Superfast quenching.

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Abstract

We study positive solutions of a fast diffusion equation in a bounded interval with a nonlinear Neumann boundary condition,

$$\begin{cases} u_t = (u^{m-1}u_x)_x & (x,t) \in (0,L) \times (0,T), \\ (u^{m-1}u_x)(0,t) = u^m(0,t) & t \in (0,T), \\ (u^{m-1}u_x)(L,t) = 0 & t \in (0,T), \\ u(x,0) = u_0(x) & x \in [0,L], \end{cases}$$

where $m < 0$. Every positive solution quenches in a finite time. We prove that the quenching rate is not always the natural one given by homogeneity, but sometimes faster. We also study the quenching set, the asymptotic behaviour close to the quenching time and the possible continuation after that.

Key words: fast diffusion equation, nonlinear boundary conditions, quenching, asymptotic behaviour

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1 Introduction and main results

We deal with the problem

$$\begin{cases} u_t = (u^{m-1}u_x)_x & (x,t) \in (0,L) \times (0,T), \\ (u^{m-1}u_x)(0,t) = u^m(0,t) & t \in (0,T), \\ (u^{m-1}u_x)(L,t) = 0 & t \in (0,T), \\ u(x,0) = u_0(x) & x \in [0,L], \end{cases} \tag{1}$$

where $m < 0$. We assume that u_0 is a C^1 function that satisfies

(H1) $\quad u_0 \geq \delta > 0, \quad u_0' \geq 0, \quad u_0'(0) = u_0(0), \quad u_0'(L) = 0.$

The equation in (1) models for example, the diffusion of Cr or Zn and Be in GaAs [YTG], [YTG2], or the heat conduction in solid hydrogen [R].

Problem (1) can also be thought of as a model for nonlinear heat propagation, where u stands for the temperature. The boundary condition can be viewed as a particular case of a nonlinear radiation law at the boundary in which the term $u^{m-1}u_x$ represents the outgoing heat flux. This kind of boundary conditions appear also in combustion problems when the absorption happens only at the boundary of the container, for example because of the presence of a solid catalyzer, see [MV] for a justification. The choice of the prescribed flux $f(u) = u^m$ at $x = 0$ implies the invariance of the interval under the natural scaling of the problem, see (6).

Local in time existence of positive classical solutions of this problem and comparison arguments can be easily established. The time T is the maximal existence time for the solution. Our first result shows that T is always finite, in the sense that u vanishes at $x = 0$ and the heat flux at the boundary becomes singular. We say that u *quenches* in finite time. Some authors (see [K]) understand quenching when u_t becomes unbounded. In this situation this is also true:

Theorem 1 *For every initial data u_0 there exists a finite time $T > 0$ such that*

$$\liminf_{t \to T^-} u(0,t) = 0, \qquad \limsup_{t \to T^-} \|u_t(\cdot,t)\|_\infty = \infty.$$

Quenching phenomenon has deserved a great deal of attention in recent years, see for example [C], [L], [L2].

As to the velocity at which the solution tends to zero, an easy dimensional analysis of problem (1) shows that there exists a *natural quenching rate*

$$\alpha = \frac{1}{1-m} \tag{2}$$

in the sense that the following estimates

$$c_1(T-t)^{\alpha} \le u(0,t) \le c_2(T-t)^{\alpha} \tag{3}$$

would hold for every solution u that quenches at time $T > 0$. Our purpose in the present work is to prove that this is not always the case, and we could have

$$\liminf_{t\to T^-}(T-t)^{-\alpha}u(0,t) = 0.$$

In fact we will find that the limit of $(T-t)^{-\alpha}u(0,t)$ exists and is a positive constant (natural quenching rate) or

$$\lim_{t\to T^-}(T-t)^{-\alpha}u(0,t) = 0, \tag{4}$$

what we will call *superfast quenching*. In other words, for some solutions the quenching rate is *faster* than the natural one. We prove that superfast quenching depends on m and the length L of the interval. A similar result holds for the semilinear blow-up problem considered in [HV], though in that case the example of superfast blow-up appears only in large dimensions.

We remark that the upper bound in (3) is easy to derive and holds for all the solutions in all the cases $m < 0$ and $L > 0$, so quenching cannot be superslow.

On the other hand, problem (1) admits, for each $m < 0$ and some range of lengths L depending on m, solutions in self-similar form (which in this case means separated variables)

$$U(x,t) = (T-t)^{\alpha}F(x), \tag{5}$$

where α is the same as in (2). See section 3. The main result of the paper asserts that, under some restrictions on the initial data, the quenching rate for the solutions to problem (1) is natural if and only if there exist self-similar solutions. This is the reason why the natural rate could also be denominated as *self-similar rate*.

In order to characterize the quenching rates we need an extra monotonicity assumption. We assume

(H2) $\quad u_t \le 0 \quad$ for t near T.

This hypothesis holds for example for solutions with smooth compatible initial data such that $(u_0^{m-1}u_0')' \leq 0$. In some particular cases another condition on u_0 will be assumed:

(H3) The initial datum u_0 have only one intersection with $U(x,0)$, where $U(x,t)$ is a self-similar solution with the same quenching time as u, and also that $u_0(0) > U(0,0)$.

In particular we prove

Theorem 2 *Let hypotheses* $(H1)$ *and* $(H2)$ *hold.*

i) If $m < -1$ *then the quenching rate is always natural.*

ii) Let $-1 \leq m < 0$*. If* $0 < L \leq -1/m$ *then the quenching rate is natural, while there exists* $L_* = L_*(m) \geq -1/m$ *such that if* $L > L_*$ *the quenching rate is superfast.*

iii) If $-1/m < L \leq L_*$ *and* $-1/3 \leq m < 0$*, assume* $(H3)$ *holds. Then the quenching rate is natural.*

The critical length $L_*(m)$ appears as a limit case in the existence of self-similar profiles, see section 3. It satisfies $L_* > -1/m$ if $m < -1/3$ and $L_* = -1/m$ if $-1/3 \leq m < 0$. It will also be critical in the description of the quenching sets, see below. Remark that for the case $-1/m < L \leq L_*$, $-1/3 \leq m < 0$ we obtain the natural quenching rate but we have to impose some additional assumption on the initial data, that we conjecture is merely technical.

We next want to show that the asymptotic behaviour of the solutions $u(x,t)$ of problem (1) as t approaches T are described by the profiles F when they exist. Following the standard technique, we introduce the new rescaled function

$$f(x,\tau) = (T-t)^{-\alpha}u(x,t), \qquad \tau = -\log(1-t/T). \tag{6}$$

Therefore, the problem of the asymptotic behaviour of $u(x,t)$ near a finite quenching time $T > 0$ is reduced to the problem of the stabilization of $f(x,\tau)$ as $\tau \to \infty$. We prove the following results.

Theorem 3 *Let* u *be a solution to problem* (1) *satisfying* $(H1)$*.*

i) If the quenching rate is natural then there exist a sequence $t_n \to T$ *such that,*

$$\frac{u(x,t_n)}{(T-t_n)^\alpha} \to F(x) \qquad \text{as } n \to \infty,$$

where F *is one of the stationary profiles constructed in theorems 12 and 13. The profile is unique, and convergence holds for every sequence, if* $L < -1/m$*.*

ii) If the quenching rate is superfast and $(H2)$ *holds, then*

$$\frac{u^m(x,t)}{u^m(0,t)} \to V(x) = (1+mx)_+ \qquad \textit{as } t \to T.$$

Next we deal with the study of the points in $[0,L]$ where u vanishes, which coincides with the set, $Q(u)$, of points where u_t blows up, see corollary 8. As a consequence of the previous theorem, we obtain

Theorem 4 *Let* u *be a solution to problem* (1) *satisfying* $(H1)$*, and assume that the quenching rate is natural. Then, if* $-1 \le m < 0$ *the quenching is always global,* $Q(u) = [0,L]$*, while if* $m < -1$ *three different cases appear:*

i) If $0 < L \le -1/m$*, then the quenching is global.*
ii) If $-1/m < L \le L_*$*, then global quenching or regional quenching may occur depending on the initial data. Moreover,* $[0,-1/m] \subsetneq Q(u)$*.*
iii) If $L > L_*$*, then the quenching is regional and* $[0,-1/m] \subsetneq Q(u) \subseteq [0, 2/(1-m)]$*.*

Theorem 5 *Let* u *be a solution to problem* (1) *satisfying* $(H1)$ *and* $(H2)$*, and assume that the quenching rate is superfast. Then quenching is always regional, and moreover* $Q(u) = [0,-1/m]$*.*

Observe that in the last theorem we must have $m \ge -1$. Also, in theorem 4 the quenching is always natural if $m < -1$ or $L < -1/m$.

An important aspect of quenching problems is the possibility of having a nontrivial extension of the solution for times $t > T$. If such a continuation exists we say that quenching is *incomplete*; otherwise, it is called *complete*, see [FG]. A natural way of obtaining a continuation consists of approximating the flux nonlinearity u^m in problem (1) by a sequence of functions $f_n(u)$ such that the corresponding solution is well defined and positive for every $t > 0$. See the precise definition in section 5.

We then obtain a sequence of global solutions $\{u_n\}$, and we want to extend our original solution $u(x,t)$ for $t > T$ as the limit

$$\overline{u}(x,t) = \lim_{n\to\infty} u_n(x,t), \tag{7}$$

since for $t < T$ they coincide. We prove that the above limit becomes identically zero after T, obtaining complete quenching. This has to be contrasted with what happens with the heat equation where quenching is incomplete, see [FG].

Theorem 6 *Problem* (1) *has complete quenching, that is*

$$\overline{u}(x,t) \equiv 0, \qquad \textit{for every } x \in [0,L],\ t > T.$$

Organization of the paper. In section 2 we establish some preliminary results. In section 3 we study the self-similar profiles. Theorems 2, 3, 4 and 5 are proved in section 4. Finally, theorem 6 is proved in section 5.

2 Preliminaries

Let us first prove that quenching always happens for the solutions to problem (1).

Proof of Theorem 1. We consider the total mass of the solution $u(t)$, that is

$$M(t) = \int_0^L u(x,t)\,dx.$$

Differentiating and using the boundary conditions, we get

$$M'(t) = \int_0^L u_t(x,t)\,dx = \int_0^L (u^{m-1}u_x)_x\,dx = -u^m(0,t). \tag{8}$$

Since the initial data is bounded a comparison argument gives that $u(x,t) \leq \|u_0\|_\infty$. Hence, using that $m < 0$ we get

$$M'(t) \leq -K.$$

Then the mass $M(t)$ should vanish at some finite time $t_0 > 0$, a contradiction if we assume that the solution $u(t)$ is positive for every $t > 0$. Finally, (8) implies that $\int_0^L u_t \to -\infty$ and therefore u_t cannot be bounded. □

We now prove that the set of points where u vanishes coincides with the set of points where u_t blows up. This is an immediate consequence of the following lower bound for u.

Lemma 7 *Let* $m < 0$*, then*

$$\limsup_{t\to T} \frac{u(0,t)}{T-t} = \infty.$$

Proof. We consider the function

$$U(x,\tau) = \frac{u(x,t)}{T-t}, \qquad \tau = (T-t)^m,$$

which verifies the following problem,

$$\begin{cases} -mU_\tau = (U^{m-1}U_x)_x + \frac{1}{\tau}U, \; (x,t) \in (0,L) \times (T^m,\infty), \\ U^{m-1}U_x(0,\tau) = U^m(0,\tau), \quad \tau \in (T^m,\infty) \\ U^{m-1}U_x(L,\tau) = 0, \qquad\qquad \tau \in (T^m,\infty) \\ U(x,T^m) = \frac{1}{T}u_0(x), \qquad\quad x \in (0,L). \end{cases}$$

Now, in order to get a contradiction we assume that $U(0,\tau) \le C$, and define,

$$I(\tau) = \int_0^L U(x,\tau)\,dx.$$

From the problem for U, we get

$$-mI'(\tau) = \frac{1}{\tau}I(\tau) - U^m(0,\tau) \le \frac{1}{\tau}I(\tau) - C^m.$$

Then by integration,

$$I(\tau) \le \tau(C_1 - C_2\log(\tau)),$$

and $I(\tau)$ should vanish at some finite time τ_0. This implies that u must vanish at some point and at some time $T' < T$, which is a contradiction. □

Observe that this implies that quenching cannot be too fast: even in the case of superfast quenching, if we have $u(0,t) \sim g(T-t)$, it must be g sublinear.

Corollary 8 *In the above hypotheses,*

$$\begin{aligned} Q(u) &= \{\, 0 \le x \le L \,:\, \exists\,(x_n,t_n) \to (x,T) \text{ such that } u(x_n,t_n) \to 0 \,\} \\ &= \{\, 0 \le x \le L \,:\, \exists\,(x_n,t_n) \to (x,T) \text{ such that } u_t(x_n,t_n) \to -\infty \,\}. \end{aligned}$$

Proof. First, if u is bounded from below at some point $0 \le x_0 \le L$, standard regularity theory asserts that u_t remains bounded in a neighbourhood of x_0.

Assume by contradiction that $u_t(x_n,t_n) \ge -C > -\infty$, while $u(x_n,t_n) \to 0$. Then, integrating in $[t,t_n]$ we get

$$u(x_n,t) - u(x_n,t_n) = -\int_t^{t_n} u_t(x_n,s)\,ds \le C(t_n - t).$$

Taking limits, for $t > t_0$ we have

$$u(0,t) \leq u(x,t) \leq C(T-t),$$

and this is a contradiction with lemma 7. $\square$

Lemma 9 *Assume hypothesis* $(H2)$ *holds. Then for every* $0 \leq x \leq -1/m$ *and every* $0 < t < T$*, the following inequality holds:*

$$u^m(x,t) \geq u^m(0,t)(1+mx).$$

Proof. By the mean value theorem we have, for some $\xi \in (0,x)$,

$$\frac{u^m(x,t) - u^m(0,t)}{x} = (u^m)_x(\xi,t) \geq (u^m)_x(0,t) = mu^m(0,t).$$

The results follows. $\square$

Corollary 10 *The quenching set always contains the interval* $[0,-1/m]$.

Observe that then single-point quenching is not possible for problem (1). Compare with the situation for the Dirichlet problem considered in [FPQR], in which the quenching set is always $\{x=0\}$ if $-1 < m < 0$ and L large enough. This apparent contradiction, if the quenching rate for problem (1) is natural, motivates our study of the possible superfast quenching phenomenon.

3 The self-similar profiles

In this section we construct the profiles corresponding to the self-similar solutions (5). See also [FPQR], [FV], [CFQ], for the construction in related problems.

We look for solutions to the following problem,

$$\begin{cases} (F^{m-1}F')' + \alpha F = 0, \qquad \text{for } \ 0 < x < L, \\ F^{m-1}F'(0) = F^m(0), \\ F^{m-1}F'(L) = 0. \end{cases} \tag{1}$$

As in [FPQR], we consider the following variables

$$X(z) = F^{m-1}(x), \quad Y(z) = (F^{m-1})'(x), \quad dz = \frac{dx}{X}, \tag{2}$$

and study the trajectories in the fourth quadrant $\Theta = \{X \geq 0, Y \leq 0\}$ satisfying

$$\begin{cases} \dfrac{dX}{dz} = XY, \\ \dfrac{dY}{dz} = X + \alpha Y^2. \end{cases} \tag{3}$$

Observe that the equation for F implies $Y \leq 0$. The condition at $x = 0$ is translated into shooting from the line $\Lambda = \{X + \alpha Y = 0\}$. The condition at $x = L$ means that the trajectories end at the horizontal axis $Y = 0$.

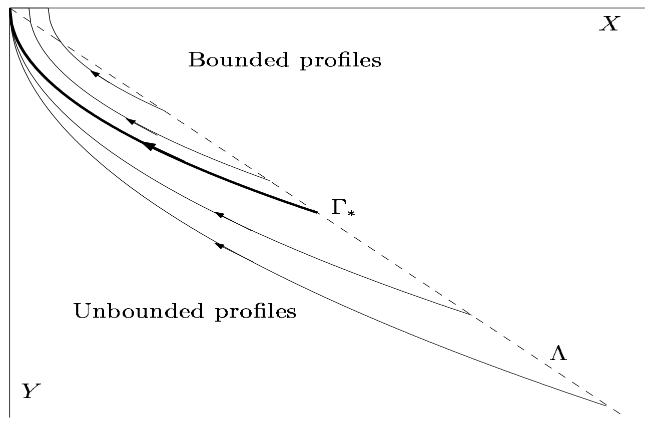

Fig. 1. The trajectories in the XY–plane for $m < -1$.

In the case $m < -1$, the separatrix between the two behaviours observed in Fig. 1 is the explicit trajectory

$$\Gamma_* = \{\, X - \frac{1-2\alpha}{2} Y^2 = 0 \,\}, \tag{4}$$

which gives the explicit profile

$$F_0(x) = \mu \left(\frac{2}{1-m} - x \right)^{-2/(1-m)}, \qquad \mu^{1-m} = \frac{2(m+1)}{m-1}. \tag{5}$$

Observe that F_0 is not bounded but has zero flux at $x = L = 2/(1-m)$, and therefore it satisfies the boundary condition at that point. On the other hand, the trajectories entering the origin below Γ_* satisfy $|Y(X)| \approx X^\alpha$. They have length

$$L = \int_0^{F^{m-1}(0)} \frac{dX}{|Y(X)|} < \infty$$

and the corresponding (unbounded) profiles satisfy

$$F(x) \approx (L-x)^{1/m}.$$

This means that they do not satisfy the boundary condition at $x = L$. However they will be useful in the sequel in comparison and asymptotic arguments. In a previous work [FPQR], these unbounded self-similar profiles also appear as possible limits of the solutions of a related Dirichlet problem. They produce compactly supported solutions in pressure variable $v = u^{m-1}$, considered in viscosity sense. We do not need to use these concepts in the present work.

In the case $-1 \leq m < 0$, the phase-plane picture is the same as in Fig. 1, but for the explicit trajectory Γ_* which has moved to the vertical axis. No trajectories in this quadrant enter the origin. In particular this means that all the profiles F are bounded.

We now proceed to characterize the length of the self-similar profiles constructed before. To do that we observe that there exists a first integral equation of (1), giving a constant energy

$$E(x) = \frac{1}{2}(F^{m-1}F'(x))^2 + \frac{1}{1-m^2}F^{m+1}(x) = E, \tag{6}$$

if $m \neq -1$. The case $m = -1$ contains a logarithmic term and needs easy modifications. At $x = 0$ we have

$$E(0) = \frac{1}{2}F^{2m}(0) + \frac{1}{1-m^2}F^{m+1}(0).$$

Observe that if $m > -1$ this implies that F is bounded. On the other hand, if $m < -1$, the profiles with negative energy are bounded, while there exists unbounded profiles with nonnegative energy. The limit case $E = 0$ corresponds to the explicit unbounded profile (5).

Since F must be nondecreasing (see above), we get from (6) that the profile F is given by the implicit formula

$$\int_{F(0)}^{F(x)} \frac{s^{m-1}}{\sqrt{2E - \frac{2}{1-m^2}\, s^{m+1}}}\, ds = x. \tag{7}$$

We thus get the following expression for the length L in terms of the values $A = F(0)$ and $B = F(L)$,

$$L = \int_A^B \frac{s^{m-1}}{\sqrt{2E - \frac{2}{1-m^2}\, s^{m+1}}}\, ds. \tag{8}$$

We want to draw the graph of L in terms of the value of F at the origin, $L = L(A)$. Allowing for B to take the value $B = \infty$ in the case $m < -1$, we include in the graph also the unbounded profiles. Observe that we have the energy given in terms of A,

$$2E = A^{2m} + \frac{2}{1-m^2}\, A^{m+1},$$

As we say above, in the case $m < -1$, the unbounded profiles correspond to $E \geq 0$, i.e. $0 < A \leq A_* = (\frac{2}{m^2-1})^{1/(1-m)}$, while the bounded profiles imply $E < 0$, i.e. $A > A_*$.

Lemma 11 *The function $L(A)$ is continuous in $(0, \infty)$ and satisfies*

1. $\lim_{A\to 0} L(A) = -1/m, \quad \lim_{A\to\infty} L(A) = 0.$
2. *If $-1/3 \leq m < 0$, L is strictly decreasing.*
3. *If $m < -1/3$, then L is first increasing and then decreasing.*

Proof. Assume first $m > -1$. In this case all the profiles are bounded, therefore the energy at $x = L$ gives us the relation

$$2E = \frac{2}{1-m^2} B^{1+m},$$

and then (8) can be written as

$$L = \sqrt{\frac{1-m^2}{2}}\, B^{(m-1)/2}\, I(A/B), \tag{9}$$

where

$$B = B(A) = \left(A^{1+m} + \frac{1-m^2}{2} A^{2m}\right)^{1/(1+m)},$$

$$I(z) = \int_z^1 \frac{s^{m-1}}{\sqrt{1-s^{m+1}}}\, ds.$$

The limits in *1.* are immediate from the fact that $I(z)$ converges at $z = 1$ and the behaviour $I(z) \approx \frac{-1}{m} z^m$ for $z \approx 0$. To see that L can only have one maximum we differentiate the expression for L to get

$$L'(A) = \frac{m-1}{2} \frac{B'(A)}{B(A)} (L(A) - H(A)),$$

where

$$H(A) = \frac{(1-m)}{m(m-1) - A^{1-m}}.$$

The point A_0 at which $B' = 0$ coincides with the vertical asymptote of H. Thus if at a point A_1 we have $L'(A_1) = 0$, this means $L(A_1) = H(A_1)$. We end with the observation that H is monotone increasing for $0 < A < A_0$ and H is negative for $A > A_0$. The fact that this maximum of L indeed exists depends on the behaviour near $A = 0$. For $A \approx 0$ we have

$$L(A) + \frac{1}{m} \approx \begin{cases} D_1(m) A^{1-m} & \text{if } -1 < m < -1/2, \\ \\ -4A^{3/2} \log A & \text{if } m = -1/2, \\ \\ D_2(m) A^{(1-m)/(1+m)} & \text{if } -1/2 < m < 0,\ m \neq -1/3, \\ \\ -\dfrac{27}{4} A^{4/3} & \text{if } m = -1/3, \end{cases} \tag{10}$$

where

$$D_1(m) = \frac{1}{m(1-m)(2m+1)},$$

$$D_2(m) = \left(\frac{2}{1-m^2}\right)^{-m/(m+1)} \sum_{j=0}^{\infty} \frac{(-1)^j \begin{pmatrix} -1/2 \\ j \end{pmatrix}}{m + (m+1)j}.$$

Observe that $D_1(m) > 0$, while $D_2(m) > 0$ if and only if $m < -1/3$. In fact the sum appearing in D_2 is monotone decreasing in m and vanishes for $m = -1/3$.

Assume now $m < -1$. If we restrict ourselves to bounded profiles, the above arguments hold word by word just by replacing the point $A = 0$ by the point $A = A_*$ and the limit $\lim_{A \searrow A_*} L(A) = 2/(1-m)$. The unbounded profiles are studied in a similar way by considering formula (8) in the form

$$L(A) = \sqrt{\frac{m^2-1}{2}} \int_A^{\infty} \frac{s^{m-1}}{\sqrt{s^{m+1} - A^{m+1} + \frac{m^2-1}{2} A^{2m}}} \, ds,$$

which is an increasing curve for $0 < A < A_*$, and satisfies $\lim_{A \nearrow A_*} L(A) = 2/(1-m)$ and $\lim_{A \searrow 0} L(A) = -1/m$. $\square$

Theorem 12 *Assume* $-1 \leq m < 0$.

(1) *If* $-1/3 \leq m < 0$, *there exists a unique bounded profile if* $L < -1/m$, *while for* $L \geq -1/m$ *no profiles exist.*
(2) *If* $-1 \leq m < -1/3$, *there exists a critical length* $L_* > -1/m$, *depending on* m, *such that:*

i) if $0 < L \le -1/m$ or $L = L_*$, there exists a unique bounded profile;
ii) if $-1/m < L < L_*$, there exist two bounded profiles;
iii) if $L > L_*$, there exist no profiles.

Theorem 13 *Assume $m < -1$. There exist a critical length, $L_* > 2/(1-m)$, depending on m, such that:*

i) if $0 < L \le -1/m$ or $L = L_*$, there exists a unique bounded profile F solution to problem (1)*;*
ii) if $2/(1-m) < L < L_*$, there exist two bounded profiles;
iii) if $L > L_*$, there exist no bounded profiles.
iv) for every $L > -1/m$ and $-1/m < \overline{L} \le \min\{2/(1-m), L\}$, there exists a unique profile defined in $[0, \overline{L})$, and satisfying $\lim_{x \to \overline{L}} F(x) = \infty$.

4 Asymptotic behaviour

We begin with the problem of characterizing the quenching rates. We always assume throughout this section that hypotheses (H1) and (H2) hold.

Lemma 14 *Let $m < 0$, then*

$$u(0,t) \le c_2(T-t)^{\alpha}.$$

Proof. In order to use lemma 9, we fix a point $0 < x_0 < -1/m$ and consider the function

$$I(t) = \int_0^{x_0} u(x,t)\,dx,$$

which verifies

$$I'(t) = u^{m-1}u_x(x_0,t) - u^{m-1}u_x(0,t) \ge -u^m(0,t).$$

We have used the fact that u is increasing in the space variable. Also, using lemma 9 we have that

$$cu(0,t) \le I(t) \le C(x_0)u(0,t). \tag{1}$$

Summing up we obtain the following inequality

$$I'(t) \ge -CI^m(t),$$

and by integration we obtain the desired upper bound. □

Lemma 15 *Let $L < -1/m$ or $L = -1/m$ and $m < -1$. Then*

$$u(x,t) \geq c_1(T-t)^\alpha.$$

Proof. If $L < -1/m$ we can take $x_0 = L$ in the above proof and then we have (1) and also $I'(t) = -u^m(0,t)$. Following the same argument the lower bound is deduced. In the case $L = -1/m$ we use, instead of (1), the estimate

$$I(t) \leq I(0) \int_0^{-1/m} (1+mx)^{1/m} dx = CI(0),$$

if $m < -1$. □

In order to complete the range of parameters for which the natural rate holds, we use an extra hypothesis on the initial data, denoted (H3) in the introduction.

Lemma 16 *Let L be such that there exists a self-similar solution $U(x,t)$ with quenching time T. Asume that the initial datum u_0 have only one intersection with $U(x,0)$ and also that $u_0(0) > U(0,0)$. Then*

$$u(0,t) \geq U(0,t) = C(T-t)^\alpha.$$

Proof. Arguing by contradiction, assume that there exists a first time $t_0 < T$ such that $u(0,t_0) = U(0,t_0)$. By intersection comparison (that we can apply since u and U are strictly positive in $[0,t_0]$), at this time t_0 we must have $u(x,t_0) \leq U(x,t_0)$ for every $0 \leq x \leq L$, and $u(x,t_0) \not\equiv U(x,t_0)$, therefore u and U must quench at different times, a contradiction. □

Lemma 17 *Assume that $L \geq -1/m$, then there exits a constant $c > 0$ such that, for every $-1/m \leq x \leq L$,*

$$u(x,t) \geq c(T-t)^\alpha.$$

Proof. Define the function

$$J(t) = \int_0^{-1/m} u(x,t)(1+mx)\, dx.$$

First, using that $u_x \geq 0$ we have

$$J(t) \leq u(-1/m,t) \int_0^{-1/m} (1+mx)\, dx = cu(-1/m,t).$$

Now, an easy integration by parts shows that

$$J'(t) = -u^m(-1/m, t).$$

Therefore, integrating between t and T and using that $u_t \le 0$, we get

$$J(t) = \int_t^T u^m(-1/m, s)\, ds \ge u^m(-1/m, t)(T - t).$$

These two estimates together imply

$$u(-1/m, t) \ge c(T - t)^\alpha,$$

and we conclude using again that u is nondecreasing in x. $\square$

Corollary 18 *If quenching is superfast, then, for every $-1/m \le x \le L$ there exists a sequence $t_n \to T$ such that,*

$$\frac{u(x, t_n)}{u(0, t_n)} \to +\infty.$$

Theorem 19 *Assume that there exists a point $-1/m \le x_0 < L$, and a sequence $t_n \to T$, such that*

$$\frac{u(x_0, t_n)}{u(0, t_n)} \to +\infty. \qquad (2)$$

Then

$$Q(u) \subseteq [0, x_0].$$

Proof. We want to prove that $u(y, t)$ is bounded from below for any $x_0 < y < L$. To this end we define the function

$$K(y, t) = \int_0^y u(x, t)(1 + mx)\, dx.$$

We easily have

$$K(y, t) \ge -cu(y, t)$$

and

$$K_t(y, t) = u^{m-1} u_x(y, t)(1 + my) - u^m(y, t) \le 0.$$

Therefore $-cu(y,t) \le K(y,t) \le K(y,t_n)$, for every $t \ge t_n$. On the other hand, we can split this last integral in the form $K(y,t_n) = I_1 + I_2 + I_3$, where

$$I_1 = \int_0^{-1/m-\varepsilon} u(x,t_n)(1+mx)\,dx \le c(\varepsilon)u(0,t_n);$$
$$I_2 = \int_{-1/m-\varepsilon}^{x_0+\varepsilon} u(x,t_n)(1+mx)\,dx \le 0;$$
$$I_3 = \int_{x_0+\varepsilon}^{y} u(x,t_n)(1+mx)\,dx \le -C(n)u(0,t_n),$$

where $C(n) \to \infty$. We have used lemma 9 to estimate I_1 and hypothesis (2) to deal with I_3. In this way we can fix $\varepsilon > 0$ small and take n large enough in order to have $I_1 + I_3 \le -C < 0$. This proves that $u(y,t)$ must be bounded below. □

From this result, the proof of theorem 5 follows easily, since we can take $x_0 = -1/m$ in (2). In fact, the converse also holds if $L > -1/m$.

Lemma 20 *If $L > -1/m$ and the quenching set is $Q(u) = [0, -1/m]$, then*

$$\lim_{t\to T}(T-t)^{-\alpha}u(0,t) = 0,$$

and therefore quenching is superfast.

Proof. By contradiction, assume that there exists a sequence $t_n \to T$ such that

$$(T-t_n)^{-\alpha}u(0,t_n) \ge C.$$

Let

$$F(x,t) = \frac{1}{(T-t)}\int_t^T \frac{u^m(x,s)}{u^m(0,s)}\,ds.$$

We observe that $0 < F \le 1$ and $F_{xx} \ge 0$. Moreover, if $0 < x < -1/m$, and $t = t_n$, thanks to lemma 9,

$$\begin{aligned} F_{xx}(x,t_n) &= \frac{m}{(T-t_n)}\int_{t_n}^T \frac{u_t(x,s)}{u^m(0,s)}\,ds \ge \frac{m(1+mx)}{(T-t_n)}\int_{t_n}^T \frac{u_t(x,s)}{u^m(x,s)}\,ds \\ &= \frac{m(1+mx)}{(1-m)(T-t_n)}\,u^{1-m}(x,t_n) \ge C(x) > 0. \end{aligned}$$

Also,

$$\int_0^L (F_x)^2(x,t_n)\,dx \le -F(0,t_n)F_x(0,t_n) = -m.$$

Thus the sequence $F(x,t_n)$ is bounded in $H^1(0,L)$, and hence there exists a subsequence t_{n_k} such that

$$F(x,t_{n_k}) \to \xi(x),$$

weakly in H^1 and uniformly in $[0,L]$. As the quenching set verifies $Q(u) = [0,-1/m]$, and ξ is continuous, we obtain $\xi(x)=0$ for every $-1/m \le x \le L$.

We finally have

$$0 < \int_0^{-1/m} F_{xx}(x,t_{n_k})(1+mx)\,dx = -mF(-1/m,t_{n_k}) \to -m\xi(-1/m) = 0.$$

This leads to a contradiction. $\square$

Lemma 21 *Let $m < -1$, $L > -1/m$ and asume that the quenching rate is superfast. Then there exists $\delta > 0$ such that the quenching set satisfies $Q(u) \supset [0,-1/m+\delta]$.*

Proof. We consider the function

$$h(x,t) = A(T-t)^{\alpha}(1+mx)^{-2\alpha}, \qquad A^{\alpha} = \frac{2m^2(m+1)}{m-1},$$

which is a solution of the equation in problem (1). As u has superfast quenching rate, and thanks to lemma 9, there exists $t_0 < T$ such that $h(x,t) > u(x,t)$ for $0 < x < -1/m$ and $t_0 < t < T$. Moreover, the rescaled function $h_\lambda(x,t) = \lambda h(\lambda^{(1-m)/2}x,t)$, with $\lambda < 1$, $\lambda \sim 1$, also satisfies $h_\lambda(x,t) > u(x,t)$ for $0 < x < -1/m$ and $t_0 < t < T$. In order to prove that this inequality holds in the whole interval of definition of h_λ, i.e., for every $0 \le x \le \overline{L}$, we consider the problem

$$\begin{cases} w_t = (w^{m-1}w_x)_x & (x,t) \in (0,\overline{L}) \times (t_0,T), \\ w(0,t) = h_\lambda(0,t) & t \in (t_0,T), \\ w(\overline{L},t) = u(\overline{L},t) & t \in (t_0,T), \\ w(x,t_0) = u(x,t_0) & x \in [0,\overline{L}]. \end{cases}$$

We remark that h_λ is a supersolution while u is a subsolution. Therefore, $h_\lambda(x,t) > u(x,t)$ for $0 < x < \overline{L}$ and $t_0 < t < T$, and the result follows. $\square$

This gives the proof of theorem 2, *i*).

Lemma 22 *Let $m \ge -1$ and suppose that the quenching rate is the natural one. Then, $f(x,\tau)$ is bounded for every $0 \le x \le L$, $\tau > 0$. Therefore, quenching is global.*

Proof. First of all, if the rate is natural, we have that $f(0,\tau) \geq c_1 > 0$ for every $\tau > 0$. Assume by contradiction that there exists a point $0 < x_1 \leq L$ such that there exists a sequence $\tau_j \to \infty$ with $\lim_{j\to\infty} f(x_1,\tau_j) = \infty$. Fix $M > 0$ large and j_0 such that $f(x_1,\tau_{j_0}) \geq 2M$. We want to perform a comparison argument in order to show that this implies that $f(x,\tau) > M/2$ for every $x_0 < x < x_1$ and τ large, for some $x_0 > 0$. To this end we first consider the function H solution to the following problem

$$\begin{cases} 0 = (H^{m-1}H')' + \alpha H, & x \in (0,x_1), \\ H(0) = c_1, \\ H(x_1) = M. \end{cases}$$

In [FPQR] it has been proved that this problem has a unique positive bounded solution with a maximum located at a point $0 < x_0 < x_1$. Moreover x_0 tends to zero as M tends to infinity. We now consider the evolution problem

$$\begin{cases} h_\tau = (h^{m-1}h_x)_x + \alpha h, & (x,\tau) \in (0,x_1) \times (\tau_0,\infty), \\ h(0,\tau) = c_1, & \tau \in (\tau_0,\infty), \\ h^{m-1}h_x(x_1,\tau) = H^{m-1}H'(x_1), & \tau \in (\tau_0,\infty), \\ h(x,\tau_0) = h_0(x), & x \in (0,x_1). \end{cases}$$

One can check that if $h_0 \leq H$ then the solution h to this problem converges to the above stationary solution H. Indeed, a Lyapunov function for this problem is the following

$$L_h(\tau) = \frac{1}{2}\int_0^L (h^{m-1}h_x(x,\tau))^2\,dx - \frac{1}{1-m^2}\int_0^L h^{m+1}(x,\tau)\,dx - \frac{k}{m}h^m(x_1,\tau),$$

where $k = H^{m-1}H'(x_1)$. It satisfies

$$\frac{d}{d\tau}L_h(\tau) = -\frac{4}{(m+1)^2}\int_0^L \left|(h^{(m+1)/2})_\tau(x,\tau)\right|^2\,dx \leq 0, \tag{3}$$

and

$$-C_1 \leq L_h(\tau) \leq L_h(0) \leq C_2. \tag{4}$$

This implies the convergence in a rather standard way, see for instance [ACP]. Since f is a supersolution to the problem for h if we take $h_0(x) \leq f(x,\tau_{j_0})$, we have $f(x,\tau) \geq M/2$ in $x_0 < x < x_1$ for every τ large. We finally choose

M large enough such that $x_0 < -1/m$, thus getting a contradiction with lemma 9. $\square$

We now prove the convergence results.

Proof of theorem 3. We first assume that the quenching is natural. The proof is based on the existence of the Lyapunov function

$$L_f(\tau) = \frac{1}{2}\int_0^L (f^{m-1}f_x(x,\tau))^2\,dx - \frac{1}{1-m^2}\int_0^L f^{m+1}(x,\tau)\,dx + \frac{1}{2m}f^{2m}(0,\tau),$$

if $m \neq -1$. For $m = -1$ the proof will follow with the obvious logarithmic corrections. It is easy to see that (3) holds for L_f, and also the corresponding upper bound in (4). As to the lower bound, we have to distinguish two cases, since the second term in L_f changes its sign as m crosses the value $m = -1$. If $m < -1$, this term is positive and we are done, while for $m > -1$, we need to obtain a bound from above of $\int f^{m+1}$. This comes from lemma 22. The convergence follows. This proves part i).

Assume now that the quenching is superfast. We follow here the technique developed by [GK] to obtain blow-up rates in semilinear problems.

We define $M = M(t) = u^m(0,t)$ and the function

$$\phi_M(x,s) = \frac{1}{M}u^m(x, M^{-\delta}s + t), \qquad \delta = \frac{m-1}{m}.$$

We want to prove the convergence

$$\frac{u^m(x,t_j)}{u^m(0,t_j)} = \phi_{M_j}(x,0) \to V(x) = (1+mx)_+,$$

through a sequence $M_j = M(t_j) \to \infty$. The function ϕ_M is a solution of the following problem:

$$\begin{cases} (\phi_M)_s = \phi_M^\delta(\phi_M)_{xx}, & (x,s) \in (0,L)\times(-M^\delta t, 0),\\ (\phi_M)_x(0,s) = m\phi_M(0,s), & s \in (-M^\delta t, 0),\\ (\phi_M)_x(L,s) = 0, & s \in (-M^\delta t, 0). \end{cases}$$

Moreover, using that $u_t \leq 0$ and $u_x \geq 0$, we get that $0 \leq \phi_M \leq 1$ and $\phi_M(0,0) = 1$. Since the functions ϕ_M are uniformly bounded we have that every sequence ϕ_{M_j} is equicontinuous on $[0,L]\times[S,0]$ for every $S < 0$, *cf.* [BPU]. Then, $\phi_{M_j} \to \Phi$ as $M_j \to \infty$ uniformly on $[0,L]\times[S,0]$. It is also easy to see that $(\phi_{M_j})_s \to \Phi_s$ in a neighbourhood of $(0,0)$.

We claim that there exists a sequence $M_j \to 0$ such that

$$(\phi_{M_j})_s(0,0) \to 0. \tag{5}$$

If not, i.e., $(\phi_{M_j})_s(0,0) \geq C$, we get $M'(t)/M^{1-\delta}(t) \geq C$. Integrating and taking into account that $M(t) = u^m(0,t)$, we obtain

$$u(0,t) \geq c(T-t)^\alpha,$$

and then the quenching is not superfast.

Using now (5), we see that $\Phi_s(0,0) = 0$.

On the other hand, since $u_t \leq 0$, and using Hopf's Lemma, we obtain that the function $w = \Phi_s$ satisfies $w \equiv 0$. Then Φ is a solution of

$$\Phi'' = 0, \qquad \Phi(0) = 1, \quad \Phi'(0) = m,$$

and hence $\Phi(x) = V(x)$. This proves part *ii*), and finishes the proof. □

As immediate consequences of the above proof, we obtain theorem 2, *ii*), since there exist no self-similar profiles if $L > L_*$, and also we complete the description of the quenching sets, theorem 4.

5 Complete quenching

In this section we prove that solutions of problem (1) have complete quenching. We consider the sequence u_n of solutions to the approximate problems

$$\begin{cases} (u_n)_t = (u_n^{m-1}(u_n)_x)_x & (x,t) \in (0,L) \times (0,\infty), \\ (u_n^{m-1}(u_n)_x)(0,t) = f_n(u_n(0,t)) & t \in (0,\infty), \\ (u_n^{m-1}(u_n)_x)(L,t) = 0 & t \in (0,\infty), \\ u_n(x,0) = u_0(x) & x \in [0,L], \end{cases}$$

where

$$f_n(s) = \begin{cases} s^m & \text{if } s \geq 1/n, \\ 1/n & \text{if } 0 < s \leq 1/n. \end{cases}$$

Consider also the limit,

$$\overline{u}(x,t) = \lim_{n\to\infty} u_n(x,t).$$

Proof of Theorem 6. First, let us observe that, for a given $\tau > 0$

$$u_n(x,t) = u(x,t), \qquad [0,L] \times [0, T-\tau],$$

for every n large enough. This proves that we recover the solution u as limit of the approximate solutions u_n for times $t < T$.

Now let us prove that given $\varepsilon > 0$, $u_n(0,t) \le \varepsilon$ for all $t > T$ and n large enough. To see this fact, let

$$v_\varepsilon(x) = \varepsilon(1+mx)^{1/m}.$$

This function v_ε is a solution of

$$\begin{aligned} &(v_\varepsilon^{m-1}(v_\varepsilon)')' = 0, \qquad x \in [0, -1/m), \\ &(v_\varepsilon^{m-1} v_\varepsilon)'(0) = v_\varepsilon^m(0), \\ &v_\varepsilon(-1/m) = +\infty. \end{aligned}$$

Since $Q(u) \supseteq [0, -1/m]$, we have

$$u_n(x,t) \le v_\varepsilon(x)$$

for some $t_0 < T$ and large n. Thus $u_n(0,t) \le v_\varepsilon(0) = \varepsilon$. This implies that $\overline{u}$ satisfies

$$\overline{u}(0,t) = 0, \qquad t > T,$$

and also the equation at any point where $\overline{u}$ is positive. Hence

$$\overline{u}(x,t) \equiv 0, \qquad t > T,$$

since solutions to the Dirichlet problem with bounded initial data become extinct instantaneously. This is easily seen by comparison with solutions to our problem. □

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