A BIFURCATION PROBLEM GOVERNED BY THE BOUNDARY CONDITION II *

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Abstract

In this work we consider the problem $\Delta u = a(x)u^p$ in $\Omega$, $\frac{\partial u}{\partial \nu} = \lambda u$ on $\partial \Omega$, where $\Omega$ is a smooth bounded domain, $\nu$ is the outward unit normal to $\partial \Omega$, $\lambda$ is regarded as a parameter and $0 < p < 1$. We consider both cases where $a(x) > 0$ in $\Omega$ or $a(x)$ is allowed to vanish in a whole subdomain $\Omega_0$ of $\Omega$. Our main results include existence of nonnegative nontrivial solutions in the range $0 < \lambda < \sigma_1$, where $\sigma_1$ is characterized by means of an eigenvalue problem, uniqueness and bifurcation from infinity of such solutions for small $\lambda$, and the appearance of dead cores for large enough $\lambda$.

1. INTRODUCTION

The aim of the present work is to complete the study initiated in [11] of the nonnegative solutions to the following boundary-value problem:

\[
\begin{aligned}
\Delta u &= a(x)u^p \quad \text{in} \quad \Omega \\
\frac{\partial u}{\partial \nu} &= \lambda u \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]

where $\Omega$ is a $C^{2,\gamma}$ domain of $\mathbb{R}^N$, $N \geq 2$, $a \in C^\gamma(\overline{\Omega})$ is a nonnegative weight function, $p > 0$ and $\lambda$ is a real parameter. While in [11] the case $p > 1$ was treated, we are focusing our attention here in the complementary range $0 < p < 1$. In addition, a mixed problem closely related to (1.1) will be also discussed in detail (see Section 5).

The main novelty in problem (1.1) is that the parameter appears precisely in the boundary condition. With respect to this parameter, we will perform a complete analysis of the

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bifurcation diagram of nonnegative solutions to (1.1), which will be shown to be entirely different from the case $p > 1$ (see Figures 1, 2 and 4).

In the course of the exposition, we are considering in first place the case of a positive weight $a(x) > 0$ in $\Omega$, and then we will deal with the situation in which $a(x)$ vanishes in a smooth nonempty subdomain $\Omega_0$ of $\Omega$ (see [17], [6], [8], [16], [10], [7] for problems with Dirichlet or Robin boundary conditions, none of them depending on parameters). It should be remarked that dropping the connectedness of $\Omega_0$ does not lead to genuinely new features regarding the material here analyzed and we are therefore omitting its discussion. Also for the sake of brevity, we are restricting our analysis to treat the cases where either $\Omega_0$ is strongly contained in $\Omega$ or where $\Omega_0$ “touches” $\partial \Omega$ in a nontrivial way (see hypothesis (H)). On the other hand, we point out that due to the smoothness of the main domains $\Omega, \Omega_0, \Omega^+$ involved in this work (see details below), their boundaries can only exhibit a finite number of connected components.

An important feature to stress with regard to problem (1.1) is that the strong maximum principle is not applicable and thus nonnegative nontrivial solutions need not be positive in $\Omega$. Indeed, it will be seen that solutions develop a dead core for large enough $\lambda$ (see for instance [3] or [9] for “dead core” phenomenology), and this entails in some domains multiplicity of solutions. Nevertheless, we can still ensure that solutions are unique for small enough $\lambda$ and moreover that a bifurcation from infinity at $\lambda = 0$ takes place. Additionally, particular properties of problem (1.1) in the ball are also stated (see below).

We come now to give precise statements of our results. First of all we consider the case of positive weights.

**Theorem 1.** Assume $\Omega$ is a $C^{2,\gamma}$ bounded domain of $\mathbb{R}^N$, $a \in C^{\gamma}(\overline{\Omega})$, $a(x) > 0$ in $\Omega$ and $0 < p < 1$. Then problem (1.1) possesses the following features:

(i) If $\lambda \leq 0$, then problem (1.1) does not have nonnegative nontrivial solutions. For $\lambda > 0$, there is always a nonnegative nontrivial solution $u \in C^{2,\gamma_1}(\overline{\Omega})$, $\gamma_1 = \min\{\gamma, p\}$.

(ii) There exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, problem (1.1) has a unique classical nonnegative solution $u_\lambda$. In addition, $u_\lambda$ is positive in that range, $u_\lambda \in C^{2,\gamma}(\overline{\Omega})$ while the mapping $\lambda \to u_\lambda$ regarded as attaining its values in $C^{2,\gamma}(\overline{\Omega})$ is real analytic and decreasing. Moreover,

$$
\lim_{\lambda \to 0^+} \lambda^{\frac{1}{1-p}} u_\lambda = \left( \frac{1}{|\partial \Omega|} \int_{\Omega} a(x) \right)^{\frac{1}{1-p}}
$$

in $C^{2,\gamma}(\overline{\Omega})$. In particular, $u_\lambda \to +\infty$ uniformly in $\overline{\Omega}$ as $\lambda \to 0^+$.

(iii) There exist positive constants $C$ and $\lambda_1$ such that for all nonnegative nontrivial solutions $u \in C^{2,\gamma_1}(\overline{\Omega})$ to (1.1) with $\lambda \geq \lambda_1$ we have

$$
u \leq C \lambda^{-\frac{2}{1-p}}. \quad (1.2)
$$

(iv) There exists $\lambda_2 > 0$ such that all nonnegative nontrivial solutions $u_\lambda$ to (1.1) for $\lambda \geq \lambda_2$ develop a dead core $O_\lambda := \{u_\lambda = 0\}$, $O_\lambda \to \Omega$ uniformly as $\lambda \to +\infty$ in the sense that
A bifurcation problem

for \( \lambda \) large \( \{d(x) \geq d_\lambda\} \subset \mathcal{O}_\lambda \) with \( d_\lambda \to 0 \) as \( \lambda \to \infty \), \( d(x) = \text{dist}(x, \partial \Omega) \). Moreover, \( d_\lambda \) can be chosen as

\[
    d_\lambda = \frac{K}{\lambda},
\]

for certain \( K > 0 \) provided \( a > 0 \) on \( \partial \Omega \).

Figure 1. Comparison between the bifurcation diagrams of cases \( p > 1 \) and \( 0 < p < 1 \) for \( a > 0 \) in \( \Omega \). Continuous pieces in the diagrams stand for uniqueness, the corresponding branches being smooth curves. Multiplicity is vaguely depicted by more than one branch in dotted lines.

Part (iv) of Theorem 1 implies that nonnegative solutions are nontrivial only near the boundary of \( \Omega \) for \( \lambda \) large. By means of a mixed version of problem (1.1) (see (5.1) and Theorem 11) it is possible to show that several solutions can be constructed if we assume that \( \partial \Omega \) consists in more than one connected component. We remark that this phenomenon is not present in the equation \( \Delta u = \lambda u^p \) if the boundary condition is \( u = 1 \) on \( \partial \Omega \), the problem treated for instance in [3].

Theorem 2. Assume \( \Omega \) is a bounded \( C^{2,\gamma} \) domain such that \( \partial \Omega \) has \( k \) connected components. Then problem (1.1) has at least \( 2^k - 1 \) nonnegative nontrivial solutions for large enough \( \lambda \).

We now turn to consider problem (1.1) in a specific domain: the unit ball \( B \) in \( \mathbb{R}^N \). In this case one should expect that a radial solution exists provided \( a \) is also radial. We show that this is indeed the case, and moreover the radial solution is unique for large \( \lambda \). It could be thought indeed that there are no nonradial solutions, since in contrast with Theorem 2, \( \partial B \) is connected and that result can not be applied. Surprisingly, we prove that this is not the case, by constructing a second solution for large \( \lambda \) which is not radial. For simplicity, we are only considering the case \( a(x) = 1 \).

Theorem 3. Assume \( \Omega = B, \ a(x) = 1 \) and \( 0 < p < 1 \). Then

(i) For every \( \lambda > 0 \), there exists at least a radially symmetric nonnegative nontrivial solution to (1.1).

(ii) There exist \( \lambda_0, \lambda_3 > 0 \) such that problem (1.1) has a unique radially symmetric nonnegative nontrivial solution \( u_\lambda \) both for \( 0 < \lambda < \lambda_0 \) and \( \lambda > \lambda_3 \).
(iii) For $\lambda > \lambda_3$ the radial solution $u_\lambda$ has a dead core $O_\lambda = \{x \in B : |x| \leq r(\lambda)\}$, where 
\[
r(\lambda) \sim 1 - \frac{\alpha}{\lambda} \quad \text{as } \lambda \to +\infty
\]
and $\alpha = 2/(1 - p)$. Moreover, $u_\lambda(1) \sim A^{\alpha-\alpha} \lambda^{-\alpha}$, where $A = [\alpha(\alpha - 1)]^{-1} 1^{1-p}$.

(iv) There exists $\lambda_4 > 0$ such that problem (1.1) has at least a nonnegative nontrivial solution $v_\lambda$ for $\lambda > \lambda_4$, which is not radial.

Finally, we are treating some features of problem (1.1) when the weight function $a(x)$ is allowed to vanish in some nonempty subdomain $\Omega_0$ of $\Omega$. Observe that the connectedness of $\Omega$ together with $a \not\equiv 0$ entail $\partial \Omega_0 \cap \Omega \neq \emptyset$. We are restricting our discussion here to the case where $\Omega_0 \subset \Omega$ is a $C^{2,\gamma}$ smooth domain such that, defining $\Gamma_1 = \partial \Omega_0 \cap \partial \Omega$, $\Gamma_2 = \partial \Omega_0 \cap \Omega$, the following condition holds (cf. [11]):
\[
\Gamma_2 \subset \Omega.
\]

(H) implies that the part of $\partial \Omega_0$ meeting $\Omega$ necessarily consists of a closed manifold ($\Gamma_2 = \Gamma_2$) entirely contained in $\Omega$, thus lying at finite distance from the remaining part $\Gamma_1$ of $\partial \Omega_0$, located on $\partial \Omega$.

It turns out that the first eigenvalue $\sigma_1$ of the problem
\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega_0 \\
\frac{\partial u}{\partial \nu} &= \sigma u \quad \text{on } \Gamma_1 \\
u &= 0 \quad \text{on } \Gamma_2,
\end{align*}
\]
will be determinant in the existence issue of solutions. The main features concerning $\sigma_1$ were studied in [11] (cf. Theorem 6). Some of them are recalled in Remarks 4 and Theorem 8 of the present work. In such statement, and for our purposes here, problem (1.3) is further analyzed in the more ambitious case where (H) fails and $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ (i.e., the “genuine” mixed problem).

To fix the notation we are setting in what follows $\sigma_1 = \infty$ if $\Gamma_1 = \emptyset$, i.e. $\Omega_0 \subset \subset \Omega$. After these preliminaries, we can state the following result.

**Theorem 4.** Assume $\Omega$ is a $C^{2,\gamma}$ bounded domain of $\mathbb{R}^N$, $a \in C^\gamma(\overline{\Omega})$ and $0 < p < 1$. Then problem (1.1) possesses the following features:

(i) Problem (1.1) does not have nonnegative nontrivial solutions when $\lambda \leq 0$. For $0 < \lambda < \sigma_1$, there always exists a nonnegative nontrivial solution $u \in C^{2,\gamma_1}(\overline{\Omega})$, $\gamma_1 = \min\{p, \gamma\}$.

(ii) There exists $\lambda_0$, $0 < \lambda_0 < \sigma_1$ such that for $0 < \lambda < \lambda_0$, problem (1.1) has a unique classical nonnegative solution $u_\lambda$. The mapping $\lambda \rightarrow u_\lambda$ exhibits the same properties as in Theorem 1 (ii). In particular
\[
\lim_{\lambda \to 0^+} \lambda^{\frac{1}{1-p}} u_\lambda = \left(\frac{1}{|\partial \Omega|} \int_{\Omega} a(x)\right)^{\frac{1}{1-p}}
\]
in $C^{2,\gamma}(\overline{\Omega})$ and so $u_\lambda \to +\infty$ uniformly in $\overline{\Omega}$ as $\lambda \to 0^+$. 

(iii) If $\sigma_1 = +\infty$, then there exist positive constants $C$, $K$ and $\lambda_2$ such that for all nonnegative nontrivial solutions $u_\lambda \in C^{2,\gamma_1}(\overline{\Omega})$ to (1.1) with $\lambda \geq \lambda_2$ we have

$$u_\lambda \leq C\lambda^{-\frac{2}{p}}.$$  

In addition $u$ develops a dead core $O_\lambda$ with $\{\text{dist}(x, \partial\Omega) \geq d_\lambda\} \subset O_\lambda$ where $d_\lambda \to 0^+$ as $\lambda \to +\infty$. Moreover, $d_\lambda = K/\lambda$, for a constant $K > 0$, provided $a > 0$ on $\Gamma_1$.

An important difference of problem (1.1) in the range $0 < p < 1$ with respect to $p > 1$ arises when $\lambda \geq \sigma_1$. Specifically, it was proved in [11] that no positive solutions of (1.1) with $p > 1$ exist when $\lambda \geq \sigma_1$, provided $\sigma_1 < +\infty$ (see Figure 2). We are showing next that this is indeed the case if $\Omega^+ := \{x \in \Omega: a(x) > 0\} \subset \subset \Omega$, but things are quite different otherwise.

![Figure 2. Bifurcation diagrams $p > 1$ versus $0 < p < 1$. In both regimes drawings for cases $\sigma_1 = \infty$ and $\sigma_1 < \infty$ but $\tilde{\sigma}_1 = \infty$ are superposed. The meaning of continuous and dotted or dashed arcs is the same as in Figure 2.](image)

In explaining the discrepancy when $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$ the eigenvalue problem (1.3) –now observed in $\Omega^+–$ has again an important rôle. Let $\{\Omega_i^+\}$ be the set of (finitely many) connected pieces of $\Omega^+$. Observe that $\partial\Omega^+ \cap \Omega = \partial\Omega_0 \cap \Omega = \Gamma_2$ (as defined before) and so $\Gamma_2 = \cup_i (\partial\Omega_i^+ \cap \Omega)$. Notice again that, due to $a \not\equiv 0$ and the connectedness of $\Omega$ each $\Gamma_{2,i} := \partial\Omega_i^+ \cap \Omega$ is nonempty. In addition $\partial\Omega^+ \cap \partial\Omega = \partial\Omega \setminus \Gamma_1 = \cup_i (\partial\Omega_i^+ \cap \partial\Omega)$. If $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$ some $\partial\Omega_i^+$ meet $\partial\Omega$. Precisely for all those components $\Omega_i^+$ define $\sigma = \tilde{\sigma}_{1,i}$ as the first eigenvalue to the problem,

$$\left\{ \begin{array}{ll}
\Delta u = 0 & x \in \Omega_i^+ \\
\partial u / \partial \nu = \sigma u & x \in \partial\Omega_i^+ \cap \partial\Omega \\
u = 0 & x \in \Gamma_{2,i} = \partial\Omega_i^+ \cap \Omega.
\end{array} \right.$$  

(1.4)

Designate $\tilde{\sigma}_1 = \min \tilde{\sigma}_{1,i}$ if $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$ setting $\tilde{\sigma}_1 = +\infty$ otherwise (i.e. the case $\partial\Omega = \Gamma_1$).

The new features concerning (1.1) in the case $a$ vanishing in $\Omega$ are next described.
Theorem 5. Assume $\sigma_1 < +\infty$. Then:

(i) If $\tilde{\sigma}_1 = +\infty$, then there are no nonnegative nontrivial solutions to (1.1) with $\lambda \geq \sigma_1$.

(ii) If $\tilde{\sigma}_1 < +\infty$ and there exists a nonnegative nontrivial solution to (1.1) with $\lambda \geq \sigma_1$, then $\lambda > \tilde{\sigma}_1$. In particular, if $\tilde{\sigma}_1 \geq \sigma_1$, then there are no solutions for $\lambda \in [\sigma_1, \tilde{\sigma}_1]$.

(iii) When $\tilde{\sigma}_1 < +\infty$, there exists $\lambda_2 > 0$ such that for $\lambda \geq \lambda_2$ problem (1.1) has at least a nonnegative nontrivial solution, which develops a dead core $O_\lambda$. In this case $\{\text{dist}(x, \Gamma^+) \geq d_\lambda\} \subset O_\lambda$ with $d_\lambda \rightarrow 0^+$ as $\lambda \rightarrow +\infty$, $\Gamma^+ = \partial \Omega^+ \cap \partial \Omega$. Similarly, the choice $d_\lambda = K/\lambda$, $K > 0$, is again possible when $a > 0$ on $\Gamma^+$.

Figure 3. A possible configuration for the domains $\Omega, \Omega_0, \Omega^+$. In this example, $\Omega^+$ possesses two connected pieces.

Remarks 1.

(a) By means of examples it is possible to show that both options $\sigma_1 \leq \tilde{\sigma}_1$ and $\tilde{\sigma}_1 < \sigma_1$ can be occur. Notice that such relative positions depend only on the support of the weight $a$, not on its size. See Remark 8.

(b) Regarding (ii) it is also possible to produce examples where $\tilde{\sigma}_1 < \sigma_1$ and either no solutions $u \geq 0$ ($u \neq 0$) exist for $\sigma_1 \leq \lambda \leq \tilde{\sigma}_1 + \varepsilon$, or a nonnegative nontrivial solution exists for all $\lambda \geq \sigma_1 - \varepsilon$, $\varepsilon > 0$ small. See also Remark 8.

Figure 4. A bifurcation diagram for the case $\sigma_1 < \tilde{\sigma}_1 < +\infty$. Solutions disappear converging to zero at $\sigma_1$ and spontaneously appear beyond $\lambda = \tilde{\sigma}_1$. 
(c) As already noticed and in contrast with the case \( p > 1 \), (1.1) can support nonnegative nontrivial solutions for \( \lambda > \sigma_1 \), provided \( \tilde{\sigma}_1 < \sigma_1 \). However, such solutions are “degenerated” in the sense that they must vanish in \( \Omega_0 \) together with all those (if any) connected pieces \( \Omega_1^+ \) of \( \Omega^+ \) such that \( \Omega_1^+ \subset \subset \Omega \) (cf. Section 7).

(d) According to (ii)-(iii) the set of solutions to (1.1) undergoes a discontinuity at \( \lambda = \sigma_1 \) to—roughly speaking—arise again at some \( \lambda \) after \( \tilde{\sigma}_1 \) (suppose \( \sigma_1 \leq \tilde{\sigma}_1 \)). It should be remarked that the latter arising is necessarily “spontaneous” in the sense that when solutions appear are bounded away from zero. In other words, they are not generated by a bifurcation from the trivial solution \( u = 0 \). See Remark 7.

Finally, we conclude our analysis by determining the asymptotic behavior of solutions near \( \lambda = \sigma_1 \) in the critical situation where solutions cease to exist at that value. We assert that \( (\lambda, u) = (\sigma_1, 0) \) defines in that case a bifurcation point for solutions to (1.1).

**Theorem 6.** Assume \( \sigma_1 < +\infty \), with either \( \sigma_1 \leq \tilde{\sigma}_1 \) or \( \tilde{\sigma}_1 = +\infty \), or even \( \tilde{\sigma}_1 < \sigma_1 \) but in this case also supposing that (1.1) has only \( u = 0 \) as solution at \( \lambda = \sigma_1 \). Then every family of nonnegative solutions \( \{u_\lambda\} \) with \( \lambda \) close to \( \sigma_1 \) verifies

\[
u_\lambda \to 0,
\]

in \( C^{2,\beta}(\bar{\Omega}) \), \( 0 < \beta < \gamma_1 \), as \( \lambda \to \sigma_1^- \). In addition, there exists \( \delta > 0 \) such that every nonnegative nontrivial solution \( u_\lambda \) to (1.1) for \( \sigma_1 - \delta \leq \lambda < \sigma_1 \) develops a dead core \( \Omega_\lambda \). More importantly, such dead core must satisfy \( \Omega_\lambda \subset \subset \Omega^+ \) together with \( \Omega_\lambda \to \Omega^+ \) uniformly as \( \lambda \to \sigma_1^+ \).

**Remark 2.** Compared with the case \( \lambda \geq \sigma_1 \) (Remark 1 (c)) it follows from the preceding statement that nonnegative solutions to (1.1) with \( \lambda < \sigma_1, \lambda \sim \sigma_1 \), exhibit the opposite behavior in \( \Omega_0 \). Namely, all those solutions are strictly positive in \( \Omega_0 \), aside of converging to zero as \( \lambda \to \sigma_1^- \).

The paper is organized as follows: in Section 2, we state some preliminaries which will be used in the paper. Section 3 is devoted to existence of nonnegative solutions and uniqueness for small \( \lambda \), while in Section 4 the asymptotic behavior as \( \lambda \to +\infty \) is elucidated. In Sections 5 and 6 we prove Theorems 2 and 3, respectively. A mixed problem, closely related to (1.1) is also studied in Section 5 (Theorem 11). Finally, Section 7 deals with problem (1.1) when the weight \( a(x) \) vanishes in a subdomain \( \Omega_0 \) of \( \Omega \) (Theorems 4, 5 and 6).

## 2. Preliminaries

In this section we collect some results which will be needed for the proofs of our theorems. We begin by proving that weak solutions to (1.1) are indeed classical solutions. It should be stressed that this result is not contained in [1]. However, as will be seen at once, it can be proved by means of their estimates.

**Lemma 7.** Let \( u \in H^1(\Omega) \) be a weak nonnegative solution to (1.1). Then \( u \in C^{2,\gamma_1}(\bar{\Omega}) \), \( \gamma_1 = \min\{\gamma, p\} \), and thus defines a classical solution.
Proof. For $\lambda > 0$ fixed and $\mu > 0$ large enough, the problem

$$
\begin{cases}
\Delta v - \mu v = f & \text{in } \Omega \\
\frac{\partial v}{\partial \nu} = \lambda v & \text{on } \partial \Omega
\end{cases}
$$

(2.1)

with $f = 0$ has $u = 0$ as the only weak solution in $H^1(\Omega)$. In fact, any nontrivial solution $u \in H^1(\Omega)$ to (2.1) with $f = 0$ defines a weak eigenfunction corresponding to the eigenvalue $\sigma = \lambda$ of the Steklov-type eigenvalue problem:

$$
\begin{cases}
\Delta v - \mu v = 0 & \text{in } \Omega \\
\frac{\partial v}{\partial \nu} = \sigma v & \text{on } \partial \Omega
\end{cases}
$$

(2.2)

Techniques in Section 2 of [11] can be used to show that problem (2.2) possesses, for every $\mu$, a first eigenvalue $\sigma_1 = \sigma_1(\mu)$ which is the unique associated to a positive eigenfunction in $H^1(\Omega)$ ("a posteriori" in $C^2,\gamma(\Omega)$) and variationally characterized as

$$\sigma_1 = \inf_{v \in H^1(\Omega) \setminus \{0\}} \frac{\int_\Omega \{ |\nabla v|^2 + \mu v^2 \}}{\int_{\partial \Omega} v^2}. \quad (2.3)$$

Moreover, $\sigma_1$ increases with $\mu$. We claim that $\sigma_1 \to \infty$ as $\mu \to \infty$. This implies the desired assertion if $\mu$ is so large as to have $\lambda < \sigma_1(\mu)$. To show the claim suppose that $\sigma_1(n) = O(1)$ as $n \to \infty$. If $\phi_n \in H^1(\Omega)$ stands for the positive eigenfunction corresponding to $\sigma_1(n)$ normalized so that $\int_{\partial \Omega} \phi_n^2 = 1$, then $\phi_n$ remains bounded in $H^1(\Omega)$ under the norm $|u|_{H^1}^2 = \int_\Omega |\nabla u|^2 + \int_{\partial \Omega} u^2$. Modulo a subsequence, $\phi_n \to \phi$ weakly in $H^1$ and by compactness $\phi_n \to \phi$ in both $L^2(\Omega)$ and $L^2(\partial \Omega)$. In particular $\int_{\partial \Omega} \phi^2 = 1$. However, the boundedness of $\sigma_1(n)$ leads to $\int_\Omega \phi^2 = 0$ which is not possible. Thus the claim is proved.

Since (2.1) can admit at most one classical solution, the results in [13] (Theorem 6.31 and subsequent remark) ensure that such problem has indeed a unique solution $v \in C^{2,\gamma}(\Omega)$ for every $f \in C^\gamma(\Omega)$.

Now let $u \in H^1(\Omega)$ be a nonnegative solution to (1.1). By the Sobolev embedding, we have $u, u^p \in L^r(\Omega)$ for every $1 \leq r \leq 2^* = 2N/(N-2)$ (this is only true if $N \geq 3$; for $N = 2$ the situation is even better). Take a sequence of $C^\gamma$ functions $f_n$ converging to $a(x) u^p - \mu u$ in $L^r(\Omega)$, and let $v_n$ be the unique solution to (2.1) with $f = f_n$. We now use the estimates of Agmon-Douglis-Nirenberg [1]. There exists a constant $C > 0$ such that

$$|v_n|_{W^{2,r}} \leq C |f_n|_{L^r}.\]$$

It follows that $v_n \to v$ in $W^{2,r}(\Omega)$, where $v$ is the (unique) solution to

$$
\begin{cases}
\Delta v - \mu v = a(x) u^p - \mu u & \text{in } \Omega \\
\frac{\partial v}{\partial \nu} = \lambda v & \text{on } \partial \Omega
\end{cases}
$$

Thus $v = u$, and so $u \in W^{2,r}(\Omega)$. Repeating this argument a finite number of times, we arrive at $u \in W^{2,q}(\Omega)$ for some $q > N$, and hence $u \in C^{1,\eta}(\Omega)$ for a certain $0 < \eta < 1$.\]
Next we choose a sequence $f_n \in C^{\gamma_1}(\overline{\Omega})$ so that $f_n \to a(x)u^p - \mu u$ in $C^{\gamma_1}(\overline{\Omega})$, and denote by $v_n$ the $C^{2,\gamma_1}(\overline{\Omega})$ solutions to (2.1) corresponding to $f = f_n$. We now use the Schauder estimates provided by Theorem 6.30 in [13] to obtain

$$|v_n|_{C^{2,\gamma_1}} \leq C(|f_n|_{C^{\gamma_1}} + |v_n|_{C^{\gamma_1}}).$$

From which it similarly follows that $v_n \to u$ in $C^{2,\gamma_1}(\overline{\Omega})$, and thus $u \in C^{2,\gamma_1}(\overline{\Omega})$.

Remark 3. Observe that as a consequence of the last part of the proof of Lemma 7 positive solutions $u$ to (1.1) are slightly more regular and directly lie in $C^{2,\gamma_1}(\Omega)$, while this is not necessarily the case if such solutions vanish somewhere in $\Omega$.

We are now introducing an eigenvalue problem under mixed boundary conditions of Dirichlet and Steklov type, located on zones $\Gamma$ and $\Gamma'$ of the boundary, which will play an important rôle in some of our forthcoming proofs. Theorem 8 in [11] analyzed this problem in full detail when the zones carrying different boundary conditions are different—and hence separated away—components of the boundary, providing class $C^{2,\gamma}$ eigenfunctions. This is precisely the more frequently used version in this work. Our next result deals with the more adverse situation in which $\Gamma$ and $\Gamma'$ meet on a $n - 2$ dimensional closed manifold $\gamma$. Such scenario will be required for showing the estimate (1.2) in Section 4. Notice that in this case results in Section 2 of [11] still provide $H^1$ eigenfunctions (see Remark 8 there). However, their optimal degree of regularity is an issue belonging to the subtle realm of smoothness of weak solutions to mixed problems (see for instance [14, 15]). Accordingly, we only provide next the amount of weak smoothness strictly required for our purposes here. Its proof, an straightforward consequence of Section 2 in [11] and Theorem B in [4], is omitted.

**Theorem 8.** Let $D \subset \mathbb{R}^N$ a bounded domain of class $C^3$ such that $\partial D = \Gamma \cup \Gamma' \cup \gamma$, $\Gamma$, $\Gamma'$, $\gamma$ pair-wise disjoint, $\Gamma$, $\Gamma'$ relatively open in $\partial D$, $\gamma$ a closed $n - 2$ dimensional manifold while $\Gamma \cup \gamma$ and $\Gamma' \cup \gamma$ define $n - 1$ dimensional manifolds of class $C^3$ with common boundary $\gamma$. Then the eigenvalue problem

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in} \quad D \\
\frac{\partial u}{\partial \nu} &= \sigma u \quad \text{on} \quad \Gamma' \\
u &= 0 \quad \text{on} \quad \Gamma,
\end{aligned}
\]  

admits a principal eigenvalue $\sigma_1$, i.e. an eigenvalue with a one-signed eigenfunction, which is characterized by:

$$\sigma_1 = \inf_{u \in H_{\Gamma'}^1(D)} \frac{\int_D |\nabla u|^2}{\int_{\Gamma'} u^2},$$

where $H_{\Gamma'}^1(D)$ is the subset of functions of $H^1(D)$ which vanish on $\Gamma'$. Moreover, $\sigma_1$ is the unique principal eigenvalue, simple and the smallest of all eigenvalues. Moreover, every associated eigenfunction $\phi$ verifies $\phi \in H^1(D) \cap W^{2,q}(D)$ for all $q < 4/3$.

**Remarks 4.**

(a) For $\lambda > 0$ designate by $D_\lambda$ the scaled copy of $D$, i.e. $D_\lambda = \{\lambda x : x \in D\}$. If $\sigma_1(D_\lambda)$ stands for the principal eigenvalue of (2.4) in $D_\lambda$ one directly finds that $\sigma_1(D_\lambda) = \frac{1}{\lambda} \sigma_1(D)$. 


(b) A suitable use of the $L^p$ estimates in [1] proves that every eigenfunction $\phi$ associated to $\sigma_1$ satisfies $\phi \in C^{2,\eta}(D \cup T), 0 < \eta < 1$ arbitrary, where $T$ is any relatively open part of $\partial D$ strongly contained either in $\Gamma$ or $\Gamma'$. Thus, as a consequence of Hopf's maximum principle, every positive eigenfunction satisfies $\frac{\partial \phi}{\partial \nu} < 0$ on $\Gamma'$ where $\nu$ is the outward unit normal on $\Gamma'$. On the other hand, although not in our framework here, smoothness of such eigenfunctions $\phi$ is enlarged provided $\Gamma$ and $\Gamma'$ meet in a convenient nonsmooth way on $\gamma$. For instance, under a suitable –not too large– angle (cf. [14, 15] and their references). In any case, it should be recalled that eigenfunctions achieve the full $C^{2,\gamma}$ regularity assumed that $\partial \Omega$ is only $C^{2,\gamma}$ when $\Gamma$ and $\Gamma'$ are disjoint closed manifolds (see [11]).

Finally, we are introducing some results from [12] concerning a singular initial value problem, which will be needed in Section 6. It should be remarked that such results were established there for the harder framework of the $p$-Laplacian operator. As a very special case of them we consider the Cauchy problem:

$$\begin{cases}
(r + d)^{N-1}u' = (r + d)^{N-1}u^p & r \in (0, \infty) \\
u(0) = 0, \quad u'(0) = 0,
\end{cases} \quad (2.6)$$

which arises after some normalization when one considers the radial version of (1.1) with $a(x) = 1$. As a consequence of Theorems 2.3, 2.5 and 2.6 (see also Corollary 2.4) in [12] and a globalizing argument, we can state the following theorem.

**Theorem 9.** For each $d \geq 0$, problem (2.6) has a unique nontrivial solution $u(r, d)$ defined in $[0, +\infty)$, in the sense that $u(r, d) > 0$ for $r > 0$. Moreover, $u(\cdot, d) \to u_0(\cdot)$ as $d \to \infty$ in $C^1_{\text{loc}}[0, +\infty)$, where $u_0(\cdot)$ is the nontrivial solution to (2.6) corresponding to $N = 1$, which is explicitly given by $u_0(r) = Ar^\alpha$, where $\alpha = 2/(1-p)$ and $A = (\alpha(\alpha - 1))^{-\frac{1}{1-p}}$. In addition, $u(r, d)$ is differentiable with respect to $d$ and verifies

$$\frac{\partial u}{\partial d}(\cdot, d) \to 0, \quad \text{as } d \to \infty \quad (2.7)$$

in $C^1_{\text{loc}}[0, +\infty)$.

**Remark 5.** Notice that problem (2.6) always has the trivial solution $u \equiv 0$. It follows that this is the only nonnegative solution when $p \geq 1$. However, when $0 < p < 1$ there is a unique positive solution and infinitely many nonnegative (nontrivial) solutions, all of them expressible in terms of the positive one (cf. Corollary 2.4 in [12]).

### 3. Existence of nonnegative solutions. Uniqueness for small $\lambda$

In this section we are considering the issue of existence of nonnegative nontrivial solutions to (1.1) when $a(x) > 0$ in $\Omega$. It will be shown that for every $\lambda > 0$ there always exists at least one solution (while for $\lambda \leq 0$ they cease to exist). In addition, we will show that the solution
is unique provided \( \lambda \) is sufficiently small. In fact, a bifurcation from infinity at \( \lambda = 0 \) takes place.

**Proof of Theorem 1 (i).** According to Lemma 7, all weak solutions in \( H^1(\Omega) \) belong to \( C^{2,\gamma_1}(\Omega) \). Assume first \( \lambda \leq 0 \). Integrating the equation (1.1) in \( \Omega \):

\[
\int_{\Omega} a(x) u^p = \int_{\Omega} \Delta u = \int_{\partial \Omega} \frac{\partial u}{\partial \nu} = \lambda \int_{\partial \Omega} u \leq 0,
\]

from which \( u \equiv 0 \) follows.

Now assume \( \lambda > 0 \). We use in \( H^1(\Omega) \) the norm \( |u|_{H^1}^2 = \int_{\Omega} |\nabla u|^2 + \int_{\partial \Omega} u^2 \) which is equivalent to the usual one. Define

\[
M := \left\{ u \in H^1(\Omega) : \int_{\Omega} a(x)|u|^{p+1} = 1 \right\}
\]

and for \( u \in M \) the functional

\[
J(u) = \int_{\Omega} |\nabla u|^2 - \lambda \int_{\partial \Omega} u^2.
\]

We claim that \( J \) is coercive on \( M \). Indeed, assume on the contrary that there exists a sequence \( \{u_n\} \subset M \) such that \( |u_n|_{H^1} \to +\infty \) and \( J(u_n) \leq C \), that is

\[
\int_{\Omega} |\nabla u_n|^2 \leq \lambda \int_{\partial \Omega} u_n^2 + C. \tag{3.1}
\]

Denoting by \( t_n = |u_n|_{L^2(\partial \Omega)} \), it follows from (3.1) that \( t_n \to +\infty \). Letting \( v_n = u_n/t_n \), we obtain again from (3.1) a bound for the \( H^1 \) norms of \( v_n \). Passing to a subsequence, we obtain \( v_n \rightharpoonup v \) weakly in \( H^1(\Omega) \), strongly in \( L^2(\Omega) \) and in \( L^2(\partial \Omega) \). In particular, \( |v|_{L^2(\partial \Omega)} = 1 \). On the other hand, since \( p < 1 \) and \( u_n \in M \), we have that \( \int_{\Omega} a(x)|v|^{p+1} = 0 \), and so \( v = 0 \), a contradiction. Thus \( J \) is coercive.

Since \( M \) is weakly closed, and \( J \) is (sequentially) weakly lower semicontinuous, it follows from standard results that \( J \) achieves its minimum in \( M \) (see [18]). Also, since \( J(|u|) = J(u) \) and \( |u| \in M \) whenever \( u \in M \), we may assume that the minimum is achieved at a nonnegative (nontrivial) function \( u \). By the Lagrange’s multipliers rule, there exists \( \mu \in \mathbb{R} \) such that

\[
\int_{\Omega} \nabla u \nabla \varphi - \lambda \int_{\partial \Omega} u \varphi = \mu \int_{\Omega} a(x)u^p \varphi
\]

for every \( \varphi \in H^1(\Omega) \). Taking in particular \( \varphi = u \), we arrive at \( J(u) = \mu < 0 \), since there is a constant function which belongs to \( M \) for which \( J \) is negative. Setting \( v = |\mu|^{1-p}u \), we obtain a nonnegative nontrivial weak solution to (1.1), which is, as remarked earlier, a classical solution in \( C^{2,\gamma_1}(\Omega) \).

**Remark 6.** Nonnegative solutions can also be obtained by applying the Mountain Pass theorem to the functional

\[
\tilde{J}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\lambda}{2} \int_{\partial \Omega} (u_+)^2 + \frac{1}{p+1} \int_{\Omega} a(x)|u|^{p+1},
\]

in \( H^1(\Omega) \), where \( u_+ = \max\{u, 0\} \) denotes the positive part of \( u \).
We now proceed to show Theorem 1 (ii). The proof makes use of local bifurcation theory. As a first step in this approach, we begin by characterizing the behavior of all nonnegative solutions for $\lambda$ approaching zero.

**Lemma 10.** Assume $\lambda_n \to 0$, and let $u_n$ be a corresponding sequence of nonnegative non-trivial solutions to (1.1). Then

$$
\begin{cases}
\lambda_n = \mu_n t_n \\
u_n = \frac{1}{\theta_n}(1 + t_n w_n)
\end{cases}
$$

where $\theta = 1/(1 - p)$, $t_n = \left(\frac{1}{|\partial\Omega|} \int_{\partial\Omega} u_n\right)^{p-1} \to 0$, $w_n \in Y := \{u \in C^{2,\gamma}(\overline{\Omega}) : \int_{\partial\Omega} u = 0\}$. Moreover,

$$
\mu_n \to \mu_0 = \frac{1}{|\partial\Omega|} \int_{\Omega} a(x),
$$

(3.2)

and $w_n \to w_0$ in $C^{2,\beta}(\overline{\Omega})$ for all $0 < \beta < \gamma$, where $w_0$ is the unique solution in $Y$ of the linear equation

$$
\begin{cases}
\Delta w = a(x) \text{ in } \Omega \\
\frac{\partial w}{\partial \nu} = \mu_0 \text{ on } \partial\Omega.
\end{cases}
$$

(3.3)

**Proof.** First of all, notice that (3.3) has a unique solution in $Y$, according to the compatibility condition given by the value of $\mu_0$ in (3.2).

We begin by showing that $|u_n|_{\infty}$ tends to infinity. Assume on the contrary that for a subsequence (still labelled by $u_n$) there holds $|u_n|_{\infty} \leq C$. Observing that problem (2.1) corresponding to $\mu = 0$ has a unique solution for $\lambda$ small and by employing the Agmon-Douglis-Nirenberg estimates (see [1]), we have for every $q > 1$

$$
|u_n|_{W^{2,q}} \leq C_1(|u_n|_{L^p} + |u_n|_{L^\gamma}) \leq C_2,
$$

(3.4)

$C_1 > 0$ being certain constant depending on $q$. Selecting $q > N$, we have by the Sobolev imbedding (passing to a subsequence) that $u_n \to u$ in $C^{1,\eta}(\overline{\Omega})$ for some $0 < \eta < 1$. It follows that $u$ is a nonnegative weak (hence classical by Lemma 7) solution to (1.1) with $\lambda = 0$. Thus, $u \equiv 0$ (in particular, $u_n \to 0$ in $H^1(\Omega)$). We claim that this is impossible. Indeed, let $v_n = u_n/|u_n|_{H^1}$. There exists a subsequence (labelled once more by $v_n$) such that $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$, in $L^2(\partial\Omega)$ and almost everywhere in $\Omega$. It follows from (1.1) that

$$
\int_{\Omega} |\nabla v_n|^2 = \lambda_n \int_{\partial\Omega} v_n^2 - |u_n|_{H^1}^{p-1} \int_{\Omega} a(x)v_n^{p+1} \leq \lambda_n \int_{\partial\Omega} v_n^2 \to 0,
$$

so by lower semicontinuity we deduce that $v$ is a constant. As a consequence, it follows that $v_n \to v$ strongly in $H^1(\Omega)$. On the other hand, again by (1.1), we have that

$$
|u_n|_{H^1}^{p-1} \int_{\Omega} a(x)v_n^{p+1} = \lambda_n \int_{\partial\Omega} v_n^2 - \int_{\Omega} |\nabla v_n|^2 \to 0,
$$
and since $0 < p < 1$, we deduce

$$\int_{\Omega} a(x)v_n^{p+1} \to 0.$$  \hfill (3.5)

Now since $v_n \to v$ in $L^2(\Omega)$, we have $v_n \to v$ in $L^{p+1}(\Omega)$, and thus $v = 0$ in $\Omega$, contradicting that $|v_n|_1 = 1$. Thus $|u_n|_\infty \to +\infty$.

Setting again $v_n = u_n/|u_n|_\infty$, and recalling that $v_n$ satisfies

$$\begin{cases}
\Delta v_n = a(x)v_n^p|u_n|_\infty^{p-1} & \text{in } \Omega \\
\frac{\partial v_n}{\partial \nu} = \lambda_n v_n & \text{on } \partial \Omega,
\end{cases}$$

we obtain arguing as before an estimate like (3.4), getting thus a bound in $C^{1,\eta}(\Omega)$ for the solutions $v_n$ for some $0 < \eta < 1$. Applying Theorem 6.30 in [13] gives $C^{2,\gamma}(\Omega)$ bounds, and thus $v_n \to v$ in $C^2(\Omega)$, $0 < \beta < \gamma_1$, where $v$ is a unique classical solution to

$$\begin{cases}
\Delta v = 0 & \text{in } \Omega \\
\frac{\partial v}{\partial \nu} = \lambda v & \text{on } \partial \Omega,
\end{cases}$$

with $|v|_\infty = 1$. Hence $v \equiv 1$. Since $v_n$ becomes positive for large $n$ we furthermore find that $v_n \in C^{2,\gamma}(\Omega)$ (Remark 3) and $v_n \to 1$ in $C^{2,\beta}(\Omega)$ for all $0 < \beta < \gamma$.

We now split $u_n = c_n + c_n z_n$, where $c_n = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} u_n$ and $z_n \in Y$. It follows that

$$\frac{c_n}{|u_n|_\infty} \to 1 \quad \text{and} \quad z_n \to 0 \quad \text{in } C^{2,\beta}(\Omega).$$

Next, notice that $z_n$ solves

$$\begin{cases}
\Delta z_n = a(x)c_n^{p-1}(1 + z_n)^p & \text{in } \Omega \\
\frac{\partial z_n}{\partial \nu} = \lambda_n(1 + z_n) & \text{on } \partial \Omega.
\end{cases} \quad \hfill (3.6)$$

By integrating in (3.6) we obtain:

$$\int_{\Omega} a(x)(1 + z_n)^p = \lambda_n c_n^{1-p}|\partial \Omega|.$$  

Setting $t_n = c_n^{p-1}$ this means that $\lambda_n = t_n \mu_n$ where $\mu_n \to \mu_0$, $\mu_0$ given by (3.2). Next, using that $\lambda_n = O(t_n)$ and from Schauder’s estimates we conclude that $z_n = O(t_n)$ in $C^{2,\gamma}(\Omega)$. Writing $z_n = t_n w_n$ with $w_n$ bounded in $C^{2,\gamma}(\Omega)$ we conclude that $w_n \to w_0$ in $C^{2,\beta}(\Omega)$, $w_0$ being the unique classical solution in $Y$ to (3.3). This proves the lemma. \hfill \Box

Proof of Theorem 1 (ii). According to Lemma 10, all nonnegative nontrivial solutions to (1.1) for small $\lambda$ are of the form:

$$\begin{cases}
\lambda = t \mu \\
u = \frac{1}{t^p}(1 + tw)
\end{cases}$$
where \( t > 0 \) is small and \((\mu, w) \in \mathbb{R} \times Y\) is close to \((\mu_0, w_0)\), \(\mu_0 \in \mathbb{R}, w_0 \in Y\) given by (3.2) and (3.3) respectively. Thus, for small \( \lambda > 0 \) solving (1.1) is equivalent to solving:

\[
\begin{cases}
\Delta w = a(x)(1 + tw)^p & \text{in } \Omega \\
\frac{\partial w}{\partial \nu} = \mu(1 + tw) & \text{on } \partial \Omega,
\end{cases}
\]

(3.7)
together with the compatibility condition:

\[
\int_{\Omega} a(x)(1 + tw)^p = \mu \int_{\partial \Omega} (1 + tw),
\]

(3.8)

for \(|t|, |\mu - \mu_0|\) and \(|w - w_0|_Y\) small, where \(Y = \{u \in C^{2,\gamma}(\overline{\Omega}) : \int_{\partial \Omega} u = 0\}\) is endowed with its natural norm.

We are accordingly showing such uniqueness by means of the Implicit Function theorem as follows. Setting \(X = C^{\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})\), define \(l \in X^*\) as:

\[
l(f, g) = \int_{\Omega} f - \int_{\partial \Omega} g, \quad (f, g) \in X.
\]

According to Schauder’s theory, for each \((f, g) \in \ker(l)\) the problem:

\[
\begin{cases}
\Delta u = f & x \in \Omega \\
\frac{\partial u}{\partial \nu} = g & x \in \partial \Omega,
\end{cases}
\]

has a unique solution \(u \in Y\), denoted \(u = K(f, g)\). Hence, \(K : \ker(l) \to Y\) defines a linear topological isomorphism. Observe in addition that \((f, g) \mapsto (f, g + \frac{1}{|\partial \Omega|} l(f, g))\) defines a projection from \(X\) onto \(\ker(l)\). Thus \(K\) is always defined at \((f, g + \frac{1}{|\partial \Omega|} l(f, g))\) for every \((f, g) \in X\).

Now introduce the Nemytskii operators \(F(t, w) = a(1 + tw)^p, G(t, w) = 1 + tw\). For small and positive \(\varepsilon, \delta\), \(F : (-\varepsilon, \varepsilon) \times B_Y(w_0, \delta) \to C^{\gamma}(\overline{\Omega})\), and \(G : (-\varepsilon, \varepsilon) \times B_Y(w_0, \delta) \to C^{1,\gamma}(\overline{\Omega})\) define real analytic mappings, \(B_Y(w_0, \delta)\) standing for the open ball in \(Y\) with radius \(\delta\) and center \(w_0\). Thus,

\[
\mathcal{F} : (-\varepsilon, \varepsilon) \times (\mu_0 - \varepsilon, \mu_0 + \varepsilon) \times B_Y(w_0, \delta) \to \mathbb{R} \times Y
\]

\[
(t, \mu, w) \mapsto (\mathcal{F}_1(t, \mu, w), \mathcal{F}_2(t, \mu, w)),
\]

where

\[
\mathcal{F}_1(t, \mu, w) = l(F(t, w), \mu G(t, w)),
\]

and

\[
\mathcal{F}_2(t, \mu, w) = w - K(F(t, w), \mu G(t, w) + \frac{1}{|\partial \Omega|} F_1(t, \mu, w))
\]

is also a real analytic mapping.
On the other hand, solving (3.7), (3.8) for \((t, \mu, w)\) close to \((0, \mu_0, w_0)\) in \(\mathbb{R}^2 \times Y\) is equivalent to solving:

\[
\mathcal{F}(t, \mu, w) = 0, \tag{3.9}
\]

with \((t, \mu, w) \in (-\varepsilon, \varepsilon) \times (\mu_0 - \varepsilon, \mu_0 + \varepsilon) \times B(w_0, \delta)\) with \(\varepsilon, \delta\) small enough. However, \(\mathcal{F}(0, \mu_0, w_0) = 0\), while

\[
\mathcal{F}_1(0, \mu, w) = |\partial\Omega|(\mu_0 - \mu),
\]

and,

\[
\mathcal{F}_2(0, \mu, w) = w - K(a, \mu_0) = w - w_0,
\]

Thus, the Fréchet derivative \(L\) acting on \((\hat{\mu}, \hat{w}) \in \mathbb{R} \times Y\) is given by

\[
L(\hat{\mu}, \hat{w}) = D_{(\mu, w)}\mathcal{F}(0, \mu_0, w_0)(\hat{\mu}, \hat{w}) = (-|\partial\Omega|\hat{\mu}, \hat{w}).
\]

Since \(L\) defines an isomorphism from \(\mathbb{R} \times Y\) into itself, the real analytic version of the Implicit Function theorem ([5]) ensures that (3.9) is uniquely solvable with \(\mu(t), w(t, \cdot)\) real analytic in \(t\) and \(\lambda = t\mu(t), u = t^{-\theta}(1 + tw(t, \cdot))\) define the unique nonnegative solutions to (1.1) for small \(\lambda > 0\). The remaining assertions in (ii) follow from this representation. \(\square\)

4. Behavior for large \(\lambda\)

In this section, we study the properties of the nonnegative nontrivial solutions to (1.1) for large positive \(\lambda\), assuming \(a(x) > 0\) in \(\Omega\).

Proof of Theorem 1 (iii). Assume that (1.2) does not hold. Let \(\lambda_n \to +\infty\) and \(u_n\) be a sequence of corresponding nonnegative nontrivial solutions to (1.1), such that

\[
\lambda_n^{\frac{1}{2}} M_n \to +\infty \tag{4.1}
\]

as \(n \to +\infty\), where \(M_n := \max_{\Omega} u_n\). Choose points \(\tilde{x}_n \in \partial \Omega\) such that \(u_n(\tilde{x}_n) = M_n\) (notice that the functions \(u_n\) are subharmonic), and assume (extracting a subsequence if necessary) that \(\tilde{x}_n \to \tilde{x}_0 \in \partial \Omega\). By means of a translation and a rotation, it can be assumed that \(\tilde{x}_0 = 0\) while \(\{x_N = 0\}\) defines the tangent hyperplane to \(\partial \Omega\) at such point. By hypothesis, there exist \(R > 0\) and \(\varphi \in C^{2,\gamma}(B(0, R) \cap \{x_N = 0\})\), \(\varphi(0) = 0, \nabla \varphi(0) = 0\) such that by writing \(x \in \mathbb{R}^n\) as \(x = (x', x_N)\), \(x' = (x_1, \ldots, x_{N-1})\) then \(\Omega \cap B(0, R) = \{x : x_N > \varphi(x')\}\) and \(\partial \Omega \cap B(0, R) = \{x : \tilde{x}_0 = \varphi(x')\}\). Then, the standard \(C^{2,\gamma}\) diffeomorphism \(y = h(x)\) given by \(y' = x', y_N = x_N - \varphi(x')\) maps \(B(0, R)\) onto a neighborhood \(V\) of \(y = 0\) in \(\mathbb{R}^N\), \(\Omega \cap B(0, R)\) onto \(V^+ = V \cap \mathbb{R}^N_+\), \(\partial \Omega \cap B(0, R)\) onto \(V \cap \partial \mathbb{R}^N_+\). Problem (1.1) in \(\Omega \cap B(0, R)\) is transformed in:

\[
\begin{cases}
\Delta u + \sum_{i=1}^{N-1} a_i(y) \frac{\partial^2 u}{\partial y_i \partial y_N} + b(y) \frac{\partial u}{\partial y_N} = a(y) u^p, & y \in V^+ \tag{4.2} \\
\nabla u \cdot \nu_1(y) = \lambda u, & y \in V \cap \partial \mathbb{R}^N_+,
\end{cases}
\]

with

\[
a_i = -2 \frac{\partial \varphi}{\partial x_i}, \quad b(y) = -\Delta \varphi, \quad \nu_1 = (\nu', -\nu' \nabla \varphi + \nu_n),
\]
\( \nu(x) = (\nu'(x), \nu_N(x)) \) the outer unit normal on \( \partial \Omega \), all such functions being evaluated at \( x = h^{-1}(y) \).

On the other hand, setting \( \tilde{y}_n = h(x_n) \) then \( \tilde{y}_n \to 0 \) and a positive \( R_1 \), not depending on \( n \), can be found so that the translations \( w_n(y) = u_n(y + \tilde{y}_n) \) are all defined in \( B(0, R_1) \cap \mathbb{R}^N_+ \). Setting \( U = B(0, R_1) \cap \mathbb{R}^N_+ \), \( U_n = \lambda_n U \) and performing the change

\[
v_n(y) = M_n^{-1} u_n(\lambda_n^{-1} y + \tilde{y}_n),
\]

the functions \( v_n \) define solutions to:

\[
\begin{align*}
\Delta v + \sum_{i=1}^{N-1} a_{i,n}(y) \frac{\partial^2 v}{\partial y_i \partial y_N} + \lambda_n^{-1} b_n(y) \frac{\partial v}{\partial y_n} &= \lambda_n^{-2} M_n^{-1} a_n(y) v^p, & y \in U_n^+ \\
\nabla v \cdot \nu_{1,n}(y) &= \lambda v, & y \in U_n \cap \partial \mathbb{R}^N_+,
\end{align*}
\]

where \( a_{i,n}(y) = a_i(\lambda_n^{-1} y + \tilde{y}_n), b_n(y) = b(\lambda_n^{-1} y + \tilde{y}_n), a_n(y) = a(\lambda_n^{-1} y + \tilde{y}_n), \nu_{1,n}(y) = \nu_1(\lambda_n^{-1} y + \tilde{y}_n). \)

Observe that \( U_n \to \mathbb{R}^N_+, U_n \to \overline{\mathbb{R}^N_+} \). Since \( |v_n|_{\infty, U_n} = 1 \) for all \( n \), then the interior version of the \( L^p \) and Schauder estimates imply that (modulo a subsequence) \( v_n \to v \) in \( C^{2,\beta}(\mathbb{R}^N_+) \) for all \( 0 < \beta < \gamma_1 \). Moreover, by employing the “up to the boundary” version of such estimates in the region \( B(0, R_2) \cap \mathbb{R}^N_+, R_2 > 0 \) arbitrary and \( n \) large, leads to the validity of such convergence in \( C^{2,\beta}_{\text{loc}}(\mathbb{R}^N_+) \). That is why \( v \) defines a nonnegative solution to:

\[
\begin{cases}
\Delta v = 0 & x \in \mathbb{R}^N_+ \\
- \frac{\partial v}{\partial y_N} = v & x \in \partial \mathbb{R}^N_+,
\end{cases}
\]

such that \( 0 \leq v \leq 1, v(0) = 1 \). We are now showing that this is impossible. Consider in fact a \( C^3 \) bounded subdomain \( D \) of \( \mathbb{R}^N_+ \) such that \( \Gamma := \partial D \cap \partial \mathbb{R}^N_+ = \overline{B(0,1)} \cap \partial \mathbb{R}^N_+ \) and set \( \Gamma' = \partial D \cap \mathbb{R}^N_+ \). Define as \( D_n \) the magnified version \( D_n = nD \) of \( D \), \( \Gamma_n = n\Gamma \), \( \Gamma'_n = n\Gamma' \). According to Theorem 8 the eigenvalue problem

\[
\begin{cases}
\Delta u = 0 & x \in D_n \\
- \frac{\partial u}{\partial y_N} = \sigma u & x \in \Gamma_n \\
u = 0 & x \in \Gamma'_n,
\end{cases}
\]

admits a first eigenvalue \( \sigma = \sigma_{1,n} \) with a positive associated eigenfunction \( \phi_n \in H^1(D_n) \cap W^{2,q}(D_n) \cap C^{2,\gamma}(D_n \cup T) \), for all \( 1 < q < 4/3 \) and any closed \( T \subset B(0, n) \cap \partial \mathbb{R}^N_+ \). Moreover, \( \sigma_{1,n} \to 0 \) (Remark 4, (a)).

Multiplying the equation in (4.4) by \( \phi_n \), integrating and taking into account Remark 4, (b) gives

\[
0 = (1 - \sigma_n) \int_\Gamma_n \nu \phi_n - \int_{\Gamma'_n} \nu \frac{\partial \phi}{\partial \nu} \geq (1 - \sigma_n) \int_{\Gamma_R} \nu \phi_n,
\]

which is not possible provided \( n \) is large.

In conclusion, (4.1) can not hold, and this proves the theorem. \qed
As a corollary, we obtain that solutions \( u_\lambda \) to (1.1) develop a dead core \( \mathcal{O}_\lambda = \{ x \in \Omega : u_\lambda(x) = 0 \} \) as \( \lambda \) grows. It turns out that this dead core covers \( \Omega \) with a speed that can be estimated if \( a > 0 \) on \( \partial \Omega \). For the sake of completeness, we are next providing a direct proof of these facts (see also [3] and [9]).

**Proof of Theorem 1 (iv).** Let us begin with the case \( a > 0 \) on \( \partial \Omega \). Choose \( x \in \Omega \) arbitrary and let \( a_0 > 0 \) be the infimum of \( a(x) \) in \( \Omega \). For \( y \in B \) (the unit ball), define
\[
v(y) = d(x)^{-\alpha} a_0^{-\frac{1}{1-p}} u(x + d(x)y),
\]
where \( \alpha = 2/(1-p) \). Then \( v \) satisfies \( \Delta v \geq v^p \) in \( B \), \( v \leq \varepsilon := C a_0^{-\frac{1}{1-p}} (d(x)\lambda)^{-\alpha} \) on \( \partial B \). It follows that \( v \leq z \), which is the unique solution to the Dirichlet problem:
\[
\begin{align*}
\Delta z &= z^p \quad \text{in} \quad B \\
z &= \varepsilon \quad \text{on} \quad \partial B.
\end{align*}
\]
On the other hand, it can be directly checked that the radial function
\[
\tilde{z}(r) = \begin{cases} 
0, & r \leq \theta \\
A_N(r - \theta)\alpha, & \theta < r < 1
\end{cases}
\]
with \( r = |x| \), \( A_N = [\alpha(\alpha + N - 2)]^{-1/(1-p)} \) (the value \( A \) in Theorem 3 (iii) corresponds to \( A_N \) with \( N = 1 \)), \( \theta = 1 - (\varepsilon/A_N)^{1/\alpha} \), defines a supersolution to (4.6) which equals \( \varepsilon \) at \( r = 1 \). Since \( \tilde{z} = 0 \) is a comparable subsolution it follows by the method of sub and supersolutions (cf. [2]) and the uniqueness of solutions to (4.6) that \( u \) vanishes in \( \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq (\varepsilon/A_N)^{1/\alpha} = K \lambda^{-1} \} \). The last assertion in (iv) is thus proved.

For the general case one repeats the argument by replacing above \( d(x) \) with \( d(x)/2 \) and \( a_0 \) with the minimum of \( a \) in \( \{ y \in \Omega : \text{dist}(y, \partial \Omega) \geq d(x)/2 \} \) to prove that \( \mathcal{O}_\lambda \neq \emptyset \). Then it suffices with observing that \( u = 0 \) in \( \{ d(y) \geq \delta \} \) provided \( u = 0 \) on \( \{ d(y) = \delta \} \). This finishes the proof.

\[\square\]

5. Multiplicity

The fact that solutions develop a dead core leads to a fail in uniqueness of nonnegative solutions for large \( \lambda \). Indeed, we are proving in this section that for domains \( \Omega \) with boundary consisting in more than one connected piece, nonnegative solutions are not unique.

The proof of this fact relies in constructing solutions whose support for large \( \lambda \) is concentrated near a prefixed connected piece of the boundary. To succeed in this proposal we need to study the following auxiliary version of problem (1.1).

**Theorem 11.** Let \( \Omega \subset \mathbb{R}^N \) be a class \( C^{2,\gamma} \) bounded domain with \( \partial \Omega = \Gamma \cup \Gamma' \), \( \Gamma, \Gamma' \) nonempty and verifying \( \Gamma \cap \Gamma' = \emptyset \), while \( a \in C^\gamma(\Omega) \), \( a(x) > 0 \) for each \( x \in \Omega \). Then, the boundary value problem,
\[
\begin{align*}
\Delta u &= a(x) u^p \quad x \in \Omega \\
\frac{\partial u}{\partial \nu} &= \lambda u \quad x \in \Gamma \\
u &= 0 \quad x \in \Gamma'
\end{align*}
\]

...
satisfies the following properties:

(i) Nontrivial and nonnegative solutions to (5.1) are only possible if $\lambda > \sigma_1$, $\sigma_1$ the principal eigenvalue of (2.4). In addition, (5.1) admits a nonnegative nontrivial solution $u \in C^{2,\gamma_1}(\Omega)$, $\gamma_1 = \min\{\gamma, p\}$, for each $\lambda > \sigma_1$.

(ii) There exists $\lambda_1 > \sigma_1$ such that (5.1) admits a unique positive solution $u_\lambda \in C^{2,\gamma_1}(\Omega)$ for each $\sigma_1 < \lambda < \lambda_1$ where the mapping $\lambda \rightarrow u_\lambda$ as attaining values in $C^{2,\gamma_1}(\Omega)$ is real analytic and,

\[
u_\lambda = \left(\frac{\bar{\mu}_0}{\lambda - \sigma_1}\right)^{\frac{1}{p-1}} \left(\phi_1 + \frac{1}{\bar{\mu}_0} w(\cdot, \lambda)(\lambda - \sigma_1)\right),
\]

where $w \in C^{2,\gamma_1}(\Omega)$, $w|_\Gamma = 0$, $\int_\Omega w\phi_1 = 0$, $w = w_0$ at $\lambda = \sigma_1$, with $\bar{\mu}_0 = \frac{\int_\Omega \phi_1^{p+1}}{\int_{\partial\Omega} \phi_1^p}$ and $w_0 \in C^{2,\gamma_1}(\Omega)$ is the unique solution to the problem

\[
\begin{cases}
\Delta w = a(x)\phi_1^p & x \in \Omega \\
\frac{\partial w}{\partial \nu} = \sigma_1 w + \bar{\mu}_0 \phi_1 & x \in \Gamma \\
w = 0 & x \in \Gamma'.
\end{cases}
\]

There exist positive constants $\lambda_2, C$ such that every nonnegative solution $u$ satisfies

\[
u \leq C\lambda^{-\frac{2}{p-2}},
\]

for $\lambda \geq \lambda_2$. In addition every nonnegative nontrivial solution $u_\lambda$ to (5.1) for $\lambda \geq \lambda_2$ exhibits a dead core $O_\lambda = \{u_\lambda(x) = 0\}$. Moreover, its support concentrates near $\Gamma$ as $\lambda \rightarrow \infty$ in the sense that $\{x : \text{dist}(x, \Gamma) \geq d_\lambda\} \subset O_\lambda$ where $d_\lambda \rightarrow 0^+$ as $\lambda \rightarrow +\infty$. Again, $d_\lambda = K/\lambda$, $K > 0$, if $a > 0$ on $\Gamma$.

Proof. Concerning (i) we first observe that the conclusions of Lemma 7 also hold true for weak solutions $u \in H_{\Gamma}^p(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma'} = 0\}$ to (5.1). Accordingly, any nonnegative solution $u \neq 0$ does not vanish identically on $\Gamma$. Since such solution satisfies $\int_\Omega |\nabla u|^2 < \lambda \int_\Omega u^2$ one gets from (2.5) that $\lambda > \sigma_1$.

The existence of a nonnegative nontrivial solution for $\lambda > \sigma_1$ follows as in Theorem 1 by minimizing the functional

\[
J(u) = \int_\Omega |\nabla u|^2 - \lambda \int_{\partial\Omega} u^2
\]

in the set $M = \{u \in H^1_{\Gamma}(\Omega) : \int_\Omega a(x)|u|^{p+1} = 1\}$. The crucial point now is that $J$ is negative at the infimum only when $\lambda > \sigma_1$ (otherwise, $J \geq 0$ and the infimum is attained at $u = 0$).

As for (iii) observe that the proof of the estimate $u_\lambda \leq C\lambda^{-\frac{2}{p-2}}$ does not require any changes regarding the corresponding one in the case of Theorem 1 since the maximum of $u_\lambda$ must necessarily be attained on $\Gamma$. Therefore and by the same arguments as in Section 4, the support of $u_\lambda$ for large $\lambda$ is contained in the set $\{\text{dist} \{x, \partial \Omega\} < d_\lambda\}$ with $d_\lambda \rightarrow 0^+$. However, we further assert that the support of $u_\lambda$ is indeed contained in $\{\text{dist} \{x, \Gamma\} < d_\lambda\}$. In fact, notice that $u_\lambda$ verifies $\Delta u \geq 0$ in $D = \{0 < \text{dist} \{x, \Gamma\} < d_\lambda\}$, together with $u_\lambda = 0$ on $\partial D$ and so $u_\lambda = 0$ in $D$. Hence, the support of $u_\lambda$ has to be contained in $\{\text{dist} \{x, \Gamma\} < d_\lambda\}$. 
To finish we sketch the proof of (ii). To begin with, it can be shown by a similar reasoning as in Lemma 10 that any possible sequence \((\lambda_n, u_n)\) of nonnegative nontrivial solutions to (5.1) with \(\lambda_n \to \sigma_1+\) satisfies \(|u_n|_\infty \to \infty\) and can be written as:

\[
\lambda_n = \sigma_1 + \tilde{\mu}_n t_n, \quad u_n = t_n^{1 \over \gamma} (\phi_1 + t_n \tilde{w}_n)
\]

where \(\tilde{\mu}_n \to \tilde{\mu}_0, \; t_n \sim |u_n|_\infty^{-1} \to 0+\), \(\tilde{w}_n \to \tilde{w}_0\) in \(C^{2,\beta}(\Omega)\) for every \(0 < \beta < \gamma_1\) where \(\tilde{\mu}_0, \tilde{w}_0 \in C^{2,\gamma_1}(\Omega)\) are given in the statement of the theorem.

On the other hand, for \(\lambda\) close to \(\sigma_1\) and by writing solutions \((\lambda, u)\) as \(\lambda = \sigma_1 + \mu t, u = t^{-1/(1-p)} \phi_1 + tw\) with \(t \to 0\), \(\mu \sim \tilde{\mu}_0\) and \(w\) close to \(\tilde{w}_0\) in the space \(Z := \{u \in C^{2,\gamma_1}(\Omega) : u_{|\Gamma'} = 0, \int_{\Omega} u \phi_1 = 0\}\), problem (5.1) is transformed into,

\[
\begin{cases}
\Delta w = a(x)(\phi_1 + tw)^p & x \in \Omega \\
{\partial w \over \partial \nu} - \sigma_1 w = \mu(\phi_1 + tw) & x \in \Gamma \\
w = 0 & x \in \Gamma'.
\end{cases}
\tag{5.4}
\]

Taking into account that the linear problem \(\Delta u = f\) in \(\Omega, {\partial u \over \partial \nu} - \sigma_1 u = g\) on \(\Gamma, u_{|\Gamma'} = 0\) is uniquely solvable in \(Z\) for \((f, g) \in X := C^{\gamma_1}(\Omega) \times C^{\gamma_1,1}(\Omega)\) provided \(\tilde{l}(f, g) := \int_{\Omega} \phi_1 - \int_{\Gamma} g \phi_1 = 0\) (cf. the Proof of Lemma 10) then the corresponding solution operator \(u = K_1(f, g)\) defines an isomorphism \(K_1 : \text{ker } \tilde{l} \to Z\). By observing that \((f, g) \to (f, g + {\tilde{l}(f, g) \over \int_{\Gamma} \phi_1})\) defines a projection from \(X\) onto \(\ker \tilde{l}\), solving (5.4) in \((t, \mu, w)\) close to \((0, \tilde{\mu}_0, \tilde{w}_0)\) in \(\mathbb{R}^2 \times Z\) amounts to solving the equation,

\[
H(t, \mu, w) = 0,
\tag{5.5}
\]

with \(H = (H_1, H_2), H_1(t, \mu, w) = \tilde{l}(F(t, \mu, w), \mu G(t, w)), F = a(\phi_1 + tw)^p, G = \phi_1 + tw\), and

\[
H_2(t, \mu, w) = w - K_1 \left( F(t, \mu, w), \mu G(t, w) + {H_1(t, \mu, w) \over \int_{\Gamma} \phi_1} \right).
\]

That (5.5) is uniquely solved in \((-\varepsilon, \varepsilon) \times (\tilde{\mu}_0 - \varepsilon, \tilde{\mu}_0 + \varepsilon) \times 2_B(\tilde{w}_0, \delta)\) for \(\varepsilon, \delta > 0\) small follows again from the Implicit Function theorem. As a preliminary checking of regularity observe that \(F\) defines a real analytic Nemytskii operator near \((t, \tilde{\mu}_0, \tilde{w}_0)\) in \(\mathbb{R}^2 \times Z\) with values in \(C^{\gamma_1}(\Omega)\). In this regard \(F\) must be written as

\[
F(t, \mu, w) = a(x)\phi_1^p \left( 1 + t \frac{w}{\phi_1} \right)^p,
\]

and keep \(t\) small for \(w \in B_Z(\tilde{w}_0, \delta)\). By using that both \(\phi_1, w\) vanish at \(\Gamma'\) and the fact that \({\partial \phi_1 \over \partial \nu} < 0\) on \(\Gamma'\) it follows that the mapping \(w \to \frac{w}{\phi_1}\) from \(Z\) to \(C^1(\Omega)\) defines a bounded linear operator. This finishes the proof. \(\Box\)

Remark 7. By an argument similar to the one used in the beginning of the proof of Lemma 10 it can be shown that any sequence of solutions \((\lambda_n, u_{\lambda_n})\) to (5.1) is bounded in \(L^\infty\) provided
\( \lambda_n \) does not accumulate at \( \lambda = \tilde{\sigma}_1 \). It is also shown by the same means that no sequence can exhibit a subsequence \((\lambda'_{n'}, u_{\lambda'_{n'}})\) with \( \lambda'_{n'} \to \lambda \in (\tilde{\sigma}_1, +\infty) \) and \( u_{\lambda'_{n'}} \to 0 \) in \( H^1(\Omega^+) \). In particular, (5.1) does not admits \((\lambda, u) = (\lambda, 0)\) as a bifurcation point to nonnegative solutions for any \( \lambda > \tilde{\sigma}_1 \).

The next statement deals with a fixed component \( \Gamma_i \) of \( \partial \Omega \). It is an immediate consequence of applying Theorem 11 to \( \Omega \) under the choice \( \Gamma = \Gamma_i, \Gamma' = \partial \Omega \setminus \Gamma_i \) for the boundary conditions.

**Corollary 12.** Let \( \Gamma_i \) be any fixed connected component of \( \partial \Omega \). Then for large \( \lambda \) problem (1.1) has at least a nonnegative nontrivial solution \( u \) such that \( u = 0 \) in \( \{ \text{dist}(x, \Gamma_i) \geq d \lambda \} \), \( d \lambda \to 0 \) as \( \lambda \to +\infty \). The rate of convergence \( d \lambda \) can be chosen as \( K/\lambda \) for certain \( K > 0 \) if \( a > 0 \) on \( \Gamma_i \).

We can now proceed to the proof of Theorem 2.

**Proof of Theorem 2.** If \( \Gamma_1, \ldots, \Gamma_k \) are the connected components of \( \partial \Omega \) then Corollary 12 implies the existence of \( k \) nonnegative nontrivial solutions \( u_{\lambda,k} \) whose support is localized near \( \Gamma_k \) for large \( \lambda \). The supports will indeed be disjoint for large \( \lambda \). This allows us to conclude that every sum of the form \( u_{\lambda,i_1} + \ldots + u_{\lambda,i_r} \) is again a solution to (1.1). Since there are \( 2^k - 1 \) such sums, we have shown that there exist at least \( 2^k - 1 \) different nonnegative nontrivial solutions to (1.1) for large \( \lambda \).

---

**6. The case of the ball**

In this section we are considering a special case of problem (1.1) by analyzing its behavior in the unit ball \( B \) of \( \mathbb{R}^N \). To simplify the exposition we are focussing our interest in the autonomous case represented by the coefficient \( a(x) = 1 \). We begin our treatment by searching only for nonnegative radially symmetric solutions \( u = u(r) \), \( r = |x| \). They satisfy

\[
\begin{align*}
  (r^{N-1}u')' &= r^{N-2}u^p, & 0 \leq r < 1, \\
  u'(0) &= 0, \quad u'(1) = \lambda u(1) \quad \text{(6.1)}
\end{align*}
\]

The scaling \( v(r) = \lambda^\alpha u(\lambda^{-1}r), \alpha = 2/(1-p) \), transforms this problem into,

\[
\begin{align*}
  (r^{N-1}v')' &= r^{N-2}v^p, & 0 \leq r < \lambda, \\
  v'(0) &= 0, \quad v'(\lambda) = v(\lambda) \quad \text{(6.2)}
\end{align*}
\]

Modifying the proof of point (i) in Theorem 1 by both restricting \( J \) and \( M \) to the class of radial functions in \( H^1(B) \), one can show the existence of a nontrivial nonnegative radial solution to (1.1) for all \( \lambda > 0 \), hence the equivalent assertion for problem (6.1), which is point (i) in Theorem 3. On the other hand, it has been already shown (Theorem 1 (iv)) that all nonnegative solutions exhibit a dead core for large \( \lambda \) which completely fills \( B \) as \( \lambda \to +\infty \). This implies the existence of \( d = d(\lambda) > 0 \), \( \lambda - K \leq d < \lambda \) and so \( d \to \infty \) as
\( \lambda \to \infty \), such that any nonnegative and nontrivial solution \( v \) to (6.2) satisfies \( v = 0 \) in \([0, d]\), \( v > 0 \) in \((d, \lambda]\). Therefore, setting \( w(r) = v(d + r) \), \( w \) defines a solution to

\[
\begin{cases}
((r + d)^{N-1}w')' = (r+d)^{N-1}w^p & 0 \leq r < \lambda - d \\
w(0) = 0, \ w'(0) = 0, \ w'(\lambda - d) = w(\lambda - d).
\end{cases}
\]  

(6.3)

Such solution is nontrivial in the sense that \( w(r) > 0 \) for \( r \in (0, \lambda - d) \) (observe that solutions to (2.6) are nontrivial, see Section 2).

Exploiting these remarks we are now showing items (ii) and (iii) in Theorem 3. (compare with the proof of Theorem 3.4 in [12]).

**Proof of Theorem 3 (ii) and (iii).** To prove (ii) notice that the uniqueness assertion in Theorem 1 (ii) furnishes a unique radial positive solution \( u_\lambda \) for \( 0 < \lambda < \lambda_0 \) satisfying the properties stated there.

To show the uniqueness for large \( \lambda \), consider the Cauchy problem

\[
\begin{cases}
((r + d)^{N-1}w')' = (r+d)^{N-1}w^p & \text{in } (0, \delta) \\
w(0) = 0, \ w'(0) = 0
\end{cases}
\]

which, by Theorem 9, has a unique nontrivial solution \( w(r, d) \) in \([0, +\infty)\) for every \( d \geq 0 \). Moreover, as \( d \to +\infty \), \( w \) converges in \( C^1_{\text{loc}}[0, +\infty) \) to \( w_0(r) = Ar^\alpha \), where \( A \) and \( \alpha \) are as in the statement of the theorem.

By the preceding discussion, showing the uniqueness amounts to prove the existence as \( d \to \infty \) of a unique positive zero \( r = T \) of the equation,

\[ w'(r, d) - w(r, d) = 0, \]  

(6.5)

such that \( T = O(1) \) while \( T + d = \lambda \) is uniquely solvable in \( d \) as \( d \to \infty \).

The solvability with uniqueness of (6.5) for \( d \) large follows from the fact that the limit equation \( w''_0(r) - w_0(r) = 0 \) has \( r = \alpha \) as a unique simple zero. In fact observe that \( w''_0(\alpha) - w'_0(\alpha) = -A\alpha^{\alpha-1} \). Thus (6.5) admits a root \( T(d) \) such that \( T(d) \to \alpha \) as \( d \to +\infty \).

We now claim that \( T(d) \) is a \( C^1 \) function for large \( d \) and verifies \( T'(d) \to 0 \) as \( d \to +\infty \). This is a consequence of the Implicit Function theorem, by first noticing that \( w(r, d) \) is \( C^1 \) in both variables (Theorem 9), while in addition

\[
T'(d) = \frac{w_d(T(d), d) - w'_d(T(d), d)}{w''(T(d), d) - w'(T(d), d)}
\]

(the subscript \( d \) denotes derivation with respect to \( d \) and ' stands for the derivative with respect to \( r \)). As already seen, the denominator is negative for large \( d \) since it converges to \( w''_0(\alpha) - w'_0(\alpha) = -A\alpha^{\alpha-1} \), while the numerator tends to zero thanks to (2.7) in Theorem 9. This proves the claim.

Finally, notice that solutions to (6.3) arise whenever there is a solution to the equation \( d + T(d) = \lambda \). Since \( d + T(d) \) is increasing for large \( d \), it follows that for large \( \lambda \) there is a unique \( d = d(\lambda) \) so that \( w(r, d) \) is a solution to (6.3). The unique solution to (6.1) for large \( \lambda \) is then obtained by setting

\[
u_\lambda(r) := \begin{cases} 
0 & \text{if } 0 \leq r \leq \frac{d}{\lambda} \\
\lambda^{-\alpha}w(\lambda r - d) & \text{if } r > \frac{d}{\lambda}.
\end{cases}
\]  

(6.6)
This proves part (ii).

Part (iii) follows at once by noticing that \( w(T(d), d) \to w_0(\alpha) = A\alpha^\alpha \) and \( r(\lambda) = d(\lambda)/\lambda = 1 - T(d(\lambda))/\lambda \sim 1 - \alpha/\lambda \).

**Corollary 13.** Let \( u_\lambda \) be the unique nonnegative nontrivial radial solution to (1.1) for large \( \lambda \). Then

\[
\int_B u_\lambda^{p+1} \sim \omega_N A^{p+1} \frac{\alpha^{(p+1)+1}}{\alpha(p+1)+1} \lambda^{-\alpha(p+1)-1}, \quad \text{as } \lambda \to +\infty,
\]

where \( \omega_N \) is the surface measure of the unit ball of \( \mathbb{R}^N \), and \( A, \alpha \) are given in Theorem 3.

**Proof.** According to (6.6), we have

\[
\int_B u_\lambda^{p+1} = \omega_N \lambda^{-\alpha(p+1)} \int_0^\lambda \frac{r^{N-1} w(\lambda r - d)^{p+1} dr}{\lambda^{N-1}} \int_0^{\lambda-d} \frac{(s+d)^{N-1} w(s)^{p+1} ds}{\lambda^{N-1}} \sim \omega_N \lambda^{-\alpha(p+1)-1} \int_0^{\alpha} w_0(s)^{p+1} ds = \omega_N A^{p+1} \frac{\alpha^{(p+1)+1}}{\alpha(p+1)+1} \lambda^{-\alpha(p+1)-1},
\]

as \( \lambda \to \infty \).

Of course one could ask the question whether there are nonradial solutions, or if the radial one is the unique one. Since \( \partial B \) is connected we can not obtain multiplicity as in Section 5. However, we prove next that besides the radial solution there is indeed another nontrivial nonnegative and nonradial solution for large \( \lambda \).

**Proof of Theorem 3 (iv).** We are showing that the solution constructed in Theorem 1 (i) is not the radial one. Denote, as in that theorem,

\[
J(u) = \int_B |\nabla u|^2 - \lambda \int_{\partial B} u^2, \quad u \in M = \left\{ u \in H^1(B) : \int_B |u|^{p+1} = 1 \right\}.
\]

For the radial solution \( u_\lambda \) we have

\[
J(u_\lambda) = -\int_B u_\lambda^{p+1},
\]

so that \( v_\lambda = u_\lambda/|u_\lambda|_{L^{p+1}} \in M \) verifies \( J(v_\lambda) = -|u_\lambda|_{L^{p+1}}^{\frac{1-p}{p+1}} \). Then by Corollary 13, there exists a positive constant \( C \) so that

\[
J(v_\lambda) \sim -C\lambda^{\frac{p+1}{p+1}}, \quad \text{as } \lambda \to +\infty.
\]

(6.7)

We are next using (6.7) to prove that \( J(v_\lambda) \) is not the minimum of \( J \) in \( M \) for large \( \lambda \). For this aim, we construct a family of functions \( \psi_\lambda \in M \) so that \( J(\psi_\lambda) < J(v_\lambda) \) for large \( \lambda \).

We first claim that a large radius \( R > 0 \) and a function \( \psi \in C_0^\infty(B_R), B_R = B(0, R) \), can be found so that

\[
\int_{B_R^+} |\nabla \psi|^2 - \int_{\Gamma_R} \psi^2 < 0,
\]

(6.8)
where \( B_R^+ = B_R \cap \mathbb{R}_+^N \), \( \Gamma_R = B_R \cap \partial \mathbb{R}_+^N \).

Assumed the existence of such \( \psi \) set \( \psi_\lambda(x) = C(\lambda)\psi(\lambda(x + e_1)) \in C^\infty_0(B(-e_N, \lambda^{-1}R)) \), which belongs to \( M \) provided \( C(\lambda) = K\lambda^{-\frac{N}{p+1}} \), where \( K\lambda^{-1} = |\psi|_{p+1,B_R} \). With this choice of \( \psi_\lambda \):

\[
J(\psi_\lambda) = K^2\lambda^{\frac{2N}{p+1}+2-N} \left( \int_{B_R \cap B(\lambda e_N, \lambda)} |\nabla \psi|^2 - \int_{B_R \cap \partial B(\lambda e_N, \lambda)} \psi^2 \right)
\approx K^2\lambda^{\frac{2N}{p+1}N+2} \left( \int_{B_R^+} |\nabla \psi|^2 - \int_{\Gamma_R} \psi^2 \right).
\]

Since \((1-p)N/(1+p) + 2 > (p+3)(p+1)\), we obtain in virtue of (6.7) and (6.8), that \( J(\psi_\lambda) < J(v_\lambda) \) for large \( \lambda \). As remarked before, this entails the existence of a nontrivial nonnegative and nonradial solution to (1.1).

To show the claim let \( \phi \in H^1(B_R^+) \) be a positive eigenfunction to,

\[
\begin{cases}
\Delta u = 0 & \text{in } B_R^+ \\
-\frac{\partial u}{\partial x_N} = \sigma u & \text{on } \Gamma_R \\
u = 0 & \text{on } \Gamma_R'
\end{cases}
\]

(6.9)

corresponding to the principal eigenvalue \( \sigma = \sigma_1(R) \) (Theorem 6 and Remark 8 in [11]). Provided \( R \) is big enough so that \( \sigma_1(R) < 1 \) (Remark 4 (a)) \( \phi \) satisfies:

\[
\int_{B_R^+} |\nabla \phi|^2 - \int_{\Gamma_R} \phi^2 = (\sigma_1(R) - 1) \int_{\Gamma_R} \phi^2 < 0.
\]

The searched function \( \psi \) can be now obtained from \( \phi \) by regularization. This concludes the proof.

7. Vanishing weights

In this final section, we give the proof of Theorems 4, 5 and 6, considering in problem (1.1) a weight \( a(x) \) that vanishes in a subdomain of \( \Omega \).

**Proof of Theorem 4.** We only sketch the proof of part (i), the remaining points being proved in exactly the same way as in Theorem 1 (ii), (iii) and (iv). If \( u \) is a nonnegative solution to (1.1) with \( \lambda \leq 0 \) (which, due to Lemma 7 can be supposed in \( C^2(\overline{\Omega}) \)), we deduce

\[
\int_{\Omega} a(x)u^p = 0,
\]

and so \( u = 0 \) in \( \Omega \setminus \Omega_0 \). Thus \( u \) verifies

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega_0 \\
\frac{\partial u}{\partial \nu} = \lambda u & \text{on } \Gamma_1 \\
u = 0 & \text{on } \Gamma_2,
\end{cases}
\]

where \( \Omega_0 \).
and it follows from the maximum principle that \( u = 0 \) in \( \Omega_0 \) as well. Hence, \( u \equiv 0 \).

Assume now \( 0 < \lambda < \sigma_1 \). If the functional

\[
J(u) = \int_{\Omega} |\nabla u|^2 - \lambda \int_{\Omega} u^2
\]

were not coercive in \( M := \{ u \in H^1(\Omega) : \int_{\Omega} a(x)|u|^{p+1} = 1 \} \), then we would obtain (proceeding as in the proof of Theorem 1 (i)) a function \( v \in H^1(\Omega) \) with \( |v|_{L^2(\Omega)} = 1 \) and satisfying the relations

\[
\int_{\Omega} |\nabla v|^2 \leq \lambda, \quad \int_{\Omega} a(x)|v|^{p+1} = 0.
\]

The last one implies \( v = 0 \) in \( \Omega \setminus \Omega_0 \) (in particular on \( \Gamma_2 \)). If \( \sigma_1 = +\infty \) it follows \( v = 0 \) on \( \partial \Omega \) what is impossible. If, on the contrary, \( \sigma_1 < \infty \) then (2.5) yields \( \sigma_1 \leq \lambda \), also a contradiction. Hence \( J \) is coercive and there exists a nonnegative nontrivial solution to (1.1).

Lemma 14. Assume \( \sigma_1 < +\infty \), and let \( u \in C^{2,\gamma}(\overline{\Omega}) \) be a nonnegative solution to (1.1) with \( \lambda \geq \sigma_1 \). Then \( u \equiv 0 \) in \( \Omega_0 \).

Proof. Multiplying (1.1) by the eigenfunction \( \phi_1 \) associated to \( \sigma_1 \) and integrating in \( \Omega_0 \) (notice that \( \phi_1 \in C^{2,\gamma}(\overline{\Omega}), u \in C^{2,\gamma}(\overline{\Omega}) \)), we obtain

\[
(\lambda - \sigma_1) \int_{\Gamma_1} u \phi_1 = \int_{\Gamma_2} u \frac{\partial \phi_1}{\partial \nu}.
\]

(7.1)

If \( \lambda > \sigma_1 \), we directly obtain \( u = 0 \) on \( \Gamma_1 \cup \Gamma_2 = \partial \Omega_0 \), and since \( \Delta u = 0 \) in \( \Omega_0 \), \( u \equiv 0 \) in \( \Omega_0 \) follows.

If \( \lambda = \sigma_1 \), then we have from (7.1) that \( u = 0 \) on \( \Gamma_2 \). Thus, \( u \) is an eigenfunction associated to \( \sigma_1 \), and the simplicity of \( \sigma_1 \) ([11], Theorem 6) implies \( u = c \phi \) for some nonnegative constant \( c \). If \( c > 0 \), then \( \frac{\partial u}{\partial \nu} < 0 \) on \( \Gamma_2 \), which would imply that \( u \) changes sign in \( \Omega \). Thus \( c = 0 \) and we arrive again at \( u \equiv 0 \) in \( \Omega_0 \). This completes the proof.

Proof of Theorem 5. To show (i), recall that \( \tilde{\sigma}_1 = +\infty \) is equivalent to \( \Gamma_1 = \partial \Omega \). Since by Lemma 14 we have \( u = 0 \) on \( \Gamma_1 \) for every nonnegative solution \( u \) with \( \lambda \geq \sigma_1 \) and \( u \) is subharmonic, we arrive at \( u = 0 \) in \( \Omega \).

As for (ii), suppose \( u \) is a nonnegative nontrivial solution corresponding to \( \lambda \geq \sigma_1 \). By Lemma 14, \( u \equiv 0 \) in \( \Omega_0 \) which implies \( u = 0 \) on \( \Gamma_1 \cup \Gamma_2 \). Since \( \partial \Omega^+_i \cap \Omega = \Gamma_2 \), then \( u \) satisfies:

\[
\begin{cases}
\Delta u = a(x)u^\rho & x \in \Omega^+_i \\
\frac{\partial u}{\partial \nu} = \lambda u & x \in \partial \Omega^+_i \cap \partial \Omega \\
u = 0 & x \in \partial \Omega^+_i \cap \Omega,
\end{cases}
\]

(7.2)

for each component \( \Omega^+_i \) having \( \partial \Omega^+_i \cap \partial \Omega \neq \emptyset \). Observe that \( u = 0 \) in the remaining components \( \Omega^+_j \subset \subset \Omega \) since \( u = 0 \) on \( \partial \Omega^+_j \) and is subharmonic there. Therefore, provided \( u \neq 0 \), \( u \) must define a nontrivial solution of some of the problems (7.2). By Theorem 11, this means \( \lambda > \tilde{\sigma}_1 \), \( \geq \tilde{\sigma}_1 \).

Finally, according to (iii) of Theorem 11, each of the problems (7.2) defines for \( \lambda > \tilde{\sigma}_1 \) large a nontrivial solution to (1.1). These solutions and a combination of them (in the spirit of the proof of Theorem 2) ensure the validity of (iii).
**Proof of Theorem 6.** Observe that under the assumptions of the theorem \( u = 0 \) is the only solution to (1.1).

Let now \( \lambda_n \to \sigma_1^- \) with \( u_n \) a corresponding nonnegative nontrivial solution to (1.1) at \( \lambda = \lambda_n \). We claim that \( t_n := |u_n|_{\infty} \) is bounded. If not, passing to a subsequence we can assume \( t_n \to +\infty \). Denoting \( u_n = v_n/t_n \), we have

\[
\begin{cases}
\Delta v_n = a(x) v_n^p |u_n|^{p-1} & \text{in } \Omega \\
\frac{\partial v_n}{\partial \nu} = \lambda_n v_n & \text{on } \partial \Omega.
\end{cases}
\]

It is standard to obtain that a subsequence of \( \{v_n\} \) converges in \( C^2(\overline{\Omega}) \) to a function \( v \) which satisfies \( |v|_{\infty} = 1 \) and:

\[
\begin{cases}
\Delta v = 0 & \text{in } \Omega \\
\frac{\partial v}{\partial \nu} = \sigma_1 v & \text{on } \partial \Omega.
\end{cases}
\]  

(7.3)

By directly integrating (7.3), it follows that \( v = 0 \), and this contradicts \( |v|_{\infty} = 1 \). Thus \( t_n \) is bounded, and we can obtain as before that for a subsequence \( u_n \to u \) in \( C^{2,\beta}(\overline{\Omega}) \), \( 0 < \beta < \gamma_1 \), where \( u \) is a solution to (1.1) with \( \lambda = \sigma_1 \). Therefore \( u = 0 \), and this proves the convergence \( u_{\lambda} \to 0 \).

Theorem 1 (iv)). However, such region can never reach \( \Gamma \) in \( \Omega \). We claim that a nonempty interior region 

\( \{u_{\lambda} = 0\} \) is generated in \( \Omega^+ \) as \( \lambda \to \sigma_1^- \) (cf. the proof of Theorem 1 (iv)). However, such region can never reach \( \Gamma_2 \) nor \( \Omega_0 \) for \( \lambda \) close to \( \sigma_1 \). In fact, \( u_{\lambda}(x_0) = 0 \) at \( x_0 \in \Gamma_2 \) implies \( u_{\lambda} = 0 \) in \( \Omega_0 \). Otherwise, since \( u_{\lambda} \) is harmonic in \( \Omega_0 \) we get \( \frac{\partial u_{\lambda}}{\partial \nu}(x_0) < 0 \), \( \nu \) the outward unit normal to \( \Omega_0 \) at \( x_0 \), what contradicts the nonnegativeness of \( u_{\lambda} \). Hence \( u_{\lambda} = 0 \) in \( \Omega_0 \) (notice that such behavior is immediately achieved if \( u_{\lambda} = 0 \) somewhere in \( \Omega_0 \)). This means that we get a family \( u_{\lambda} \) of nontrivial solutions to the mixed problem (5.1) in \( \Omega^+ \) such that \( u_{\lambda} \to 0 \) as \( \lambda \to \sigma_1^- \). Observe that the existence of such family is forbidden if \( \sigma_1 \leq \tilde{\sigma}_1 \) since such behavior is not possible if \( \tilde{\sigma}_1 < \sigma_1 \), as observed in Remark 7. This shows that \( \mathcal{O}_\lambda \subset \Omega^+ \) for \( \lambda \to \sigma_1^- \). The remaining assertions follow in the same way as in the case of Theorem 1.

**Remark 8.** In order to illustrate with an example the relative values of \( \sigma_1 \) and \( \tilde{\sigma}_1 \) consider the annulus \( \Omega = \{R_1 < |x| < R_2\} \). Let \( R_1 < R < R_2 \) fixed. For \( R_1 < R < R_2 \) variable, set \( \Omega^+ = \{R_1 < |x| < R\} \), \( \Omega_0 = \{R < |x| < R_2\} \) and so \( \Gamma_1 = \{|x| = R_2\} \), \( \Gamma_2 = \{|x| = R\} \), \( \partial \Omega \setminus \Gamma_1 = \{|x| = R_1\} \). The principal eigenvalues \( \sigma_1, \tilde{\sigma}_1 \) (see (1.3), (1.4)) must in this case be associated to radial (harmonic) eigenfunctions and hence are explicitly given by:

\[
\sigma_1 = \frac{(N - 2)R^{N-2}}{R_2(R_2^{-1} - R^{-1})}, \quad \tilde{\sigma}_1 = \frac{(N - 2)R_1^{N-2}}{R_1(R_1^{N-2} - R^{-N-2})}. \tag{7.4}
\]

Observe that \( \sigma_1 = \tilde{\sigma}_1 \) at \( R = R \), \( R < R_1 < R < R_2 \) fixed. For \( R_1 < R < R_2 \), \( \sigma_1 < \tilde{\sigma}_1 \) for \( R < R < R_2 \), \( \sigma_1 > \tilde{\sigma}_1 \) if \( R < R < R_1 \).

Let us now construct an example showing the behavior described in Remark 1 (b). Let \( Q \subset \mathbb{R}^N \) be a class \( C^{3,\gamma} \) bounded domain, \( B(x_0, R) \) any large enough ball so that \( Q \subset B(x_0, R) \) and choose \( a \in C^\gamma(\mathbb{R}^N) \) such that \( a > 0 \) for \( |x| < R \), \( a = 0 \) in \( |x| \geq R \). By
observing problem (5.1) with $\Omega^+, \partial Q, \Gamma_2$ replacing now $\Omega, \Gamma$ and $\Gamma'$, Theorem 11 provides the existence of $\bar{\sigma}_1 < \lambda^* \leq \lambda_1$ (see (ii)) such that the unique positive solution $u_\lambda$ to (5.1), observed in $\Omega^+$, satisfies $\frac{\partial u_\lambda}{\partial \nu} < 0$ on $\Gamma_2$ ($\nu$ the outward unit normal to $\Omega^+$ on $\Gamma_2$) for each $\overline{\sigma}_1 < \lambda < \lambda^*$.

Finally define $\Omega = \Omega^+ \cup \{ R \leq |x| < R_2 \}$, $R_2 > R$. Since $\Omega_0 = \{ R < |x| < R_2 \}$, (7.4) allows to choose $R_2$ such that $\bar{\sigma}_1 < \sigma_1 < \lambda^*$. This means that problem (1.1) can not exhibit a nonnegative nontrivial solution $u$ at least for any $\sigma_1 \leq \lambda \leq \lambda^*$. In fact, such possible solution $u$ should vanish in $\Omega_0$ (Lemma 14) thus providing a nonnegative nontrivial solution to (5.1) with $\frac{\partial u_\lambda}{\partial \nu} = 0$ on $\Gamma_2$ what is not possible in such range for $\lambda$. This completes the announced construction.

References


