows-up.
c2) If $K_1 \le 0$ and $K_2 \le 0$ then every solution of (1.7)-(1.9) is global.

We may apply Theorem 3 to the following problem: suppose that the p_{ij} are fixed and such that case *b)* or case *c)* holds, $m = n = 1$ and the q_{ij} depends on a parameter r ($q_{ij} = q_{ij}(r)$ are increasing nonnegative functions). Then there holds

Theorem 4.
Consider the problem (1.7)-(1.9) with the p_{ij} such that case b) or case c) of Theorem 3 holds, $m = n = 1$, $q_{ij}(r)$ increasing with r. There exists a critical value $r_0 \in [0, +\infty]$ such that if $r < r_0$ the solutions are global and if $r > r_0$ every solution has a finite blow-up time.

Now we apply Theorem 3 to obtain upper and lower bounds for r_0. If we are in case *b)* we define

$$\Gamma_1(r) = \alpha_1(q_{11}(r)-1)+\alpha_2 q_{12}(r)$$
$$\Gamma_2(r) = \alpha_1 q_{21}(r)+\alpha_2(q_{22}(r)-1)$$

and in case *c)*

$$\Gamma_1(r) = \beta_1(q_{11}(r)-1)+\beta_2 q_{12}(r)$$
$$\Gamma_2(r) = \beta_1 q_{21}(r)+\beta_2(q_{22}(r)-1)$$

Γ_i are increasing fuctions of r and if we define

$$\underline{r} = \sup\{r/\Gamma_1(r) \le 0, \Gamma_2(r) \le 0\}$$
$$\overline{r} = \inf\{r/\Gamma_1(r) > 0, \Gamma_2(r) > 0\}$$

then $\underline{r} \le r_0 \le \overline{r}$.

Examples : (we suppose that the p_{ij} are in case *b)*)
1) If $q_{11} = q_{22} = r$, $q_{21} = q_{12} = 0$, then the critical value r_0 is 1.
2)If $q_{11} = q_{22} = q_{21} = q_{12} = r$, then $\frac{\alpha_1}{\alpha_1+\alpha_2} \le r_0 \le \frac{\alpha_2}{\alpha_1+\alpha_2}$

If $r = s$, $a = b$ and $f = g$ in the system (1.1)-(1.3) we can reduce the problem to a single equation. We obtain the following Theorem for a scalar equation as a consequence of Theorem 1.

Theorem 5. *Let $f>0$, $a \geq c>0$ and $r>0$ be C^2 and nondecreasing. Let u be a positive solution of the problem*

$$\begin{cases} u_t = \operatorname{div}(a(u)\nabla u) + f(u) \\ \frac{\partial u}{\partial \eta} = r(u) \end{cases}$$

Let φ be a positive solution of $\varphi'(\sigma) = r(\varphi(\sigma))$. Now Theorem 1 says that

a) If $\int^{+\infty} \frac{1}{r} < +\infty$, φ blows-up, and then the solution u also blows-up.

b) Suppose that $\int^{+\infty} \frac{1}{r} = +\infty$, then φ is global, and that

$$a(s)\{1 + r'(s)\} + a'(s)r(s) + \frac{f(s)}{r(s)} = F(s)$$

is monotone increasing or decreasing. Then the existence of global solutions depends on the convergence of the integral

$$\int^{+\infty} \frac{1}{F(s)}\, ds$$

As a Corollary we obtain

Corollary. *If $a(s) = s^{m-1}$, $f(s) = s^q$ and $r(s) = s^p$ $(m \geq 1,\, p, q \geq 0)$ in Theorem 5 we obtain*

a) If $p > 1$ then the solution u blow-up.

b) Suppose that $p < 1$, let $M = \max\{\frac{(m-1)}{(1-p)}, \frac{(q-1)}{(1-p)} + 1\}$

b1) If $M > 1$ then every positive solution u has finite time blow-up.

b2) If $M \leq 1$ then every positive solution u is global.

c) Suppose that $p = 1$, let $K = \max\{(m-1), (q-1)\}$

c1) If $K > 0$ then every positive solution u has finite time blow-up.

c2) If $K \leq 0$ then every positive solution u is global.

In the rest of the paper we prove Theorem 1 (with some examples), Theorem 2 and obtain, as a corollary, Theorem 3. Finally we prove Theorem 4.

II. Proof of the Theorems.

Def 1. Let $\varepsilon > 0$. If $(\overline{u}, \overline{v})$ is a classical solution of

$$\begin{cases} \overline{u}_t \geq \operatorname{div}(a(\overline{u})\nabla\overline{u})) + f(\overline{u}, \overline{v}) + \varepsilon \\ \overline{v}_t \geq \operatorname{div}(b(\overline{v})\nabla\overline{v})) + g(\overline{u}, \overline{v}) + \varepsilon \end{cases} \tag{2.1}$$

$$\begin{cases} \frac{\partial \overline{u}}{\partial \eta} \geq r(\overline{u}, \overline{v}) \\ \frac{\partial \overline{v}}{\partial \eta} \geq s(\overline{u}, \overline{v}) \end{cases} \tag{2.2}$$

$$(2.3) \qquad \begin{cases} \overline{u}(x,0) = \overline{u}_0(x) \\ \overline{v}(x,0) = \overline{v}_0(x) \end{cases}$$

we call it an $\varepsilon - \textit{supersolution}$ of (1.1)-(1.3).

Def 2. Let $\varepsilon > 0$. If $(\underline{u}, \underline{v})$ is a classical solution of

$$(2.4) \qquad \begin{cases} \underline{u}_t \leq \operatorname{div}(a(\underline{u})\nabla \underline{u})) + f(\underline{u}, \underline{v}) - \varepsilon \\ \underline{v}_t \leq \operatorname{div}(b(\underline{v})\nabla \underline{v})) + g(\underline{u}, \underline{v}) - \varepsilon \end{cases}$$

$$(2.5) \qquad \begin{cases} \frac{\partial \underline{u}}{\partial \eta} \leq r(\underline{u}, \underline{v}) \\ \frac{\partial \underline{v}}{\partial \eta} \leq s(\underline{u}, \underline{v}) \end{cases}$$

$$(2.6) \qquad \begin{cases} \underline{u}(x,0) = \underline{u}_0(x) \\ \underline{v}(x,0) = \underline{v}_0(x) \end{cases}$$

we call it an $\varepsilon - \textit{subsolution}$ of (1.1)-(1.3).

The followings comparison Lemmas justify the preceding definitions

Lemma 2.1. *If an $\varepsilon-$supersolution $(\overline{u}, \overline{v})$ verifies*

$$(2.7) \qquad \overline{u}_0(x) > u_0(x) \qquad \overline{v}_0(x) > v_0(x)$$

then

$$\overline{u}(x,t) > u(x,t) \qquad \overline{v}(x,t) > v(x,t)$$

(as long as they are both defined)

Proof. Suppose that there exist a time τ such that $\overline{u}(x,\tau) \leq u(x,\tau)$ for certain $x \in \Omega$.

Let t_0 be the minimum of the following set

$$\{t / \overline{u}(x(t),t) \leq u(x(t),t) \text{ or } \overline{v}(x(t),t) \leq v(x(t),t) \text{ for some } x(t) \in \overline{\Omega}\}$$

We observe that $t_0 > 0$ because of (2.7) and the continuity of $\overline{u}, \overline{v}, u$ and v up to $t = 0$.

Without loss of generality we may assume that at $(x(t_0), t_0)$, $\overline{u}(x(t_0), t_0) = u(x(t_0), t_0)$, and therefore

$$(\overline{u} - u)(x(t_0), t_0) = \min_{0<t<t_0} (\overline{u} - u)(t)$$

Now we observe that $x(t_0)$ can not belong to $\partial\Omega$ because of the strong maximun principle and the fact that,

$$\frac{\partial(\overline{u} - u)}{\partial \eta}(x(t_0), t_0) \geq (r(\overline{u}, \overline{v}) - r(u,v))(x(t_0), t_0) \geq 0$$

($(\overline{u} - u)$ is not constant).

And if $x(t_0) \in \Omega$ then substracting (1.1) from (2.1)

$$\begin{aligned} (\overline{u} - u)_t(x(t_0), t_0) &\geq (\operatorname{div}(a(\overline{u})\nabla\overline{u}) - \operatorname{div}(a(u)\nabla u) + f(\overline{u}, \overline{v}) - f(u,v) + \varepsilon)(x(t_0), t_0) \\ &\geq (a(\overline{u})\Delta\overline{u} - a(u)\Delta u + a'(\overline{u}) \mid \nabla\overline{u} \mid^2 - a'(u) \mid \nabla u \mid^2 + \varepsilon)(x(t_0), t_0) \end{aligned}$$

But $\nabla\overline{u}(x(t_0), t_0) = \nabla u(x(t_0), t_0)$ and $\Delta\overline{u}(x(t_0), t_0) \geq \Delta u(x(t_0), t_0)$ so that $(\overline{u} - u)_t(x(t_0), t_0) \geq \varepsilon$, which is a contradiction.

So that τ can not exist and the conclusion follows.

Lemma 2.2. *If* $(\underline{u}, \underline{v})$ *is an* $\varepsilon-$*subsolution and*

$$\underline{u}_0(x) < u_0(x) \qquad \underline{v}_0(x) < v_0(x) \tag{2.8}$$

then

$$\underline{u}(x,t) < u(x,t) \qquad \underline{v}(x,t) < v(x,t)$$

(as long as they are defined)

Proof. It follows by the same arguments used in Lemma 2.1.

Proof of Theorem 1.

The basic idea is to construct an $\varepsilon-$subsolution (or $\varepsilon-$supersolution) of (1.1)-(1.3) that blows-up in finite time (or exists globally) and then use the previous Lemmas 2.1 and 2.2.

We propose as the desired $\varepsilon-$subsolution ($\varepsilon-$supersolution)

$$\begin{cases} w(x,t) = \varphi(\alpha(x) + \beta(t)) \\ z(x,t) = \psi(\alpha(x) + \beta(t)) \end{cases} \tag{2.9}$$

where the pair (φ, ψ) is a solution of the ODE system (1.6). Then we compute

$$\begin{cases} w_t(x,t) = \varphi'(\sigma)\beta'(t) \\ z_t(x,t) = \psi'(\sigma)\beta'(t) \end{cases} \tag{2.10}$$

$$\begin{cases} \frac{\partial w}{\partial \eta}(x,t) = \varphi'(\sigma)\frac{\partial \alpha}{\partial \eta}(x) \\ \frac{\partial z}{\partial \eta}(x,t) = \psi'(\sigma)\frac{\partial \alpha}{\partial \eta}(x) \end{cases} \tag{2.11}$$

$$\begin{cases} \operatorname{div}(a(w)\nabla w) = a'(\varphi(\sigma))(\varphi'(\sigma))^2 \mid \nabla\alpha(x) \mid^2 \\ +a(\varphi(\sigma)) \left\{ \varphi'(\sigma)\Delta\alpha(x) + \varphi''(\sigma) \mid \nabla\alpha(x) \mid^2 \right\} \end{cases} \tag{2.12}$$

$$\begin{cases} \operatorname{div}(b(z)\nabla z) = b'(\psi(\sigma))(\psi'(\sigma))^2 \mid \nabla\alpha(x) \mid^2 \\ +b(\psi(\sigma)) \left\{ \psi'(\sigma)\Delta\alpha(x) + \psi''(\sigma) \mid \nabla\alpha(x) \mid^2 \right\} \end{cases} \tag{2.13}$$

where $\sigma = \alpha(x) + \beta(t)$.

We begin by *a)*, so we assume that (φ, ψ) blows-up at a finite time T . We have to choose $\alpha(\cdot)$, $\beta(\cdot)$ and ε in order to make (w, z) an $\varepsilon-$subsolution of (1.1)-(1.3) that verifies (2.8).

We take $\beta(t) = \kappa t$ and $\alpha(x) = \delta\|x - x_0\|^2$ ($x_0 \notin \overline{\Omega}$). By (2.11) and recalling that (φ, ψ) is a solution of (1.6), is easy to see that, choosing δ small enough, (2.5) holds. Now, taking (φ_0, ψ_0) and δ small we ensure (2.8).

In order to verify (2.4), we notice that by using (2.10), (2.12) and (2.13) it is sufficient to choose ε and κ such that

$$\kappa + \frac{\varepsilon}{\varphi'(0)} \leq 2c\delta n$$

$$\kappa + \frac{\varepsilon}{\psi'(0)} \leq 2c\delta n$$

where c is such that $a \geq c > 0$, $b \geq c > 0$ and n is the dimension of the space.

So we have an $\varepsilon-$subsolution that blows-up because of our hypothesis on (φ, ψ) and the fact that we can choose $\kappa > 0$. This complete the proof of a).

b1) Again we want to choose $\alpha(\cdot)$, $\beta(\cdot)$ and ε in order to obtain a subsolution (w, z). As before we take $\alpha(x) = \delta\|x - x_0\|^2$ and we can choose δ, φ_0, ψ_0 small enough to verify (2.5) and (2.8). In order to satisfy (2.4) it is sufficient that β verifies (recall (2.12)-(2.13))

$$\beta'(t) \leq \frac{C\left\{a'(\varphi(\sigma))(\varphi'(\sigma)^2 + a(\varphi(\sigma))\varphi'(\sigma) + a(\varphi(\sigma))\varphi''(\sigma) + f(\varphi(\sigma), \psi(\sigma))\right\} - \varepsilon}{\varphi'(\sigma)}$$

and also

$$\beta'(t) \leq \frac{C\left\{b'(\psi(\sigma))(\psi'(\sigma)^2 + b(\psi(\sigma))\psi'(\sigma) + b(\psi(\sigma))\psi''(\sigma) + g(\varphi(\sigma), \psi(\sigma))\right\} - \varepsilon}{\psi'(\sigma)}$$

We observe that the hypothesis *b1)* and the monotonicity assumption imply that $\frac{F}{\varphi'}$ and $\frac{G}{\psi'}$ must be increasing and then we can take $\beta(t)$ a positive increasing function such that

$$\beta'(t) = \min\left\{\frac{CF(\beta(t))}{\varphi'(\beta(t))} - \varepsilon_1, \frac{CG(\beta(t))}{\psi'(\beta(t))} - \varepsilon_2\right\}$$

where $\varepsilon_1 = \frac{\varepsilon}{\varphi'(0)}$ and $\varepsilon_2 = \frac{\varepsilon}{\psi'(0)}$. If *b1)* holds then $\beta(t)$ blows-up and hence, as the functions φ and ψ are increasing and tends to infinity we obtain the result.

b2) Now we look for global $\varepsilon-$supersolutions. We choose $\alpha(x)$ a C^2 function such that $\frac{\partial\alpha}{\partial\eta} \geq 1$ at $\partial\Omega$ (for instance a smooth extension of the distance to $\partial\Omega$). We can assume that $\alpha(x) > 0$ in $\overline{\Omega}$ (just add a constant). With this α (2.2) holds. To satisfy (2.7) it sufices to take φ_0 and ψ_0 big enough. It rests to choose $\beta(t)$ as a solution of

$$\beta'(t) = L\max\left\{\frac{F(\beta(t+k))}{\varphi'(\beta(t+k))} + \varepsilon_1, \frac{G(\beta(t+k)}{\psi'(\beta(t+k))} + \varepsilon_2\right\}$$

where $k = 0$ or $k = \max(\alpha)$ depending on the monotonicity of $\frac{F}{\varphi'}$ and $\frac{G}{\psi'}$, $L = \max\{| \nabla\alpha |^2, \Delta\alpha, 1\}$, ε_1, ε_2 as before. We observe that $\beta(t)$ is global because of our hypothesis.

Examples. 1- If $r(u,v) = s(u,v) = 1$ then $\varphi(\sigma) = \psi(\sigma) = \sigma$. Then if a , b are convex and $b'(\sigma) + b(\sigma) + g(\sigma,\sigma) \geq a'(\sigma) + a(\sigma) + f(\sigma,\sigma)$ for every $\sigma \geq \sigma_0$ we obtain that

b1). If $\int^{+\infty} \frac{1}{a'(\sigma)+a(\sigma)+f(\sigma,\sigma)} d\,\sigma < +\infty$ then (u,v) blows-up.

b2). If $\int^{+\infty} \frac{1}{b'(\sigma)+b(\sigma)+g(\sigma,\sigma)} d\,\sigma = +\infty$ then (u,v) is global.

2- If $r(u,v) = u$, $s(u,v) = v$ then we may take $\varphi(\sigma) = \psi(\sigma) = e^{\sigma}$. And if we choose $a = b = 1$ then if $\frac{g(s,s)}{s}$ and $\frac{f(s,s)}{s}$ are increasing $f(s,s) \geq g(s,s)$ for every s big enough, the existence of global solutions of (1.1)-(1.3) is guaranted by $\int^{+\infty} \frac{1}{f(s,s)}\, ds = +\infty$ and if $\int^{+\infty} \frac{1}{g(s,s)}\, ds < +\infty$ every solution of (1.1)-(1.3) blows-up.

If instead of $a = b = 1$ we take $a(s) = b(s) = s$ then every solution of (1.1)-(1.3) blows-up.

Proof of Theorem 2.

We make just a sketch of the proof.

Part *1)* is trivial because if $p_{11} > 1$ we observe that z_2 is increasing and so z_1 is a solution of $z_1' \geq c(z_1)^{p_{11}}$ wich has blow-up if $p_{11} > 1$.

Parts *2.2)* and *3)* are straightforward computations.

The last part *2.1)* follows by a comparison argument with $z_i(s) = c_i(S_0 - s)^{\alpha_i}$ as a subsolution (note that this subsolution blows-up at time S_0).

Proof of Theorem 3.

First we observe that, in spite of the fact that the powers involved may not be C^2, the existence result of Amann ([1]) applies here because the initial data u_0, v_0 are strictly positive. In fact we can take a C^2 modification of the power functions involved that coincide with them below $\frac{\min\{u_0,v_0\}}{2}$, and observe that the solution (u,v) remains greater than $\frac{\min\{u_0,v_0\}}{2}$ because of an easy corollary of the minimum principle (we can use a constant as subsolution).

The part *a)* is an inmediate consequence of part *a)* of Theorem 1.

To prove *b)* let us define

$$\theta_1(\sigma) = \sigma^{(\alpha_1(n-1))} + \sigma^{(\alpha_1(q_{11}-1)+\alpha_2 q_{12}+1)}$$

$$\theta_2(\sigma) = \sigma^{(\alpha_2(m-1))} + \sigma^{(\alpha_1 q_{21}+\alpha_2(q_{22}-1)+1)}$$

then there exist $C, c > 0$ and σ_0 such that, for every $\sigma > \sigma_0$

$$c\theta_1(\sigma) \leq \frac{F(\sigma)}{\varphi'(\sigma)} \leq C\theta_1(\sigma)$$

$$c\theta_2(\sigma) \leq \frac{G(\sigma)}{\psi'(\sigma)} \leq C\theta_2(\sigma)$$

And the result follows just by recalling that the convergence of the integrals involved in *b)* of Theorem 1 are just equivalent to the hypothesis on the exponent of θ_i.

It only remains item *c)*. The proof is the same as in the previous part but we have to take

$$\theta_1(\sigma) = e^{(\beta_1(n-1)\sigma)} + e^{(\beta_1(q_{11}-1)+\beta_2 q_{12})\sigma}$$
$$\theta_2(\sigma) = e^{(\beta_2(m-1)\sigma)} + e^{(\beta_1(q_{21})+\beta_2(q_{22}-1))\sigma}$$

and there holds

$$c\theta_1(\sigma) \leq \frac{F(\sigma)}{\varphi'(\sigma)} \leq C\theta_1(\sigma)$$
$$c\theta_2(\sigma) \leq \frac{G(\sigma)}{\psi'(\sigma)} \leq C\theta_2(\sigma)$$

for some constants C, c. Then we have to proceed just as before.

Proof of Theorem 4.

In this part of the paper we suppose that the p_{ij} are fixed such that *b)* or *c)* of Theorem 3 holds, $m = n = 1$ and the $q_{ij} = q_{ij}(r)$ are positive and nondecreasing.

First we prove an auxiliary lemma.

Lemma 2.3. *Given r, if for some initial datum (u_0, v_0) the problem (1.7)-(1.9) has blow-up (or global existence) then the same is valid for every positive initial datum.*

Proof. We can apply a comparison argument to show that if (w_0, z_0) is such that $w_0 > u_0$ and $z_0 > v_0$ the same inequalities holds as long as both solutions exists (see the proof of Lemma 2.1). So (w, z) has blow-up if (u, v) has. If (w_0, z_0) are not greater than (u_0, v_0) then we observe that $\inf w$ and $\inf z$ are strictly increasing and tends to infinity with t because w is a solution of

$$w_t \geq \Delta w + c_1$$
$$\frac{\partial w}{\partial \eta} \geq c_2$$

for some positive constants c_1, c_2, and then $w \geq c_1 t$. Then there exists a time τ such that $w(\tau) > u_0$ and $z(\tau) > v_0$ and we can use the comparison principle again to conclude that (w, z) has finite time blow-up.

Now we prove Theorem 4. We take $r_1 < r_2$, r_2 such that every solution with r_2 is global. We want to show that a solution with r_1 is global.

We choose $u_0 > 1$ and $v_0 > 1$ and take (u, v), (w, z) the solution to problem (1.7)-(1.9) with r_1,r_2 and initial data (u_0, v_0), $(u_0 + \delta, v_0 + \delta)$ respectively. It is enough to prove that $u < w$ and $v < z$ because then (u, v) must be global and hence every solution with r_1 has to be global by an application of Lemma 2.3.

To see this fact we suppose that it is false and take the first time, t_0, such that there exists $x_0 \in \overline{\Omega}$ with $(w-u)(x_0,t_0) = \delta/2$ or $(z-v)(x_0,t_0) = \delta/2$. We can assume that this holds for $(w-u)$. Then $x_0 \notin \partial\Omega$ because at that point (x_0,t_0)

$$\frac{\partial(w-u)}{\partial\eta} = w^{p_{11}}z^{p_{12}} - u^{p_{11}}v^{p_{12}} > 0$$

and if $x_0 \in \Omega$,

$$(w-u)_t = \Delta(w-u) + w^{q_{11}(r_2)}z^{q_{12}(r_2)} - u^{q_{11}(r_1)}v^{q_{12}(r_1)} > 0$$

a contradiction. (we are using the monotonicity of $q_{ij}(r)$).

We have proved that if the solutions with r_2 are global the same holds for every $r < r_2$.

With the same argument we can conclude that if for some r_1 the solutions have blow-up the same occurs for every $r > r_1$. From this we deduce the existence of the r_0 which is claimed in Theorem 4.

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