THE BEHAVIOR OF THE BEST SOBOLEV TRACE CONSTANT AND EXTREMALS IN THIN DOMAINS

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ABSTRACT. In this paper we study the asymptotic behavior of the best Sobolev trace constant and extremals for the immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ in a bounded smooth domain when it is contracted in one direction. We find that the limit problem, when rescaled in a suitable way, is a Sobolev type immersion in weighted spaces over a projection of Ω, $W^{1,p}(P(\Omega),\alpha) \hookrightarrow L^q(P(\Omega),\beta)$.

For the special case $p = q$, this problem leads to an eigenvalue problem with a nonlinear boundary condition. We also study the convergence of the eigenvalues and eigenvectors in this case.

1. INTRODUCTION.

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. Of importance in the study of boundary value problems for differential operators in Ω are the Sobolev trace inequalities. For any $1 < p < N$, and $1 \leq q \leq p^* = p(N-1)/(N-p)$ we have that $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ and hence the following inequality holds:

$$S\|u\|^p_{L^q(\partial\Omega)} \leq \|u\|^p_{W^{1,p}(\Omega)},$$

for all $u \in W^{1,p}(\Omega)$. This is known as the Sobolev trace embedding Theorem. The best constant for this embedding is the largest S such that the above inequality holds, that is,

$$S_{p,q}(\Omega) = \inf_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p + |u|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |u|^q\,d\sigma\right)^{p/q}}. \tag{1.1}$$

Moreover, if $1 \leq q < p^*$ the embedding is compact and as a consequence we have the existence of extremals, i.e. functions where the infimum is attained, see [10]. These extremals are weak solutions of the following problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and $\frac{\partial}{\partial\nu}$ is the outer unit normal derivative.

Our main interest in this paper is to study the dependence of the best constant $S_{p,q}(\Omega)$ and extremals on the domain. In [12] and [13] a first step in this direction was made by considering a family of domains obtained by contracting or expanding

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a fixed one, that is $\mu\Omega = \{\mu x \mid x \in \Omega\}$, and studying the limits $\mu \to 0+$ and $\mu \to \infty$. In particular, in [12] it is proved that

$$\lim_{\mu\to 0} \frac{S_{p,q}(\mu\Omega)}{\mu^{(Nq-Np+p)/q}} = \frac{|\Omega|}{|\partial\Omega|^{p/q}}. \tag{1.3}$$

In this paper we use the notation $|A|$ for the measure of the set A in its corresponding dimension, that is, if A is a set of dimension r, $|A|$ stands for the r-dimensional measure of A.

Here we consider a different family of domains. More precisely, we focus our attention on thin domains. To this end, let $N = n + k$ and define the family

$$\Omega_\mu = \{(\mu x, y) \mid (x,y) \in \Omega,\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^k\}.$$

Remark that for small values of μ, Ω_μ is a narrow domain in the x direction.

Our first result shows that, when the domain is very narrow, the problem of looking at the trace of a function is equivalent, in some sense, to the problem of the immersion of the function in the projection of the domain over the y variables. More precisely, we define the projection

$$P(\Omega) = \{y \in \mathbb{R}^k \mid \exists x \in \mathbb{R}^n \text{ with } (x,y) \in \Omega\}$$

and consider the weighted Sobolev embedding $W^{1,p}(P(\Omega),\alpha) \hookrightarrow L^q(P(\Omega),\beta)$ with associated best constant given by

$$\bar{S}_{p,q}(P(\Omega),\alpha,\beta) = \inf_{v\in W^{1,p}(P(\Omega),\alpha)} \frac{\displaystyle\int_{P(\Omega)} (|\nabla v|^p + |v|^p)\ \alpha(y)\,dy}{\left(\displaystyle\int_{P(\Omega)} |v|^q\,\beta(y)\,dy\right)^{p/q}}. \tag{1.4}$$

We prove

Theorem 1.1. *Let $1 \le q < p^*$, then there exists two nonnegative weights $\alpha, \beta \in L^\infty(P(\Omega))$ such that*

$$\lim_{\mu\to 0+} \frac{S_{p,q}(\Omega_\mu)}{\mu^{(nq-np+p)/q}} = \bar{S}_{p,q}(P(\Omega),\alpha,\beta) \tag{1.5}$$

and if we scale the extremals u_μ of (1.1) to the original domain Ω as $v_\mu(x,y) = u_\mu(\mu x, y)$, $(x,y) \in \Omega$, normalized as $\|u_\mu\|^q_{L^q(\partial\Omega_\mu)} = \mu^{n-1}$, then $v_\mu \to v = v(y)$ strongly in $W^{1,p}(\Omega)$, where $v \in W^{1,p}(P(\Omega),\alpha)$ is an extremal for $\bar{S}_{p,q}(P(\Omega),\alpha,\beta)$.

We want to remark that the weights α and β can be determined in terms of the geometry of Ω. In fact, $\alpha(y) = |\Omega_y|$ where Ω_y is the section at level y of Ω and $\beta(y)$ has a more subtle definition, see Section 4.

To clarify the content of the result, assume that Ω is a product, $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^k$. Then

$$\Omega_\mu = \mu\Omega_1 \times \Omega_2 = \{(\mu x, y) \mid x \in \Omega_1,\ y \in \Omega_2\}.$$

As in Theorem 1.1, let us call u_μ an extremal corresponding to Ω_μ and define $v_\mu(x,y) = u_\mu(\mu x, y)$. We have that $v_\mu \in W^{1,p}(\Omega)$ and

$$\frac{S_{p,q}(\Omega_\mu)}{\mu^{(nq-np+p)/q}} = \frac{\displaystyle\int_\Omega |(\mu^{-1}\nabla_x v_\mu, \nabla_y v_\mu)|^p + |v_\mu|^p \, dxdy}{\left(\displaystyle\int_{\partial\Omega_1\times\Omega_2} |v_\mu|^q \, d\sigma_x dy + \mu \int_{\Omega_1\times\partial\Omega_2} |v_\mu|^q \, dx d\sigma_y\right)^{p/q}}, \tag{1.6}$$

where $\nabla_x u = (u_{x_1}, \dots, u_{x_n})$ and $\nabla_y u = (u_{y_1}, \dots, u_{y_k})$. This notation will be followed throughout the paper. The normalization imposed in Theorem 1.1 in this case reduces to

$$\int_{\partial\Omega_1\times\Omega_2} |v_\mu|^q \, d\sigma_x dy + \mu \int_{\Omega_1\times\partial\Omega_2} |v_\mu|^q \, dx d\sigma_y = 1. \tag{1.7}$$

In this simpler case, the weight functions α, β are constants and can be computed explicitly, in fact

$$\alpha(y) = |\Omega_1| \qquad \text{and} \qquad \beta(y) = |\partial\Omega_1|.$$

Hence, Theorem 1.1 reads as follows,

Theorem 1.2. *Let $1 \le q < p^*$ and $\Omega = \Omega_1 \times \Omega_2$, then*

$$\lim_{\mu\to 0+} \frac{S_{p,q}(\Omega_\mu)}{\mu^{(nq-np+p)/q}} = \frac{|\Omega_1|}{|\partial\Omega_1|^{p/q}} \bar{S}_{p,q}(\Omega_2), \tag{1.8}$$

where $\bar{S}_{p,q}(\Omega_2) = \bar{S}_{p,q}(\Omega_2, 1, 1)$ is the usual Sobolev constant. Moreover, if we scale the extremals u_μ to the original domain Ω as $v_\mu(x,y) = u_\mu(\mu x, y)$, $x \in \Omega_1$, $y \in \Omega_2$, normalized by (1.7), then $v_\mu \to v = v(y)$ strongly in $W^{1,p}(\Omega)$, where $v \in W^{1,p}(\Omega_2)$ is an extremal for $\bar{S}_{p,q}(\Omega_2)$.

Observe, that the critical exponent for the Sobolev embedding, $W^{1,p}(\Omega_2) \hookrightarrow L^q(\Omega_2)$, valid for $1 \le q < pk/(k-p)$, is larger than the one for the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, which holds for $1 \le q < p(k+n-1)/(k+n-p)$. For the relation of the critical exponents in the general case, see the discussion in section 5.

In the special case $p = q$, problem (1.2) becomes a nonlinear eigenvalue problem. For $p = 2$, this eigenvalue problem is known as the *Steklov* problem, see [1]. In [10] it is proved, applying the Ljusternik-Schnirelman critical point Theory on C^1 manifolds, that there exists a sequence of variational eigenvalues $\lambda_j \nearrow +\infty$. Following [10] (see also [5]), a sequence of variational eigenvalues λ_j of (1.2) can be characterized by

$$\lambda_j = \inf_{C\in\mathcal{C}_j} \max_{u\in C} \frac{\|u\|^p_{W^{1,p}(\Omega)}}{\|u\|^p_{L^p(\partial\Omega)}}, \tag{1.9}$$

where

$$\mathcal{C}_j = \{\Phi(S^{j-1}) \subset W^{1,p}(\Omega) \mid \Phi : S^{j-1} \to W^{1,p}(\Omega) - \{0\} \text{ is continuous and odd}\}$$

and S^{j-1} is the unit sphere of $\mathbb{R}^j$. These eigenvalues differ slightly from the ones considered in [10]. However the same arguments used there apply proving that in fact $\{\lambda_j\}$ is an unbounded sequence of eigenvalues.

When μ goes to zero, there is a limit problem which is a weighted eigenvalue problem on the projection $P(\Omega)$. Let α and β be the weights given by Theorem 1.1 and consider the following eigenvalue problem,

$$\left\{ \begin{array}{ll} -\mathrm{div}(\alpha|\nabla v|^{p-2}\nabla v) + \alpha|v|^{p-2}v = \bar\lambda\beta|v|^{p-2}v & \text{in } P(\Omega), \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial P(\Omega). \end{array} \right. \tag{1.10}$$

For problem (1.10), one can define the sequence

$$\bar\lambda_j = \inf_{C\in\bar{\mathcal{C}}_j} \max_{u\in C} \frac{\displaystyle\int_{P(\Omega)} (|\nabla u|^p + |u|^p)\alpha\,dy}{\displaystyle\int_{P(\Omega)} |u|^p\beta\,dy}. \tag{1.11}$$

where

$$\bar{\mathcal{C}}_j = \left\{ \begin{array}{ll} \Phi(S^{j-1}) \subset W^{1,p}(P(\Omega)) \mid & \Phi : S^{j-1} \to W^{1,p}(P(\Omega)) - \{0\} \\ & \text{is continuous and odd} \end{array} \right\}.$$

Once again, applying the Ljusternik-Schnirelman critical point Theory one could check that $\{\bar\lambda_j\}$ is an unbounded sequence of eigenvalues for (1.10). However, this fact is a direct consequence of our next result.

Theorem 1.3. *Let $\lambda_{j,\mu}$ given by (1.9) in Ω_μ and let $u_{j,\mu}$ be an associated eigenfunction normalized as in Theorem 1.1. Then,*

$$\lim_{\mu\to 0} \frac{\lambda_{j,\mu}}{\mu} = \bar\lambda_j$$

where $\bar\lambda_j$ is defined by (1.11) and is an eigenvalue of (1.10). Also, along a subsequence, $v_{j,\mu}(x,y) = u_{j,\mu}(\mu x, y)$ converges strongly in $W^{1,p}(\Omega)$ to a function $\bar v_j = \bar v_j(y)$ which is an eigenfunction of (1.10) with eigenvalue $\bar\lambda_j$.

Observe that the first eigenvalue λ_1 coincides with the best Sobolev trace constant $S_{p,p}(\Omega)$. Hence, for $p = q$ and for the first eigenvalue, Theorem 1.1 and Theorem 1.3 coincides.

As before, in the case $\Omega = \Omega_1 \times \Omega_2$, the limit problem has a simpler form, i.e.,

$$\left\{ \begin{array}{ll} -\Delta_p v + |v|^{p-2}v = \dfrac{|\partial\Omega_1|}{|\Omega_1|}\bar\lambda|v|^{p-2}v & \text{in } \Omega_2, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega_2. \end{array} \right. \tag{1.12}$$

However, Theorem 1.3 conserves the same statement.

Our last result is concerned with the following fact: once the domain has been contracted in the x direction, we can now try to contract it in the y direction and see if the limit coincides with the one obtained by contracting the domain in every direction at the same time. Surprisingly, this is not the case. In fact, we obtain

Theorem 1.4. *Let $\Omega = \Omega_1 \times \Omega_2$ and consider $\Omega_{\mu,\nu} = \{(\mu x, \nu y) \mid (x,y) \in \Omega\}$, then*

$$\lim_{\nu\to 0}\left(\lim_{\mu\to 0} \frac{S_{p,q}(\Omega_{\mu,\nu})}{\mu^{(nq-np+p)/q}\nu^{(kq-kp)/q}} \right) = \frac{|\Omega|}{(|\partial\Omega_1||\Omega_2|)^{p/q}}. \tag{1.13}$$

By the result of [12], (1.3), we have

$$\lim_{\mu\to 0}\frac{S_{p,q}(\mu\Omega)}{\mu^{(Nq-Np+p)/q}}=\frac{|\Omega|}{|\partial\Omega|^{p/q}}\neq\frac{|\Omega|}{(|\partial\Omega_1||\Omega_2|)^{p/q}}.$$

This shows that the double limit $\lim_{(\mu,\nu)\to(0,0)} S_{p,q}(\Omega_{\mu,\nu})$ does not exists.

For a general domain Ω we have

$$\lim_{\nu\to 0}\left(\lim_{\mu\to 0}\frac{S_{p,q}(\Omega_{\mu,\nu})}{\mu^{(nq-np+p)/q}\nu^{(kq-kp)/q}}\right)=\frac{|\Omega|}{\left(\int_{P(\Omega)}\beta\,dy\right)^{p/q}}, \tag{1.14}$$

where β is the weight given in Theorem 1.1. To prove this fact we have to assume that the immersion $W^{1,p}(P(\Omega),\alpha)\hookrightarrow L^q(P(\Omega),\beta)$ is compact. To see in which cases this holds, see section 5.

To end this introduction, we want to comment briefly on related work. As a precedent, see [13] for a detailed analysis of the behavior of extremals and best Sobolev constants in expanding domains for $p=2$ and $q>2$. In that paper it is proved that the extremals develop a peak near the point where the curvature of the boundary attains a maximum. See also [9] where the symmetry properties of the extremals and their uniqueness is studied for $p=2$, $q>1$.

Nonlinear boundary conditions like the ones that appear in (1.2) have only been considered in recent years, see for example [2], [3], [13], [10], [11], [12], [15], [19]. In [6] and [18] a related problem in the half-space $\mathbb{R}^N_+$ for the critical exponent is studied. See also [7], [8] for other geometric problems that lead to nonlinear boundary conditions.

The paper is organized as follows, to simplify and clarify the exposition, we prove in sections 2 and 3 our main results in the case $\Omega=\Omega_1\times\Omega_2$, that is, Theorem 1.1, Theorem 1.2 for $\Omega_1\times\Omega_2$, and Theorem 1.3. In section 4 we indicate how to modify our arguments to deal with the general case. Finally, in section 5, we prove Theorem 1.4. Throughout the paper, by C we denote a constant that may vary from line to line but remains independent of the relevant quantities.

2. The best constant for thin domains

In this section we focus on the proof of Theorem 1.2, so throughout this section $\Omega=\Omega_1\times\Omega_2$.

Let us begin with the following Lemma.

Lemma 2.1. *Under the assumptions of Theorem 1.2, it follows that*

$$S_{p,q}(\Omega_\mu)\le\mu^{(nq-np+p)/q}\frac{|\Omega_1|}{|\partial\Omega_1|^{p/q}}\bar{S}_{p,q}(\Omega_2).$$

Proof. Let us recall that

$$S_{p,q}(\Omega_\mu)=\inf_{u\in W^{1,p}(\Omega_\mu)}\frac{\displaystyle\int_{\Omega_\mu}|\nabla u|^p+|u|^p\,dx}{\left(\displaystyle\int_{\partial\Omega_\mu}|u|^q\,d\sigma\right)^{p/q}}.$$

Then, taking $u = u(y)$ and observing that $\partial\Omega = (\partial\Omega_1 \times \overline{\Omega}_2) \cup (\overline{\Omega}_1 \times \partial\Omega_2)$, we get

$$\begin{aligned} S_{p,q}(\Omega_\mu) &\leq \mu^{(nq-np+p)/q} \frac{|\Omega_1| \displaystyle\int_{\Omega_2} (|\nabla u|^p + |u|^p)\,dy}{\left(|\partial\Omega_1| \displaystyle\int_{\Omega_2} |u|^q\,dy + \mu|\Omega_1| \int_{\partial\Omega_2} |u|^q\,d\sigma_y \right)^{p/q}} \\ &\leq \mu^{(nq-np+p)/q} \frac{|\Omega_1|}{|\partial\Omega_1|^{p/q}} \frac{\displaystyle\int_{\Omega_2} (|\nabla u|^p + |u|^p)\,dy}{\left(\displaystyle\int_{\Omega_2} |u|^q\,dy \right)^{p/q}}, \end{aligned}$$

and the result follows by taking infimum over all $u \in W^{1,p}(\Omega_2)$. $\square$

This Lemma shows that the ratio $S_{p,q}(\Omega_\mu)/\mu^{(nq-np+p)/q}$ is bounded. So a natural question is to determine if it converges to some value. This is answered in Theorem 1.2 that we prove next.

Proof of Theorem 1.2. Let $u_\mu \in W^{1,p}(\Omega_\mu)$ be a extremal for $S_{p,q}(\Omega_\mu)$ and define $v_\mu(x) = u_\mu(\mu x, y)$, we have that $v_\mu \in W^{1,p}(\Omega)$. Then

$$\frac{S_{p,q}(\Omega_\mu)}{\mu^{(nq-np+p)/q}} = \frac{\displaystyle\int_\Omega (|(\mu^{-1}\nabla_x v_\mu, \nabla_y v_\mu)|^p + |v_\mu|^p)\,dxdy}{\left(\displaystyle\int_{\partial\Omega_1 \times \Omega_2} |v_\mu|^q\,d\sigma_x dy + \mu \int_{\Omega_1 \times \partial\Omega_2} |v_\mu|^q\,dx d\sigma_y \right)^{p/q}} . \tag{2.1}$$

Recall that the u_μ are normalized as (1.7).

Using the previous lemma we have that v_μ is bounded in $W^{1,p}(\Omega)$. We have, for a subsequence, that

$$\begin{array}{ll} v_\mu \rightharpoonup v & \text{weakly in } W^{1,p}(\Omega), \\ v_\mu \to v & \text{in } L^p(\Omega), \\ v_\mu \to v & \text{in } L^q(\partial\Omega), \end{array}$$

and then it also happens that

$$\begin{array}{ll} v_\mu \to v & \text{in } L^q(\partial\Omega_1 \times \Omega_2), \\ v_\mu \to v & \text{in } L^q(\Omega_1 \times \partial\Omega_2). \end{array}$$

Using condition (1.7) and taking limit, we obtain that v verifies,

$$\int_{\partial\Omega_1 \times \Omega_2} |v|^q\,d\sigma_x dy = 1.$$

Hence $v \neq 0$. Returning to (2.1) and by the previous lemma we have that

$$\int_\Omega |\mu^{-1}\nabla_x v_\mu|^p\,dxdy \leq C, \tag{2.2}$$

where C is a constant that does not depend on μ, then

$$\int_\Omega |\nabla_x v_\mu|^p\,dxdy \leq \mu^p C \to 0.$$

We conclude that the limit v does not depend on x, that is, $v = v(y)$.

By (2.2) we have that $\mu^{-1}\nabla_x v_\mu$ is bounded in $L^p(\Omega)$, then there exist $w \in L^p(\Omega)$ such that,

$$\mu^{-1}\nabla_x v_\mu \rightharpoonup w \qquad \text{weakly in } L^p(\Omega).$$

Hence

$$\liminf_{\mu\to 0+} \|\mu^{-1}\nabla_x v_\mu\|^p_{L^p(\Omega)} \geq \|w\|^p_{L^p(\Omega)}. \tag{2.3}$$

Passing to the limit in (2.1), by (1.7) and (2.3) we get

$$\liminf_{\mu\to 0+} \frac{S_{p,q}(\Omega_\mu)}{\mu^{(nq-np+p)/q}} \geq \int_\Omega |(w,\nabla_y v)|^p + |v|^p\,dxdy \geq |\Omega_1|\|v\|^p_{W^{1,p}(\Omega_2)}.$$

Finally, using the fact that

$$1 = \int_{\partial\Omega_1\times\Omega_2} |v|^q\,d\sigma_x dy = |\partial\Omega_1| \int_{\Omega_2} |v|^q\,dy$$

we have that

$$\liminf_{\mu\to 0+} \frac{S_{p,q}(\Omega_\mu)}{\mu^{(nq-np+p)/q}} \geq \frac{|\Omega_1|}{|\partial\Omega_1|^{p/q}} \frac{\|v\|^p_{W^{1,p}(\Omega_2)}}{\|v\|^{p/q}_{L^q(\Omega_2)}} \geq \frac{|\Omega_1|}{|\partial\Omega_1|^{p/q}} \bar{S}_{p,q}(\Omega_2). \tag{2.4}$$

Now, we combine (2.4) with Lemma 2.1 to obtain

$$\lim_{\mu\to 0} \frac{S_{p,q}(\Omega_\mu)}{\mu^{(nq-np+p)/q}} = \frac{|\Omega_1|}{|\partial\Omega_1|^{p/q}} \bar{S}_{p,q}(\Omega_2)$$

and that v is an extremal for $\bar{S}_{p,q}(\Omega_2)$.

By the arguments just given, we conclude that $w = 0$ and that $\|v_\mu\|_{W^{1,p}(\Omega)} \to \|v\|_{W^{1,p}(\Omega)}$, so that

$$v_\mu \to v \quad \text{in } W^{1,p}(\Omega).$$

This completes the proof of the Theorem. □

3. The eigenvalue problem

In this section we consider $\Omega = \Omega_1 \times \Omega_2$. First of all let us observe that, for $1 \leq q \leq p$ the constant $\bar{S}_{p,q}(\Omega_2)$ can be computed explicitly, $\bar{S}_{p,q}(\Omega_2) = |\Omega_2|^{1-p/q}$. In fact, by Hölder's inequality,

$$\int_{\Omega_2} |u|^q\,dy \leq \left(\int_{\Omega_2} |u|^q\,dy\right)^{q/p} |\Omega_2|^{(p-q)/p}.$$

Hence,

$$\frac{\displaystyle\int_{\Omega_2} |\nabla u|^p + |u|^p\,dy}{\left(\displaystyle\int_{\Omega_2} |u|^q\,dy\right)^{p/q}} \geq \frac{\displaystyle\int_{\Omega_2} |\nabla u|^p + |u|^p\,dy}{\left(\displaystyle\int_{\Omega_2} |u|^p\,dy\right) |\Omega_2|^{(p-q)/q}} \geq |\Omega_2|^{1-p/q}.$$

Therefore

$$\bar{S}_{p,q}(\Omega_2) \geq |\Omega_2|^{1-p/q}.$$

On the other hand, taking $u \equiv 1$ as test function we get

$$\bar{S}_{p,q}(\Omega_2) \leq |\Omega_2|^{1-p/q},$$

hence $u \equiv 1$ is an extremal and the claim is proved. In particular, when $p = q$, $\bar{S}_{p,p}(\Omega_2) = 1$.

Now we turn our attention to the case $p = q$ which is a nonlinear eigenvalue problem. We recall that Theorem 1.2 says that

$$\frac{\lambda_1(\Omega_\mu)}{\mu} = \frac{S_{p,p}(\Omega_\mu)}{\mu} \to \frac{|\Omega_1|}{|\partial\Omega_1|}\bar{S}_{p,p}(\Omega_2) = \frac{|\Omega_1|}{|\partial\Omega_1|}$$

where $\frac{|\Omega_1|}{|\partial\Omega_1|} = \bar{\lambda}_1(\Omega_2)$ is the first eigenvalue of (1.12) with eigenfunction $u = 1$.

Now, we analyze the convergence of the remaining variational eigenvalues. First let us introduce the following notation

$$Q_{p,q}(u) = \frac{\displaystyle\int_\Omega (|\nabla u|^p + |u|^p)\, dxdy}{\left(\displaystyle\int_{\partial\Omega} |u|^q\, d\sigma\right)^{p/q}} \qquad \text{and} \qquad \bar{Q}_{p,q}(u) = \frac{\displaystyle\int_{\Omega_2} (|\nabla u|^p + |u|^p)\, dy}{\left(\displaystyle\int_{\Omega_2} |u|^q\, dy\right)^{p/q}}.$$

Lemma 3.1. *Let $\lambda_{j,\mu}$ be the $j-th$ variational eigenvalue of* (1.2). *Then*

$$\lambda_{j,\mu} \le \mu\bar{\lambda}_j,$$

where $\bar{\lambda}_j$ is the $j-th$ variational eigenvalue of (1.12).

Proof. First let us recall that, for $u = u(y)$

$$\begin{aligned} Q_{p,p}(u) &= \mu\frac{|\Omega_1|}{|\partial\Omega_1|}\frac{\displaystyle\int_{\Omega_2} (|\nabla u|^p + |u|^p)\, dy}{\displaystyle\int_{\Omega_2} |u|^p\, dy + \mu\frac{|\Omega_1|}{|\partial\Omega_1|}\int_{\partial\Omega_2} |u|^p\, d\sigma_y} \\ &\le \mu\frac{|\Omega_1|}{|\partial\Omega_1|}\frac{\displaystyle\int_{\Omega_2} (|\nabla u|^p + |u|^p)\, dy}{\displaystyle\int_{\Omega_2} |u|^p\, dy} = \mu\frac{|\Omega_1|}{|\partial\Omega_1|}\bar{Q}_{p,p}(u). \end{aligned}$$

Now let us observe that if we call

$$\mathcal{C}_j = \{\phi(S^{j-1}) \mid \phi : S^{j-1} \to W^{1,p}(\Omega_\mu) - \{0\}, \text{ is continuous and odd}\}$$

and

$$\bar{\mathcal{C}_j} = \{\bar{\phi}(S^{j-1}) \mid \bar{\phi} : S^{j-1} \to W^{1,p}(\Omega_2) - \{0\}, \text{ is continuous and odd}\},$$

then $\bar{\mathcal{C}_j} \subset \mathcal{C}_j$. Therefore

$$\begin{aligned} \lambda_{j,\mu} &= \inf_{C\in\mathcal{C}_j}\sup_{u\in C} Q_{p,p}(u) \le \inf_{C\in\bar{\mathcal{C}_j}}\sup_{u\in C} Q_{p,p}(u) \\ &\le \inf_{C\in\bar{\mathcal{C}_j}}\sup_{u\in C} \mu\frac{|\Omega_1|}{|\partial\Omega_1|}\bar{Q}_{p,p}(u) = \mu\bar{\lambda}_j \end{aligned}$$

as we wanted to show. □

As we know that the quotient $\lambda_{j,\mu}/\mu$ is bounded, we can assume that

$$\frac{\lambda_{j,\mu}}{\mu} \to \rho_j \le \bar{\lambda}_j \qquad \text{as } \mu \to 0$$

so a natural question is whether $\rho_j = \bar{\lambda}_j$. This is the content of our next lemma.

Lemma 3.2. *With the previous notation we have that*

$$\rho_j = \bar{\lambda}_j.$$

Proof. First we have, by Lemma 3.1, that

$$\rho_j \le \bar{\lambda}_j.$$

It remains to prove the reverse inequality. Using the variational characterization of $\lambda_{j,\mu}$ we have that for all $\varepsilon > 0$ there exists $C_\varepsilon = \phi_\varepsilon(S^{j-1}) \in \mathcal{C}_j$ such that

$$\sup_{v \in C_\varepsilon} \frac{\displaystyle\int_\Omega (|\nabla_\mu v|^p + |v|^p)\, dxdy}{\left(\displaystyle\int_{\partial\Omega_1 \times \Omega_2} |v|^p \, d\sigma_x dy + \mu \int_{\Omega_1 \times \partial\Omega_2} |v|^p \, dx d\sigma_y\right)} \le \frac{\lambda_{j,\mu}}{\mu} + \varepsilon. \tag{3.1}$$

We can assume, and we do so, that $v \in C_\varepsilon$ is normalized as (1.7).

Let us define the application $\Psi : W^{1,p}(\Omega) \to W^{1,p}(\Omega_2)$ by

$$\Psi(f)(y) = \frac{1}{|\Omega_1|} \int_{\Omega_1} f(x,y)\, dx.$$

We observe that, by Hölder's inequality,

$$|\Omega_1| \int_{\Omega_2} |\Psi(f)|^p \, dy \le \int_{\Omega_2} \int_{\Omega_1} |f|^p \, dxdy, \tag{3.2}$$

$$\nabla_y \Psi(f) = \Psi(\nabla_y f) = (\Psi(f_{y_1}), \dots, \Psi(f_{y_k})) \tag{3.3}$$

and

$$|\Psi(\nabla_y f)| \le \Psi(|\nabla_y f|), \tag{3.4}$$

so, by (3.2), (3.3) and (3.4), Ψ is a bounded linear operator.

Thanks to this application, we can obtain from ϕ_ε a function

$$\bar{\phi}_\varepsilon : S^{j-1} \to W^{1,p}(\Omega_2),$$

defined by

$$\bar{\phi}_\varepsilon(a) = \Psi(\phi_\varepsilon(a)).$$

It is immediate to check, from (3.2), (3.3) and (3.4), that $\bar{\phi}_\varepsilon$ is continuous and odd. Also, from (3.2), (3.3) and (3.4), we obtain

$$\int_\Omega (|\nabla_\mu v|^p + |v|^p)\, dxdy \ge |\Omega_1| \int_{\Omega_2} (|\nabla_y \Psi(v)|^p + |\Psi(v)|^p)\, dy.$$

Next, let us compute the integrals at the boundary.

First, let us observe that if $v \in C_\varepsilon$ then, there exists a constant C such that

$$\|\nabla_x v\|_{L^p(\Omega)} \le C\mu. \tag{3.5}$$

Hence, by Poincaré inequality, we get

$$\|v(\cdot, y) - \Psi(v)(y)\|^p_{L^p(\partial\Omega_1)} \le C \|\nabla_x v(\cdot, y)\|^p_{L^p(\Omega_1)}$$

and by (3.5)

$$\int_{\partial\Omega_1 \times \Omega_2} |v - \Psi(v)|^p \, d\sigma_x dy \le C \int_\Omega |\nabla_x v|^p \, dxdy \le C\mu^p$$

from where it follows that

$$\left| \left(\int_{\partial\Omega_1 \times \Omega_2} |v|^p \, d\sigma_x dy \right)^{1/p} - \left(\int_{\partial\Omega_1 \times \Omega_2} |\Psi(v)|^p \, d\sigma_x dy \right)^{1/p} \right| \le C\mu.$$

Hence, by (1.7) with μ small enough,

$$\left| \int_{\partial\Omega_1 \times \Omega_2} |v|^p \, d\sigma_x dy - |\partial\Omega_1| \int_{\Omega_2} |\Psi(v)|^p \, dy \right| \le C\mu.$$

So we obtain that, for μ small enough,

$$|\partial\Omega_1| \int_{\Omega_2} |\Psi(v)|^p \, dy \ge 1 - \varepsilon. \tag{3.6}$$

From this fact, we get that $\Psi(v) \neq 0$ for every $v \in C_\varepsilon$. Finally, from (3.1), (3.2), (3.3), (3.4) and (3.6) we get

$$\sup_{v \in C_\varepsilon} \frac{|\Omega_1| \displaystyle\int_{\Omega_2} (|\nabla_y \Psi(v)|^p + |\Psi(v)|^p) \, dy}{|\partial\Omega_1| \displaystyle\int_{\Omega_2} |\Psi(v)|^p \, dy + \varepsilon} \le \frac{\lambda_{j,\mu}}{\mu} + \varepsilon,$$

and hence

$$\bar\lambda_j \le \sup_{\bar v \in \bar C_\varepsilon} \frac{|\Omega_1| \displaystyle\int_{\Omega_2} (|\nabla_y \bar v|^p + |\bar v|^p) \, dy}{|\partial\Omega_1| \displaystyle\int_{\Omega_2} |\bar v|^p \, dy} \le \frac{\lambda_{j,\mu}}{\mu} + \varepsilon',$$

as we wanted to show. □

By Lemma 3.2, if we knew that $\bar\lambda_j$ are in fact eigenvalues of (1.12), the proof of Theorem 1.3 would be finished. By an indirect method, we can prove this fact. This is the content of the next lemma.

Lemma 3.3. *Let ρ_j be as above. Then ρ_j is an eigenvalue of* (1.12) *and, up to a subsequence, the functions $v_{j,\mu}$ converges strongly in $W^{1,p}(\Omega)$ to an eigenfunction $\bar v_j$ of* (1.12).

Proof. First, let us observe that

$$\frac{\lambda_{j,\mu}}{\mu} = \frac{\displaystyle\int_\Omega |(\mu^{-1} \nabla_x v_{j,\mu}, \nabla_y v_{j,\mu})|^p + |v_{j,\mu}|^p) \, dxdy}{\left(\displaystyle\int_{\partial\Omega_1 \times \Omega_2} |v_{j,\mu}|^q \, d\sigma_x dy + \mu \int_{\Omega_1 \times \partial\Omega_2} |v_{j,\mu}|^q \, dx d\sigma_y \right)}.$$

As $v_{j,\mu}$ is normalized such as in (1.7), it follows that $\|v_{j,\mu}\|_{W^{1,p}(\Omega)} \le C$. Arguing exactly as in the proof of Theorem 1.2 it follows that

$$\begin{aligned} v_{j,\mu} &\rightharpoonup \bar v_j && \text{weakly in } W^{1,p}(\Omega), \\ \nabla_x v_{j,\mu} &\to 0 && \text{in } L^p(\Omega), \\ v_{j,\mu} &\to \bar v_j && \text{in } L^p(\Omega), \\ v_{j,\mu} &\to \bar v_j && \text{in } L^q(\partial\Omega). \end{aligned}$$

It remains to show that $\bar{v}_j = \bar{v}_j(y)$ is an eigenfunction of (1.12) with eigenvalue ρ_j and that the convergence is actually strong. To this end, let us consider w the solution of the following problem

$$
\begin{cases} -\Delta_p w + |w|^{p-2}w = \dfrac{|\partial\Omega_1|}{|\Omega_1|}\rho_j|\bar{v}_j|^{p-2}\bar{v}_j & \text{in } \Omega_2, \\ \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega_2, \end{cases} \tag{3.7}
$$

and proceed as follows. First, let us introduce the following notation

$$
\nabla_\mu z(x,y) = \big(\mu^{-1}\nabla_x z, \nabla_y z\big)
$$

and consider the following norm in $W^{1,p}(\Omega)$

$$
\|z\|_\mu^p = \int_\Omega |\nabla_\mu z|^p + |z|^p \, dxdy.
$$

As we are dealing with a strongly monotone operator (see [4]), we get

$$
\begin{aligned} c\|w - v_{j,\mu}\|_\mu^p \le & \int_\Omega (|\nabla_\mu w|^{p-2}\nabla_\mu w_y - |\nabla_\mu v_{j,\mu}|^{p-2}\nabla_\mu v_{j,\mu})(\nabla_\mu w - \nabla_\mu v_{j,\mu})\, dxdy \\ & + \int_\Omega (|w|^{p-2}w - |v_{j,\mu}|^{p-2}v_{j,\mu})(w - v_{j,\mu})\, dxdy \end{aligned}
$$

Using the facts that $v_{j,\mu}$ is a weak solution of (1.2), that w is a weak solution of (3.7) and taking $\phi = w - v_{j,\mu}$ as a test function, we get that the last term equals

$$
\begin{aligned} & \frac{|\partial\Omega_1|}{|\Omega_1|}\rho_j \int_\Omega |\bar{v}_j|^{p-2}\bar{v}_j(w - v_{j,\mu})\, dxdy \\ & - \frac{\lambda_{j,\mu}}{\mu}\left(\int_{\partial\Omega_1\times\Omega_2} |v_{j,\mu}|^{p-2}v_{j,\mu}(w - v_{j,\mu})\, d\sigma_x dy \right. \\ & \left. + \mu\int_{\Omega_1\times\partial\Omega_2} |v_{j,\mu}|^{p-2}v_{j,\mu}(w - v_{j,\mu})\, dxd\sigma_y \right). \end{aligned}
$$

Rearranging the terms conveniently, we get

$$
\begin{aligned} & c\|w - v_{j,\mu}\|_\mu^p \le \\ & \rho_j\left[|\partial\Omega_1|\int_{\Omega_2} |\bar{v}_j|^{p-2}\bar{v}_j(w - v_{j,\mu})\, dy - \int_{\partial\Omega_1\times\Omega_2} |v_{j,\mu}|^{p-2}v_{j,\mu}(w - v_{j,\mu})\, d\sigma_x dy\right] \\ & + \frac{|\partial\Omega_1|}{|\Omega_1|}\rho_j\int_\Omega |\bar{v}_j|^{p-2}\bar{v}_j(\bar{v}_j - v_{j,\mu})\, dxdy \\ & - \left(\frac{\lambda_{j,\mu}}{\mu} - \rho_j\right)\int_{\partial\Omega_1\times\Omega_2} |v_{j,\mu}|^{p-2}v_{j,\mu}(w - v_{j,\mu})\, d\sigma_x dy \\ & - \lambda_{j,\mu}\int_{\Omega_1\times\partial\Omega_2} |v_{j,\mu}|^{p-2}v_{j,\mu}(w - v_{j,\mu})\, dxd\sigma_y. \end{aligned}
$$

Using the convergence of $v_{j,\mu}$ to $\bar{v}_j$ in $L^p(\partial\Omega)$ and the convergence of $\lambda_{j,\mu}/\mu$ to ρ_j, one can easily verify that

$$
\|w - v_{j,\mu}\|_\mu^p \to 0 \qquad \text{as } \mu \to 0,
$$

which implies that

$$\nabla_y v_{j,\mu} \to \nabla_y w \quad \text{in } L^p(\Omega),$$

$$v_{j,\mu} \to w \qquad \text{in } L^p(\Omega),$$

and therefore $w = \bar{v}_j$ and $v_{j,\mu} \to \bar{v}_j$ strongly in $W^{1,p}(\Omega)$. Finally, by (3.7), we get that $\bar{v}_j$ is an eigenfunction of (1.12) with eigenvalue ρ_j. □

4. General geometries

In this section, we show how to modify our previous arguments in order to generalize the results when Ω is a general bounded domain in $\mathbb{R}^{n+k}$ and not necessarily a product. As we mentioned in the introduction, what we get as limit of the best Sobolev trace constant is the best constant of a weighted Sobolev type inequality.

Let $\Omega \subset \mathbb{R}^{n+k} = \{(x,y) \mid x \in \mathbb{R}^n,\ y \in \mathbb{R}^k\}$ be a general bounded smooth domain and we consider $\Omega_\mu = \{(\mu x, y) \mid (x,y) \in \Omega\}$.

As before, we define the best Sobolev trace constant in Ω_μ as

$$S_{p,q}(\Omega_\mu) = \inf_{u \in W^{1,p}(\Omega_\mu)} \frac{\displaystyle\int_{\Omega_\mu} |\nabla u|^p + |u|^p \, dxdy}{\left(\displaystyle\int_{\partial\Omega_\mu} |u|^q \, d\sigma\right)^{p/q}}$$

and we want, as in the product case, to write the integrals involved in the quotient as integrals over the projection of Ω over y. To do this, we define

$$\Omega_y = \{x \in \mathbb{R}^n \mid (x,y) \in \Omega\}, \qquad P(\Omega) = \{y \in \mathbb{R}^k \mid \exists x \in \mathbb{R}^n \text{ with } (x,y) \in \Omega\}.$$

For a given function $u \in W^{1,p}(\Omega_\mu)$, if we call $v_\mu(x,y) = u(\mu x, y)$, $v_\mu \in W^{1,p}(\Omega)$ and by Fubini's theorem,

(4.1)
$$\begin{aligned}\int_{\Omega_\mu} |\nabla u|^p + |u|^p \, dxdy &= \mu^n \int_\Omega |(\mu^{-1}\nabla_x v_\mu, \nabla_y v_\mu)|^p + |v_\mu|^p \, dxdy \\ &= \mu^n \int_{P(\Omega)} \left(\int_{\Omega_y} |(\mu^{-1}\nabla_x v_\mu, \nabla_y v_\mu)|^p + |v_\mu|^p \, dx \right) dy.\end{aligned}$$

Observe that if $v_\mu = v_\mu(y)$, by (4.1), we obtain

$$\int_{\Omega_\mu} |\nabla u|^p + |u|^p \, dxdy = \mu^n \int_{P(\Omega)} (|\nabla_y v_\mu|^p + |v_\mu|^p)|\Omega_y| \, dy.$$

To deal with the boundary, by our assumptions on the domain, $\partial\Omega$ can be locally described as the graph of a smooth function. So we have that

$$\partial\Omega = \bigcup_{i=1}^{l} S_i \cup \bigcup_{j=1}^{r} T_j \qquad \text{(disjoint union)},$$

where, after relabelling the variables if necessary,

$$S_i = \{(x,y) \mid x_1 = h_i(x', y) \text{ where } h_i : D_i \subset \mathbb{R}^{n-1} \times \mathbb{R}^k \to \mathbb{R}\}$$

and the terms labeled T_j collect the "vertical" parts of the boundary

$$T_j = \{(x,y) \mid y_1 = g_j(x, y') \text{ where } g_j : E_j \subset \mathbb{R}^n \times \mathbb{R}^{k-1} \to \mathbb{R}\}.$$

As T_j is "vertical", we can assume that the parametrization has been taken such that, in the case $y_1 = g_j(x, y')$, the function g_j satisfies $\nabla_x g_j \equiv 0$ in E_j.

Observe that

$$P(\Omega) = \bigcup_{i=1}^{l} P(D_i) \qquad \text{(not necessarily disjoint).}$$

Hence, $\partial\Omega_\mu$ is described as

$$\partial\Omega_\mu = \bigcup_{i=1}^{l} S_{i,\mu} \cup \bigcup_{j=1}^{r} T_{j,\mu} \qquad \text{(disjoint union),}$$

where

$$S_{i,\mu} = \{(x,y) \mid x_1 = \mu h_i(\mu^{-1}x', y) \text{ where } h_i : D_i \subset \mathbb{R}^{n-1} \times \mathbb{R}^k \to \mathbb{R}\}$$

and

$$T_{j,\mu} = \{(x,y) \mid y_1 = g_j(\mu^{-1}x, y') \text{ where } g_j : E_j \subset \mathbb{R}^{n} \times \mathbb{R}^{k-1} \to \mathbb{R}\}.$$

We have

$$\int_{\partial\Omega_\mu} |u|^q \, d\sigma = \sum_{i=1}^{l} \int_{S_{i,\mu}} |u|^q \, d\sigma + \sum_{j=1}^{r} \int_{T_{j,\mu}} |u|^q \, d\sigma.$$

In the first case,

$$\begin{aligned} \int_{S_{i,\mu}} |u|^q \, d\sigma &= \mu^{n-1} \int_{D_i} |v_\mu|^q \sqrt{1 + |\nabla_{x'} h_i|^2 + \mu^2 |\nabla_y h_i|^2} \, dx'dy \\ &= \mu^{n-1} \int_{D_i} |v_\mu|^q \omega_{i,\mu} \, dx'dy. \end{aligned}$$

It is easy to see that $\omega_{i,\mu} \to \omega_i$ uniformly in D_i, where

$$\omega_i = \sqrt{1 + |\nabla_{x'} h_i|^2}.$$

In the second case, using that $\nabla_x g_j \equiv 0$ in E_j, we get

$$\begin{aligned} \int_{T_{j,\mu}} |u|^q \, d\sigma &= \mu^{n} \int_{E_j} |v_\mu|^q \sqrt{1 + \mu^{-2} |\nabla_x g_j|^2 + |\nabla_{y'} g_j|^2} \, dxdy' \\ &= \mu^{n} \int_{E_j} |v_\mu|^q \sqrt{1 + |\nabla_{y'} g_j|^2} \, dxdy' \\ &= \mu^{n} \int_{E_j} |v_\mu|^q \gamma_j \, dxdy'. \end{aligned}$$

Collecting all these facts, we have that

$$
\begin{aligned}
\frac{Q_{p,q}(u)}{\mu^{(nq-np+p)/q}} &= \frac{1}{\mu^{(nq-np+p)/q}} \frac{\displaystyle\int_{\Omega_\mu} |\nabla u|^p + |u|^p \, dxdy}{\left(\displaystyle\int_{\partial\Omega_\mu} |u|^q \, d\sigma\right)^{p/q}} \\
&= \frac{\displaystyle\int_{\Omega} |(\mu^{-1}\nabla_x v_\mu, \nabla_y v_\mu)|^p + |v_\mu|^p \, dxdy}{\left(\displaystyle\sum_{i=1}^{l} \int_{D_i} |v_\mu|^q \omega_{i,\mu} \, dx'dy + \mu \sum_{j=1}^{r} \int_{E_j} |v_\mu|^q \gamma_j \, dxdy'\right)^{p/q}}
\end{aligned} \tag{4.2}
$$

Once these observations had been made, all the arguments given in the previous sections follow without any change.

To conclude with the proof of the theorems, it only remains to show that if in the quotient (4.2) we take a function $u = u(y)$ we get

$$
\frac{\displaystyle\int_{P(\Omega)} (|\nabla_y v_\mu|^p + |v_\mu|^p)|\Omega_y| \, dy}{\left(\displaystyle\sum_{i=1}^{l} \int_{P(D_i)} |v_\mu|^q \left(\int_{(D_i)_y} \omega_{i,\mu} \, dx'\right) dy + \mu \sum_{j=1}^{r} \int_{E_j} |v_\mu|^q \gamma_j \, dxdy'\right)^{p/q}}
$$

$$
= \frac{\displaystyle\int_{P(\Omega)} (|\nabla_y v_\mu|^p + |v_\mu|^p)|\Omega_y| \, dy}{\left(\displaystyle\int_{P(\Omega)} |v_\mu|^q \left(\sum_{i=1}^{l} \int_{(D_i)_y} \omega_{i,\mu} \, dx'\right) dy + \mu \sum_{j=1}^{r} \int_{E_j} |v_\mu|^q \gamma_j \, dxdy'\right)^{p/q}}
$$

So, if the sequence $v_\mu \to v$ strongly in $W^{1,p}(\Omega)$, passing to the limit we arrive at

$$
\frac{\displaystyle\int_{P(\Omega)} (|\nabla_y v|^p + |v|^p)|\Omega_y| \, dy}{\left(\displaystyle\int_{P(\Omega)} |v|^q \left(\sum_{i=1}^{l} \int_{(D_i)_y} \omega_i \, dx'\right) dy\right)^{p/q}}
$$

and hence, the weights in Theorem 1.1 are given by

$$
\alpha(y) = |\Omega_y| \qquad \text{and} \qquad \beta(y) = \sum_{i=1}^{l} \int_{(D_i)_y} \omega_i \, dx'.
$$

5. Proof of Theorem 1.4

In order to prove (1.13), observe that in Theorem 1.1 we have proved that

$$
\lim_{\mu\to 0} \frac{S_{p,q}(\Omega_{\mu,\nu})}{\mu^{(nq-np+p)/q}\nu^{(kq-kp)/q}} = \frac{\bar{S}_{p,q}(\nu P(\Omega), \alpha(\nu y), \beta(\nu y))}{\nu^{(kq-kp)/q}}.
$$

Remark that this is valid for a general domain Ω. To study the limit $\nu \to 0$ we argue as in Theorem 1.1. In fact, taking $u \equiv 1$ as a test function, we get

$$\begin{aligned}\bar{S}_{p,q}(\nu P(\Omega), \alpha(\nu y), \beta(\nu y)) &\le \frac{\displaystyle\int_{\nu P(\Omega)} \alpha(\nu y)\, dy}{\left(\displaystyle\int_{\nu P(\Omega)} \beta(\nu y)\, dy\right)^{p/q}} \\ &= \nu^{(kq-kp)/q} \frac{\displaystyle\int_{P(\Omega)} \alpha(y)\, dy}{\left(\displaystyle\int_{P(\Omega)} \beta(y)\, dy\right)^{p/q}} \\ &= \nu^{(kq-kp)/q} \frac{|\Omega|}{\left(\int_{P(\Omega)} \beta(y)\, dy\right)^{p/q}}.\end{aligned}$$

On the other hand, if $u_\nu \in W^{1,p}(\nu P(\Omega), \alpha(\nu y))$ is an extremal for $\bar{S}_{p,q}$ (which exists by Theorem 1.1), then

$$\bar{S}_{p,q}(\nu P(\Omega), \alpha(\nu y), \beta(\nu y)) = \frac{\displaystyle\int_{\nu P(\Omega)} (|\nabla u_\nu|^p + |u_\nu|^p)\alpha(\nu y)\, dy}{\left(\displaystyle\int_{\nu P(\Omega)} |u_\nu|^q \beta(\nu y)\, dy\right)^{p/q}},$$

calling $v_\nu(y) = u_\nu(\nu y)$ and changing variables we get

$$\frac{\bar{S}_{p,q}(\nu P(\Omega), \alpha(\nu y), \beta(\nu y))}{\nu^{(kq-kp)/q}} = \frac{\displaystyle\int_{P(\Omega)} (\nu^{-p}|\nabla v_\nu|^p + |v_\nu|^p)\alpha(y)\, dy}{\left(\displaystyle\int_{P(\Omega)} |v_\nu|^q \beta(y)\, dy\right)^{p/q}}. \tag{5.1}$$

Now, we follow exactly the same arguments given in the proof of Theorem 1.1 as long as the immersion

$$W^{1,p}(P(\Omega), \alpha) \hookrightarrow L^q(P(\Omega), \beta)$$

is compact. To see this fact, first let us assume that $\Omega = \Omega_1 \times \Omega_2$, then

$$C \ge \alpha(y) = |\Omega_y| = |\Omega_1| \ge c > 0. \tag{5.2}$$

Hence, the compactness of the immersion is straightforward because we have

$$W^{1,p}(P(\Omega), \alpha) = W^{1,p}(P(\Omega))$$

with equivalent norms and the weight β lies in $L^\infty(P(\Omega))$.

Once this compactness has been established, we can extract a subsequence, such that

$$v_\nu \rightharpoonup v = \text{constant}, \qquad \text{weakly in } W^{1,p}(P(\Omega), \alpha),$$

and so, taking limits in (5.1), we arrive to

$$\liminf_{\nu\to 0} \frac{\bar{S}_{p,q}(\nu P(\Omega), \alpha(\nu y), \beta(\nu y))}{\nu^{(kq-kp)/q}} = \frac{\displaystyle\int_{P(\Omega)} \alpha(y)\,dy}{\left(\displaystyle\int_{P(\Omega)} \beta(y)\,dy\right)^{p/q}} = \frac{|\Omega|}{(|\partial\Omega_1||\Omega_2|)^{p/q}},$$

and hence the convergence is actually strong. This finishes the proof of Theorem 1.4, proving (1.13).

To deal with the general case, observe that the arguments remains valid if $\alpha(y)$ is bounded from below. However, we can not expect this to hold for any bounded smooth domain Ω. In fact, we only have that $\alpha(y) = |\Omega_y|$ verifies a lower bound of the form,

$$C \geq \alpha(y) \geq c\,(\operatorname{dist}(y, \partial P(\Omega)))^{n/2}. \tag{5.3}$$

In order to see this, given $y \in P(\Omega)$, we take $y_0 \in \partial P(\Omega)$ such that $|y - y_0| = \operatorname{dist}(y, \partial P(\Omega))$. Now take $x, x_0 \in \mathbb{R}^n$ such that $(x, y) \in \Omega$, $(x_0, y_0) \in \partial\Omega$ and there exists an inner tangent ball B_r with $(x, y) \in B_r \subset \Omega$ and $\overline{B_r} \cap \overline{\Omega} = \{(x_0, y_0)\}$.

Hence, $\Omega_y \supset (B_r)_y$ and then $\alpha(y) = |\Omega_y| \geq |(B_r)_y| = (r^2 - |y|^2)^{n/2}$. The claim follows noticing that $r^2 - |y|^2 \sim \operatorname{dist}(y, \partial P(\Omega))$.

Now, the compactness follows from the following theorem, that can be found in [17].

Theorem 5.1 ([17], Theorems 19.11 and 19.24). *Let $d(x) = dist(y, \partial P(\Omega))$. The compactness of the immersion*

$$W^{1,p}(P(\Omega), d^{n/2}) \hookrightarrow L^q(P(\Omega))$$

holds in any of the following cases:

(1) *If $p \leq n/2$ then $q < p$.*
(2) *If $n/2 < p \leq (n+2)/2$ then $q < \dfrac{kp}{k + \frac{n}{2} - p}$.*
(3) *If $p > (n+2)/2$ then $q < \dfrac{kp}{k + 1 - 2p}$.*

The rest of the proof runs as in the previous case. □

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