EXISTENCE, ASYMPTOTIC BEHAVIOR AND UNIQUENESS FOR LARGE SOLUTIONS TO $\Delta u = e^{q(x)u}$

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Abstract. In this paper we consider existence, asymptotic behavior near the boundary and uniqueness for solutions to $\Delta u = e^{q(x)u}$ in a bounded smooth domain Ω with the boundary condition $u(x) \to +\infty$ as $\text{dist}(x, \partial\Omega) \to 0$. The exponent $q(x)$ is assumed to be a Hölder continuous function which is either positive on $\partial\Omega$ or is positive in a neighborhood of $\partial\Omega$ maybe vanishing on $\partial\Omega$. When dealing with nonnegative exponents q we are allowing nonempty interior regions $\Omega_0 \subset \Omega$ where q vanishes. Changing sign exponents q will be also considered.

1. Introduction

We will be concerned in this paper with the study of solutions to

$$\begin{cases} \Delta u(x) = e^{q(x)u(x)} & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and the boundary condition is to be interpreted in the sense that $u(x) \to +\infty$ as $d(x) := \text{dist}(x, \partial\Omega) \to 0$. The exponent $q(x)$ will be assumed to be a Hölder continuous function which is either nonnegative or exhibits both signs.

Problems like (1.1) are usually known in the literature as boundary blow-up problems, and their solutions are also named "large solutions". A great amount of work has been dedicated to study such problems, see [1], [2], [3], [4], [6], [7], [8], [9], [10], [11], [12], [13], [18], [19], [20], [21], [22], [24], [25], [26], [27] and [28]. We refer the reader to [12] and [27] for a more complete list of references. In most of these references, the nonlinearities considered are generalizations of the two model ones: $f(u) = u^p$ with $p > 1$ and $f(u) = e^u$. Up to our knowledge, this is the first time where a variable exponent $q(x)$ is studied in an exponential (see also [23] and specially [14] for further results concerning a power nonlinearity with variable exponent $f(u) = u^{q(x)}$).

In the case where $q(x)$ is a constant $q = q_0 > 0$, problem (1.1) was first considered by Bieberbach in [4], who proved existence of a solution and later by Rademacher in [26], who showed uniqueness of solutions with the additional requirement that $|u + 2\log d|$ is bounded in Ω. It was finally shown by Lazer and McKenna in [20] that the solution is unique with no further restriction. On the other hand, if $q(x) \equiv 0$ then it is easily seen that there is no solution to (1.1).

One of the main novelties of the present work is that $q(x)$ is going to be a general nonconstant exponent. In a first group of results q will restricted to be nonnegative but allowing that q vanishes inside Ω. To precisely describe our hypotheses concerning nonnegative exponents, we assume the existence of a smooth open subset $\Omega_0 \subset \Omega$ such that $\overline{\Omega}_0 = \{x \in \Omega : \; q(x) = 0\}$ (of course, Ω_0 may be empty). Smooth is understood in the usual sense, i. e., Ω_0 is located at one side of its boundary $\partial\Omega_0$ which is a closed smooth submanifold of $\mathbb{R}^N$. This means that Ω_0 possesses finitely many connected pieces which are in turn smooth subdomains of Ω. We will designate by $\Omega_{0,1}$ the union of those connected components of Ω_0 whose boundaries have common points with the boundary of Ω. Similarly, we set $\Omega_{0,2} = \Omega_0 \setminus \Omega_{0,1}$ and thus $\overline{\Omega}_{0,2} \subset \Omega$. Finally we are naming $\Omega_+ := \Omega \setminus \overline{\Omega}_0 = \{x \in \overline{\Omega} : \; q(x) > 0\}$ and setting $\Gamma_1 = \partial\Omega_0 \cap \partial\Omega$, $\Gamma_2 = \partial\Omega_0 \cap \Omega$.

To avoid unnecessary technicalities we are supposing that Γ_1 and Γ_2 are "far away" from each other, that is, $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. This means that whenever $\partial\Omega_0$ and $\partial\Omega$ touch, the tangency is along a whole closed submanifold of $\partial\Omega_0$ (or, to be more precise, a submanifold of $\partial\Omega_{0,1}$). On the other hand, observe that the set $\Omega_+ \cup \overline{\Omega}_{0,2} = \Omega \setminus \overline{\Omega}_{0,1}$ and so is an smooth open part of Ω. Moreover $\partial(\Omega_+ \cup \overline{\Omega}_{0,2}) = \partial\Omega_+ \setminus \partial\Omega_{0,2} = \{\partial\Omega_+ \cap \partial\Omega\} \cup \{\Gamma_2 \setminus \partial\Omega_{0,2}\}$.

The main idea to show both existence and nonexistence of solutions in most of the previous works is to prove existence of solutions to (1.1) replacing the boundary condition $u = +\infty$ by $u = M$ for finite M, and then take the limit as $M \to \infty$. Here we perform a similar approximation procedure and consider (1.1) with $u = M$ as boundary condition. Our first result describes exactly what happens with these approximations.

For immediate use we fix the following notation:

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\}.$$

Recall that we are designating by $d(x)$ the distance $\mathrm{dist}(x, \partial\Omega)$.

Theorem 1. *Assume $q \in C^\eta(\overline{\Omega})$ is nonnegative and verifies the previous hypotheses. Then for every $M > 0$ there exists a unique solution u_M to*

$$\begin{cases} \Delta u(x) = e^{q(x)u(x)} & \text{in } \Omega, \\ u(x) = M & \text{on } \partial\Omega. \end{cases}$$

Moreover, the family $\{u_M\}_{M>0}$ is increasing in M, the limit u_∞ of u_M as $M \to \infty$ exists and verifies

$$\begin{aligned} u_\infty(x) &= +\infty, & x &\in \Omega_{0,1}, \\ u_\infty(x) &< +\infty, & x &\in \Omega_+ \cup \overline{\Omega}_{0,2}. \end{aligned}$$

In addition, u_∞ is a finite solution to $\Delta u(x) = e^{q(x)u(x)}$ in $\Omega_+ \cup \overline{\Omega}_{0,2}$. Furthermore, if the exponent $q(x)$ verifies

$$\sup_{\{\mathrm{dist}(x, \partial\Omega_{0,1}) < \delta\}} q(x) = o(\delta) \qquad \text{as } \delta \to 0 \tag{1.2}$$

then

$$\lim_{M \to \infty} u_M(x) = +\infty, \qquad \text{for} \quad x \in \partial\Omega_+ \setminus \partial\Omega_{0,2},$$

and hence u_∞ is a solution to (1.1) in $\Omega_+ \cup \overline{\Omega}_{0,2}$.

As a conclusion that can be drawn at once from Theorem 1 we have: on the one hand, if u is a solution to (1.1) it follows by comparison that $u \geq u_M$, and hence no solutions to (1.1) can exist when $\Omega_{0,1} \neq \emptyset$. On the other hand, when $\Omega_{0,1} = \emptyset$, $u_\infty < \infty$ in Ω, and then it is a solution to (1.1) (observe that condition (1.2) is not needed here). Thus we have:

Corollary 2. *Assume $q \in C^\eta(\overline{\Omega})$ is nonnegative. Then problem* (1.1) *has a positive solution if and only if* $\overline{\{x \in \Omega : q(x) = 0\}} \cap \partial\Omega = \emptyset$, *i.e.,* $\Omega_{0,1} = \emptyset$.

We remark that $q = 0$ on $\partial\Omega$ is compatible with $\Omega_{0,1} = \emptyset$, as long as $q > 0$ in a region of the form $U \cap \Omega$ with U an open in $\mathbb{R}^N$ containing $\partial\Omega$.

Next, we deal with the asymptotic behavior of the solutions near the boundary. We consider two illustrative cases: the first one is when $q > 0$ on the whole $\partial\Omega$, where it is shown that the behavior of the solutions consist of a sum of a blowing-up term and a term that vanishes when we approach the boundary. Some extra regularity of q near $\partial\Omega$ is needed in this case.

Theorem 3. *Assume $q \in C^\eta(\overline{\Omega}) \cap C^2(\Omega_\delta)$ for some $\delta > 0$ and is nonnegative in Ω with $q > 0$ on $\partial\Omega$. Then if u is a solution to* (1.1) *we have*

$$\lim_{d(x)\to 0} u(x) + \frac{2}{q(x)} \log d(x) + \frac{1}{q(x)} \log q(x) - \frac{1}{q(x)} \log 2 = 0. \tag{1.3}$$

With the natural hypothesis $q > 0$ on $\partial\Omega$ imposed in Theorem 3 we obtain uniqueness of solutions. This uniqueness is a consequence of the asymptotic behavior (1.3). Nevertheless, regarding uniqueness it suffices with obtaining a not so sharp estimate as (1.3) (compare with (3.3) in the proof of Theorem 3 and see Remark 1 after the proof of Theorem 4).

Theorem 4. *Let $q \in C^\eta(\overline{\Omega}) \cap C^2(\Omega_\delta)$ for some $\delta > 0$ be a nonnegative function with $q > 0$ on $\partial\Omega$. Then there exists a unique solution to* (1.1).

Note that $q(x)$ may vanish in some interior region of Ω and we still get existence and uniqueness of solutions to (1.1) as long as $q(x) > 0$ on $\partial\Omega$.

The second boundary behavior that we are considering is when $q = 0$ on $\partial\Omega$, but it essentially behaves as a power of the distance $d(x)$ near $\partial\Omega$.

Theorem 5. *Assume $q \in C^\eta(\overline{\Omega})$ is nonnegative and there exist $Q, \gamma > 0$ such that*

$$\lim_{d(x)\to 0} \frac{q(x)}{d(x)^\gamma} = Q. \tag{1.4}$$

Then for every solution u to (1.1) *we have*

$$\lim_{d(x)\to 0} \frac{d(x)^\gamma u(x)}{-\log d(x)} = \frac{\gamma + 2}{Q}. \tag{1.5}$$

Notice that estimate (1.5) is weaker than (1.3). The limit (1.5), does not give information about the presence of other explosive terms in the asymptotic expansion of solutions u near $\partial\Omega$. However, it is still sufficient to guarantee uniqueness of solutions.

Theorem 6. *Let $q \in C^\eta(\overline{\Omega})$ with $q > 0$ in Ω, and assume that condition* (1.4) *holds. Then problem* (1.1) *admits a unique solution.*

A second main feature of this work is the handling of changing sign exponents $q(x)$ in problem (1.1). In order to properly fix our assumptions on q we first present a negative result. It is the natural extension of Corollary 2 for general exponents q.

Theorem 7. *Assume that $q \in C^\eta(\overline{\Omega})$ satisfies $q \le 0$ in a ball relative to Ω, $B(x_0,\delta) = \{x \in \Omega : |x - x_0| < \delta\}$ for certain $\delta > 0$ and $x_0 \in \partial\Omega$. Then problem* (1.1) *does not admit any solution.*

Thus as in the case of nonnegative exponents, problem (1.1) can only be solved if $q > 0$ near the boundary, i. e., in a open region $U \cap \Omega$ with U an open neighborhood of $\partial\Omega$ in $\mathbb{R}^N$. Such requirement is thus implicit in our next hypotheses on q. We are assuming that $q \in C^\eta(\overline{\Omega})$ behaves so that $\Omega_- := \{x \in \Omega : q(x) < 0\}$ is a smooth subdomain of Ω strongly contained in Ω, i. e. $\overline{\Omega}_- \subset \Omega$. Moreover, q will satisfy in addition $q > 0$ in $\Omega \setminus \overline{\Omega}_-$. This means in particular that $\{x \in \Omega : q = 0\} = \partial\Omega_-$. Actually, the forthcoming statements allow more general configurations on the nodal regions for q, in particular the presence of a larger null set $\{q = 0\}$. However, and for the benefit of the presentation, we are pointing out those possible extensions in Remark 4.

Our main existence result shows the solvability of (1.1) under conditions that involve the amplitude of the positive part q_+ of q or the corresponding amplitude of its negative part q_- or the size of Ω. If for $A \subset \mathbb{R}^N$ we set χ_A its characteristic function (i. e. $\chi_A(x) = 1$ if $x \in A$, $\chi_A(x) = 0$ otherwise) then we are using the notation $q_-(x) = -q(x)\chi_{\Omega_-}(x)$, $q_+(x) = q(x)\chi_{\Omega_+}(x)$ so that $q = q_+ - q_-$.

Theorem 8. *Assume $q \in C^\eta(\overline{\Omega})$ achieves negative values in Ω according to the structure conditions introduced above and let $q_-(x)$, $q_+(x)$ be its negative and positive parts, respectively. Then,*

i) *Problem* (1.1) *possesses a solution provided that*

$$\operatorname{diam}(\Omega)^2 \sup q_- \le \frac{8N}{e}, \tag{1.6}$$

where $\operatorname{diam}(\Omega) = \sup\{|x - y| : x, y \in \Omega\}$.

ii) *If L_Ω stands for the minimum distance between parallel hyperplanes in $\mathbb{R}^N$ enclosing Ω, then* (1.1) *admits a* positive *solution provided*

$$L_\Omega^2 \sup q_+ < 2\pi^2. \tag{1.7}$$

As a consequence of Theorem 8 a second existence result for problem (1.1) holds in any domain Ω when scaling the problem in a suitable way.

Corollary 9. *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain and for $\lambda > 0$ set $\Omega_\lambda = \{\lambda x : x \in \Omega\}$. Assume that $q \in C^\eta(\overline{\Omega})$ achieves negative values according to the preceding conditions while $q_\lambda \in C^\eta(\overline{\Omega}_\lambda)$ designates the scaled version $q_\lambda(x) = q(x/\lambda)$ of q. Then there exists a (possibly small) positive value λ_0 such that the problem*

$$\begin{cases} \Delta u = e^{q_\lambda(x) u} & x \in \Omega_\lambda, \\ u = \infty & x \in \partial\Omega_\lambda, \end{cases}$$

admits a positive *solution for all $\lambda \le \lambda_0$.*

It will be also shown in Section 5 by means of a class of one-dimensional examples that the existence of solutions to (1.1) may be lost if either q_- or

q_+, or alternatively the size of Ω, become large in amplitude. Thus, conditions (1.6) or (1.7) for existence in Theorem 8 can not be unconditionally suppressed.

Finally, we remark that our examples also show that a second solution to (1.1) exists say, whenever the size of q_-, q_+, a or Ω becomes small. Moreover, in all cases, a bifurcation from infinity occurs. See Section 5 for details.

The rest of the paper is organized as follows. Nonnegative coefficients q are treated in Sections 2 and 3. In the former we consider the existence of solutions and prove Theorem 1, while in the latter we are dealing with the boundary behavior of solutions stated in Theorems 3 and 5 and with the uniqueness of solutions to (1.1), Theorems 4 and 6. The proofs corresponding to changing-sign coefficients q are contained in Section 4. Finally, section 5 is devoted to present some one-dimensional examples.

2. Behavior of the approximations. Existence of solutions to (1.1)

This section is devoted to the proof of Theorem 1.

Proof of Theorem 1. First, we prove that for any $M \geq 0$ there exists a unique solution u_M to

$$\begin{cases} \Delta u(x) = e^{q(x)u(x)} & \text{in } \Omega, \\ u(x) = M & \text{on } \partial\Omega. \end{cases}$$

To this end we observe that a subsolution can be obtained considering $\underline{u}(x) = (|x|^2 - R^2)/2N$, where R is taken so that $|x| < R$ in Ω. Indeed, notice that $\underline{u} < 0$ in Ω, and thus $\Delta\underline{u} = 1 \geq e^{q\underline{u}}$ in Ω. Since $\overline{u} = M$ is a supersolution, a standard monotonicity argument shows that there exists a solution u_M that verifies $(|x|^2 - R^2)/2N < u_M(x) < M$ in Ω. Uniqueness of the solution is a consequence of the maximum principle. Moreover, if $M_1 > M_2$ then, by comparison we get $u_{M_1} > u_{M_2}$ in Ω. Therefore $u_M(x)$ is increasing in M and hence the limit

$$u_\infty(x) = \lim_{M\to\infty} u_M(x)$$

exists.

Now we make an important remark: notice that solutions to (1.1) need not be positive. However, since $u_0 = u_{M|M=0}$ is bounded from below, we may choose $K > 0$ such that $v = u_0 + K > 0$ in Ω, and v will solve the equation

$$\Delta v(x) = a(x)e^{q(x)v(x)}$$

where $a(x) = e^{-Kq(x)}$ is bounded and strictly positive. Hence, with no loss of generality we may assume in what follows that $u_0 > 0$ in Ω, and this will imply $u_M > 0$ in Ω for every $M > 0$.

Our next aim is to prove that u_∞ is finite in $\Omega_+ \cup \overline{\Omega}_{0,2}$. To this end, let x_0 be such that $q(x_0) > 0$. Since $q(x)$ is continuous there exists $q_0 > 0$ and $\delta > 0$ such that $B(x_0, \delta) \subset \Omega$ and

$$q(x) \geq q_0, \qquad \text{in } B(x_0, \delta).$$

Now, recall that there exists a unique solution to

$$\begin{cases} \Delta v(x) = e^{q_0 v(x)} & \text{in } B(x_0,\delta), \\ v = +\infty & \text{on } \partial B(x_0,\delta), \end{cases} \tag{2.1}$$

(cf. [20]). By comparison, $u_M \le v$ in $B(x_0,\delta)$, and it follows that

$$u_\infty(x) \le v(x) < +\infty, \qquad x \in B(x_0,\delta),$$

and hence the limit u_∞ is finite in Ω_+.

To see that it is also finite in $\overline{\Omega}_{0,2}$, we take a small δ and consider the set $V_\delta = \{x \in \Omega : \ \mathrm{dist}(x,\Omega_{0,2}) < \delta\}$. Since $\overline{\Omega}_{0,2}$ does not touch the boundary of Ω, it follows that $\partial V_\delta \subset \Omega_+$. Notice that for every $M > 0$, u_M is a subharmonic function, and hence

$$u_M \le \sup_{\partial V_\delta} u_M \le \sup_{\partial V_\delta} u_\infty$$

in V_δ. This last supremum is finite since $u_\infty < +\infty$ in Ω_+. This shows that actually $u_\infty < +\infty$ in $\Omega_+ \cup \overline{\Omega}_{0,2}$.

Now, let us show that $u_\infty = +\infty$ in $\Omega_{0,1}$. Let v_M be the solution to

$$\begin{cases} \Delta v_M(x) = 1 & \text{in } \Omega_{0,1}, \\ v_M(x) = M & \text{on } \Gamma_1, \\ v_M(x) = 0 & \text{on } \Gamma_2 \setminus \partial\Omega_{0,2}, \end{cases}$$

which is easily seen to be $v_M = M\phi + \psi$, where ϕ is the harmonic function which equals 1 on Γ_1 and 0 on $\Gamma_2 \setminus \partial\Omega_{0,2}$ and ψ is the solution to $\Delta\psi = 1$ in $\Omega_{0,1}$ with $\psi = 0$ on $\partial\Omega_{0,1}$. By comparison we obtain $u_M \ge v_M$ in $\Omega_{0,1}$ (recall that $q = 0$ in $\Omega_{0,1}$), and since $\phi > 0$ this implies $u_M \to \infty$ in $\Omega_{0,1}$, as we wanted to prove. Note that this divergence is uniform in compact subsets of $\Omega_{0,1} \cup (\Gamma_2 \setminus \partial\Omega_{0,2})$.

To conclude the proof of the theorem, we have to show that, under the additional hypothesis (1.2), $u_M \to \infty$ on the boundary of $\Omega_+ \cup \overline{\Omega}_{0,2}$. We have to show that $u_M \to \infty$ on $\Gamma_2 \setminus \partial\Omega_{0,2}$. To this end, fix a small $\delta > 0$, consider the set $U_\delta = \{x \in \Omega : \ \mathrm{dist}(x,\Omega_{0,1}) < \delta\}$ (an open part of Ω containing $\Gamma_2 \setminus \partial\Omega_{0,2}$) and let

$$q(\delta) = \sup_{\{x\in\Omega_+ : \ \mathrm{dist}(x,\Omega_{0,1}) < \delta\}} q(x).$$

Let v_δ be the solution to

$$\begin{cases} \Delta v(x) = e^{q(\delta)v(x)} & \text{in } U_\delta, \\ v(x) = M & \text{on } \Gamma_1, \\ v(x) = 0 & \text{on } \partial U_\delta \cap \Omega, \end{cases}$$

for some $M = M(\delta)$ to be chosen which will verify $M \to \infty$ as $\delta \to 0$. Now, we scale out the factor M: it turns out that $w_\delta = v_\delta / M$ verifies

$$\begin{cases} \Delta w(x) = \dfrac{e^{Mq(\delta)w(x)}}{M} & \text{in } U_\delta, \\ w(x) = 1 & \text{on } \Gamma_1, \\ w(x) = 0 & \text{on } \partial U_\delta \cap \Omega. \end{cases}$$

If we choose $M = 1/q(\delta)$, it will follow that $e^{Mq(\delta)w}/M \le C\, q(\delta) \to 0$ as $\delta \to 0$, and then it is standard to deduce that $w_\delta \to w_0$ as $\delta \to 0$, where w_0

is the harmonic function in $\Omega_{0,1}$ which equals 1 on $\Gamma_1 = \partial\Omega_{0,1} \cap \partial\Omega$ and 0 on $\partial\Omega_{0,1} \cap \Omega$. Notice in particular that $w_0 > 0$ in Ω_0 and $\partial w_0/\partial\nu < 0$ on $\partial\Omega_0$.

Now let $x_0 \in \partial\Omega_0 \cap \Omega$, and denote by $\bar{x}_{0,\delta}$ the closest point to x_0 lying on $\partial U_\delta \cap \Omega$. Thanks to the mean value theorem,

$$w_\delta(x_0) = -\delta \frac{\partial w_\delta}{\partial\nu}(\xi_\delta) \geq c\,\delta$$

for some positive constant c not depending on δ, where ν is the outward unit normal to $\partial\Omega_0$ at x_0 and ξ_δ is a point in the segment $[x_0, \bar{x}_{0,\delta}]$

We now notice that $u_M \geq v_\delta$ in U_δ by comparison. Therefore

$$u_M(x_0) \geq v_\delta(x_0) = \frac{1}{q(\delta)} w_\delta(x_0) \geq c\,\frac{\delta}{q(\delta)} \to +\infty,$$

thanks to condition (1.2). The proof concludes by recalling that u_M is increasing in M. □

3. Asymptotic behavior and uniqueness

In this section, we find the asymptotic behavior of solutions to (1.1) and, as a consequence, we prove uniqueness of such solutions both when $q > 0$ on $\partial\Omega$ and when q verifies condition (1.4).

Proof of Theorem 3. Fix a small $\varepsilon > 0$, and let us see that, for $\beta \in (0,1)$ and $K > 0$, the function

$$\overline{u}(x) = -\frac{2}{q(x)} \log(d(x)) + \frac{1}{q(x)} \log\left(\frac{2+\varepsilon}{q(x)}\right) + K d(x)^\beta$$

is a supersolution to (1.1) in a neighborhood of $\partial\Omega$ of the form

$$\Omega_\delta = \{x \in \Omega :\ 0 < d(x) < \delta\}\ .$$

We assume that δ is chosen small enough to have $d \in C^2(\Omega_\delta)$ and $|\nabla d| = 1$ there (cf. [16]). We have

$$\begin{aligned}\Delta\overline{u} &= \frac{2}{qd^2} - \frac{2}{qd}\Delta d - \frac{4}{d}\nabla q^{-1}\nabla d - \log d\ \Delta q^{-1} \\ &\quad + \Delta\{q^{-1}\log\left(\frac{2+\varepsilon}{q}\right)\} + K\beta\{(\beta-1) + d\Delta d\}d^{\beta-2}.\end{aligned}$$

More briefly,

$$\Delta\overline{u} = \frac{1}{d^2}\left\{\frac{2}{q} + K\beta\{(\beta-1) + d\Delta d\}d^\beta + O(d)\right\},$$

as $d \to 0$, where the terms grouped in $O(d)$ do not depend on K.

From this expression, it is easy to see that $\overline{u}$ is a supersolution in Ω_δ provided that

$$\frac{2}{q} + K\beta(\beta - 1 + d\Delta d)d^\beta + O(d) \leq \frac{2+\varepsilon}{q} e^{Kqd^\beta}. \tag{3.1}$$

Since $\beta \in (0,1)$, we can diminish δ so that $\beta - 1 + d\Delta d < 0$, and thus (3.1) will hold for $K > 1$ if

$$\frac{2}{q} + O(d) \leq \frac{2+\varepsilon}{q}$$

in Ω_δ. For an arbitrarily chosen $\varepsilon > 0$ such inequality is certainly true in Ω_δ (regardless the size of K) if δ is small enough.

It can be similarly checked that a subsolution is provided by

$$\underline{u}(x) = -\frac{2}{q(x)}\log(d(x)) + \frac{1}{q(x)}\log\left(\frac{2-\varepsilon}{q(x)}\right) - Kd(x)^\beta.$$

Now let u be a solution to (1.1). If K is large enough, we can achieve $\underline{u} \le u \le \overline{u}$ in $d = \delta$. We now claim that the problem

$$\begin{cases} \Delta w = e^{q(x)w} & x \in \Omega_\delta \\ w = u & d = \delta \\ w = \infty & x \in \partial\Omega, \end{cases} \tag{3.2}$$

possesses a unique solution. Therefore by using the method of sub and supersolutions (see [13]) we conclude that

$$\underline{u}(x) \le u(x) \le \overline{u}(x)$$

in Ω_δ. This means

$$\begin{aligned} &\frac{1}{q(x)}\log\left(1-\frac{\varepsilon}{2}\right) - Kd^\beta(x) \\ &\qquad \le u(x) + \frac{2}{q(x)}\log d(x) + \frac{1}{q(x)}\log q(x) - \frac{1}{q(x)}\log 2 \\ &\qquad\quad \le \frac{1}{q(x)}\log\left(1+\frac{\varepsilon}{2}\right) + Kd^\beta(x) \end{aligned}$$

for every x in Ω_δ. We take the limits $d(x) \to 0$ and $\varepsilon \to 0$ to prove (1.3).

To achieve the uniqueness of solutions to (3.2) it suffices with showing that every solution $u \in C^2(\Omega_\delta) \cap C(\Omega_\delta \cup \{d = \delta\})$ has the first-term asymptotic behavior on the boundary predicted by (1.3). Namely, that the limit

$$\lim_{d\to 0} \frac{u(x)}{\frac{2}{q(x)}\log d(x)^{-1}} = 1 \tag{3.3}$$

holds true. In fact, once it has been shown that every solution u to (3.2) satisfies (3.3) then uniqueness follows from the general argument provided in the proof of Theorem 6 to be given later (it will be omitted here for the sake of brevity).

We begin showing that

$$\overline{\lim_{d\to 0}} \frac{u(x)}{\frac{2}{q(x)}\log d(x)^{-1}} \le 1. \tag{3.4}$$

Thus assume that $\delta > 0$ is chosen so that $q(x) \ge q_0 > 0$ in $\overline{\Omega}_{2\delta}$. For $x \in \overline{\Omega}_{2\delta}$ let $B_x = B(x, \sigma d(x))$ be the ball centered at x with radius σ and $0 < \sigma < 1$ fixed. Now, for an arbitrary solution u to (1.1) set

$$v(y) = u(x + \sigma d(x) y) + \frac{1}{\overline{q}_x}\log(\sigma d(x))^2 \qquad y \in B(0,1),$$

where $\overline{q}_x := \min_{B_x} q$. Then a direct computation shows that

$$v(y) \le \frac{1}{\overline{q}_x}\left(w(y) + \log\frac{1}{\overline{q}_x}\right) \qquad y \in B(0,1),$$

where $w = w(y)$ is the unique solution to

$$\begin{cases} \Delta w = e^w & y \in B(0,1) \\ w = \infty & y \in \partial B(0,1). \end{cases}$$

By setting $y = 0$ in the latter expression we obtain

$$u(x) \leq \frac{1}{\overline{q}_x} \log d(x)^{-2} + \frac{1}{\overline{q}_x} \log(\sigma \overline{q}_x)^{-1} + \frac{1}{\overline{q}_x} w(0).$$

Then, to arrive at (3.4) notice that $\lim_{x \to \partial\Omega} q(x)/\overline{q}_x = 1$.

To check the complementary limit estimate leading to (3.3) we are using a sweeping approach from [8] (see also [13]). In fact, computations similar to those at the beginning of the proof show the existence of positive numbers δ, τ_0 such that

$$\underline{u}_\tau(x) := -\frac{2}{q(x)} \log(d(x) + \tau) - K,$$

defines a subsolution to $\Delta u = e^{q(x)u}$ in Ω_δ for every $0 < \tau < \tau_0$ and $K > 0$ an arbitrary constant. Moreover, a fixed $K > 0$ can be found so that $\underline{u}_\tau$ is a subsolution to the problem

$$\begin{cases} \Delta v = e^{q(x)v} & x \in \{x \in \Omega_\delta : d > \varepsilon\}, \\ v = u & x \in \{d = \varepsilon\} \cup \{d = \delta\}, \end{cases} \tag{3.5}$$

for all $0 < \tau < \tau_0$ and $0 < \varepsilon < \varepsilon_0$. Since (3.5) has arbitrarily large supersolutions and $v = u$ is its unique solution we conclude that $\underline{u}_\tau(x) \leq u(x)$ for all x such that $\varepsilon < d(x) < \delta$. Letting $\varepsilon \to 0$ and then $\tau \to 0$ we obtain

$$u(x) \geq -\frac{2}{q(x)} \log d(x) - K,$$

$x \in \Omega_\delta$. This readily implies that

$$\lim_{d \to 0} \frac{u(x)}{q(x)^{-1} \log d(x)^{-2}} \geq 1$$

and the proof of (3.3) is concluded. □

Now we are ready to prove the uniqueness of solutions to (1.1) stated in Theorem 4.

Proof of Theorem 4. Let u and v be solutions to (1.1). According to Theorem 3 we have that

$$u(x) - v(x) \to 0, \qquad \text{as } x \to \partial\Omega.$$

Assume that the set $G := \{u < v\}$ is nonempty. Then since $\Delta u \leq \Delta v$ in G and $u(x) - v(x) \to 0$ as $x \to \partial G$, the strong maximum principle implies $u > v$ in G, which is clearly impossible. Thus G is empty and this means $u \geq v$. The symmetric argument shows $u = v$, and uniqueness is proved. □

Remark 1. In base of the weaker asymptotic estimate (3.3) an alternative proof to show uniqueness can be produced following the argument of the forthcoming Theorem 6.

Next we determine the boundary behavior of solutions in the case where q vanishes on $\partial\Omega$, subject to condition (1.4).

Proof of Theorem 5. We will construct sub and supersolutions with a precise growth near $\partial\Omega$ and then we will proceed with a sweeping approach. Fixing $\varepsilon > 0$ there exists $\delta > 0$ such that $q(x) \geq (Q-\varepsilon)\, d(x)^\gamma$ in Ω_δ. Next we check that the function

$$\overline{u}(x) = -\frac{\gamma+2+\varepsilon}{(Q-\varepsilon)d(x)^\gamma}\log d(x) + K,$$

is a supersolution in Ω_δ for every $K > 0$ if δ is small enough. Indeed, it can be easily seen that

$$\Delta\overline{u} = \frac{\gamma+2+\varepsilon}{Q-\varepsilon}(d^{-\gamma-2}(2\gamma+1-\gamma(\gamma+1)\log d) + d^{-\gamma-1}(\gamma\log d - 1)),$$

and thus $\overline{u}$ will be a supersolution provided that

$$\frac{\gamma+2+\varepsilon}{Q-\varepsilon}((2\gamma+1-\gamma(\gamma+1)\log d) + d(\gamma\log d - 1)) \\ \leq d^{\gamma+2}\, e^{-(\gamma+2+\varepsilon)\log d + (Q-\varepsilon)d^\gamma K}.$$

Since $K > 0$, this will be achieved if

$$\frac{\gamma+2+\varepsilon}{Q-\varepsilon}d^\varepsilon((2\gamma+1-\gamma(\gamma+1)\log d) + d(\gamma\log d - 1)) \leq 1,$$

which is clearly true by diminishing δ if necessary. It follows that $\overline{u}$ is a supersolution to (1.1) in $0 < d < \delta$ for every $K > 0$. Note that the choice of δ does not depend on K.

The previous computations ensure the existence of $0 < \tau_0 < \delta$, not depending on K, such that

$$\Delta\overline{u}_\tau \leq e^{(Q-\varepsilon)(d-\tau)^\gamma\overline{u}_\tau}$$

in $\{x \in \Omega_\delta : \tau < d < \delta\}$ for every $0 < \tau < \tau_0$ where,

$$\overline{u}_\tau(x) = \overline{u}(d(x)-\tau) = -\frac{\gamma+2+\varepsilon}{(Q-\varepsilon)(d(x)-\tau)^\gamma}\log(d(x)-\tau) + K.$$

Since $(d-\tau)^\gamma < d^\gamma$ a conveniently large $K > 0$ can be found so that $\overline{u}_\tau$ defines a supersolution to

$$\begin{cases} \Delta v = e^{q(x)v} & x \in \{x \in \Omega_\delta : \tau+\varepsilon_1 < d < \delta\} \\ v = u & x \in \{d = \tau+\varepsilon_1\} \cup \{d = \delta\}\ , \end{cases}$$

for all $\varepsilon_1 > 0$ smaller than some critical size $\varepsilon^* > 0$ and every $0 < \tau < \tau_0$. However, $v = u$ is the unique solution to such problem which in turn has sufficiently small subsolutions. Thus we obtain:

$$u(x) \leq \overline{u}_\tau(x) \qquad \tau+\varepsilon_1 < d(x) < \delta.$$

Now by letting first $\varepsilon_1 \to 0$ and then $\tau \to 0$ we achieve

$$u(x) \leq \overline{u}(x) \qquad x \in \Omega_\delta. \tag{3.6}$$

It can be proved by the symmetric reasoning that

$$\underline{u}(x) = -\frac{\gamma+2-\varepsilon}{(Q+\varepsilon)d(x)^\gamma}\log d(x) - K,$$

is a subsolution in $0 < d < \delta$ for every $K > 0$. Proceeding now exactly in the same way as in the proof of Theorem 3 we arrive at the complementary estimate,

$$\underline{u}(x) \le u(x) \qquad \text{for all } x \in \Omega_\delta.$$

Summing up the conclusions, the inequalities

$$\underline{u}(x) \le u(x) \le \overline{u}(x)$$

hold true in Ω_δ. Therefore, we have,

$$\frac{\gamma+2-\varepsilon}{Q+\varepsilon} \le \liminf_{d\to 0} \frac{d(x)^\gamma u(x)}{-\log d(x)} \le \limsup_{d\to 0} \frac{d(x)^\gamma u(x)}{-\log d(x)} \le \frac{\gamma+2+\varepsilon}{Q-\varepsilon}$$

and (1.5) is obtained by letting $\varepsilon \to 0$. $\square$

Remark 2. Estimate (3.6) can be also obtained by means of the scaling approach leading to (3.4).

Our last uniqueness proof differs from that of Theorem 4 in an important issue: we are not using the monotonicity of e^{qu}. What is really needed here is that e^{qu}/u is increasing when $qu > 1$.

Proof of Theorem 6. Let u and v be solutions to (1.1). Thanks to Theorem 5 we have

$$\lim_{d(x)\to 0} \frac{u(x)}{v(x)} = 1. \tag{3.7}$$

We also observe that the boundary behavior of solutions given by (1.5) implies

$$\lim_{d(x)\to 0} q(x)u(x) = +\infty,$$

and, as $q > 0$, we can choose $K > 0$ such that $q(u+K) > 1$ in Ω. We notice that $\tilde{u} = u + K$ verifies

$$\Delta \tilde{u}(x) = a(x) e^{q(x)\tilde{u}(x)} \qquad \text{in } \Omega, \tag{3.8}$$

where $a(x) = e^{-Mq(x)}$ is a strictly positive and bounded weight. By enlarging K if necessary we can also assume that $q(v+K) > 1$ in Ω, and $\tilde{v} = v + K$ is also a solution to (3.8). We also observe that $\tilde{u}/\tilde{v} \to 1$ as $d \to 0$ thanks to (3.7). This means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(1-\varepsilon)\tilde{v} \le \tilde{u} \le (1+\varepsilon)\tilde{v} \qquad \text{in } \Omega_\delta.$$

Next we consider the problem

$$\begin{cases} \Delta w(x) = a(x) e^{q(x)w(x)} & \text{in } \Omega \setminus \Omega_\delta, \\ w = \tilde{u} & \text{on } \partial(\Omega \setminus \Omega_\delta). \end{cases} \tag{3.9}$$

Thanks to the monotonicity of the right-hand side, this problem has a unique solution, which is exactly $\tilde{u}$. Moreover, since $q\tilde{v} > 1$, it is easily seen that $(1-\varepsilon)\tilde{v}$ and $(1+\varepsilon)\tilde{v}$ are respectively a sub and a supersolution to (3.9) and thus by uniqueness

$$(1-\varepsilon)\tilde{v} \le \tilde{u} \le (1+\varepsilon)\tilde{v}, \qquad \text{in } \Omega \setminus \Omega_\delta.$$

It follows that the inequality is valid throughout Ω, and letting $\varepsilon \to 0$ we obtain $\tilde{u} = \tilde{v}$, that is, $u = v$ in Ω. This concludes the proof. $\square$

4. Changing-sign exponents

We begin this section with a proof of Theorem 7 with an argument entirely similar to the one considered when we proved Theorem 1.

Proof of Theorem 7. Suppose that (1.1) admits a solution $u \in C^2(\Omega)$ and let $q \le 0$ in $W := B(x_0, \delta) \cap \Omega$ for certain $x_0 \in \partial\Omega$ and positive δ. A smooth function ψ supported on $B(x_0, 2\delta/3) \cap \partial\Omega$ can be found such that $0 \le \psi \le 1$ and $\psi = 1$ on $B(x_0, \delta/3) \cap \partial\Omega$. Then, for $K > 0$ the solution v_K to the auxiliary problem,

$$\begin{cases} \Delta u = 1 & x \in W \\ u = K\psi & x \in \partial W, \end{cases}$$

can be written as $v_K = -\phi_1 + K\phi_2$ with $v = \phi_1$ solving $-\Delta v = 1$ in W, $v_{|\partial W} = 0$ and ϕ_2 is harmonic in W and achieves the value ψ on ∂W.

On the other hand, for small fixed $\varepsilon > 0$ set $W_\varepsilon = \{x \in W : \text{dist}(x, \partial\Omega) > \varepsilon\}$. Due to the singular boundary value of u on $\partial\Omega$ one finds by simple comparison that $u \ge v_K$ in W_ε. Letting $\varepsilon \to 0$ leads to $u \ge v_K$ in W which is not possible since K can be chosen arbitrarily large. □

As a preliminary result for the remaining proofs in this section it is convenient to introduce a lemma whose basic assertions are essentially well-known. However an "ad hoc" proof is included for later use.

Lemma 10. *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain, $q \in C^\eta(\overline{\Omega})$ such that $q(x) > 0$ for $x \in \Omega$ (but possibly $q = 0$ on $\partial\Omega$). Then there exists a value $m^* > -\infty$ such that the problem*

$$\begin{cases} \Delta u = e^{-q(x)u} & x \in \Omega \\ u = m & x \in \partial\Omega, \end{cases} \tag{4.10}$$

exhibits a classical solution $u \in C^{2,\eta}(\overline{\Omega})$ if $m > m^*$ while it does not admit solutions when $m < m^*$. Moreover, for every $m > m^*$ there exists a maximum classical solution u_m which increases in m and satisfies

$$u_m(x) = m + o(1) \qquad m \to \infty, \tag{4.11}$$

that is, u_m "bifurcates" from infinity as $m \to \infty$. In addition, u_m is smooth in m and unique for large m.

Proof. Since any possible solution u to (4.10) is subharmonic then $u \le m$ in Ω. Thus, setting $v = m - u$ the solutions to (4.10) correspond to positive solutions to:

$$\begin{cases} -\Delta v = e^{-mq(x)} e^{q(x)v} & x \in \Omega \\ v = 0 & x \in \partial\Omega. \end{cases} \tag{4.12}$$

Observe now that $\bar{v} = \phi$, the solution to $-\Delta u = 1$ in Ω, $u_{|\partial\Omega} = 0$, defines a supersolution to (4.12) if $m \ge m_c := \sup_\Omega \phi$. Since $\underline{v} = 0$ is a subsolution then (4.12) has a solution $0 < v < \phi$ for every $m \ge m_c$.

On the other hand, $m^* = \inf\{m \in \mathbb{R} : \text{(4.12) has a solution}\}$ satisfies $-\infty \le m^* < \infty$. Now observe that if v is a positive solution to $(4.12)_{m=m_1}$, it becomes a supersolution for all problems $(4.12)_m$ with $m \ge m_1$. Since

$\underline{v} = 0$ is a fixed subsolution the existence for $m > m^*$ of a minimal solution v_m to (4.12) follows from the method of sub and supersolutions.

To show the finiteness of m^* observe that for v solving (4.12) we find

$$-\Delta v > qe^{-mq}v$$

in Ω. Setting $\lambda_1^\Omega(-\Delta + V(x))$ the first Dirichlet eigenvalue of $-\Delta + V(x)$ under Dirichlet conditions we conclude from standard properties,

$$0 < \lambda_1^\Omega(-\Delta - qe^{-mq}v) < \lambda_1^{\Omega_1}(-\Delta - qe^{-mq}v) < \lambda_1^{\Omega_1}(-\Delta) - q_1 e^{-mq_1},$$

where $\Omega_1 \subset \overline{\Omega}_1 \subset \Omega$, $q_1 = \inf_{\Omega_1} q$. This inequality is not possible if $m \to -\infty$.

Regarding uniqueness and smoothness for large m (see Remark 3 below) (4.12) can be read off as a regular perturbation problem as $m \to \infty$. In fact, the function $f(x,\varepsilon) = e^{-\frac{1}{|\varepsilon|}q(x)}$ for $\varepsilon \neq 0$, $f(x,0) = 0$ is smooth and increasing in $\varepsilon > 0$ while (4.12) can be written as

$$\begin{cases} -\Delta v = f(x,\varepsilon)e^{q(x)v} & x \in \Omega \\ v = 0 & x \in \partial\Omega. \end{cases}$$

with $\varepsilon = 1/m$. The linearization of such problem at $v = 0$, $\varepsilon = 0$ has $\lambda_1^\Omega(-\Delta)$ as a first eigenvalue and so the implicit function theorem implies the existence of a unique positive solution v_ε close to zero for $0 < \varepsilon < \varepsilon^*$.

Results for (4.10) follow by the simple transcription $\varepsilon \to m^{-1}$, $v \to u$. In particular, the maximal solution $u_m = m - v_m \in C^{2,\eta}(\overline{\Omega})$ to (4.10) satisfies $m - \phi \leq u_m \leq m$ for each $m \geq m_c \geq m^*$. This shows (4.11). □

Remark 3. Equation in (4.12) is qualitatively equivalent to Gelfand's equation (see, for instance, [5], [15], [17]) $-\Delta v = \lambda e^v$, $\lambda > 0$, with $e^{-q(x)m}$ replacing λ. Following the ideas in [5] it can be shown that the maximal solution u_m is smooth in the whole range $m > m^*$.

Proof of Theorem 8. First, we find a weak supersolution $\overline{u}_m \in H^1_{\text{loc}}(\Omega) \cap C(\Omega)$ which takes the value $+\infty$ on $\partial\Omega$. In fact, choose a large enough $m > 0$ and let $u = u_{-,m}(x) \in C^{2,\eta}(\overline{\Omega}_-)$ be the solution to

$$\begin{cases} \Delta u = e^{-q_-(x)u} & x \in \Omega_- \\ u = m & x \in \partial\Omega_-, \end{cases}$$

which satisfies $m - \delta \leq u \leq m$ for $\delta > 0$ small and whose existence is obtained in Lemma 10.

On the other hand, set $u = u_{+,m} \in C^{2,\eta}(\Omega_+ \cup \partial\Omega_-)$ the minimal solution to the problem:

$$\begin{cases} \Delta u = e^{q_+(x)u} & x \in \Omega_+ \\ u = m & x \in \partial\Omega_- \\ u = \infty & x \in \partial\Omega. \end{cases}$$

The existence of $u_{+,m}$ follows by constructing the family $v_K(x) \in C^{2,\eta}(\overline{\Omega}_+)$ where for $K > 0$, $u = v_K$ is the solution to

$$\begin{cases} \Delta u = e^{q_+(x)u} & x \in \Omega_+ \\ u = m & x \in \partial\Omega_- \\ u = K & x \in \partial\Omega. \end{cases}$$

Then, it can be shown as in the proof of Theorem 1 that $v_K \to u_{+,m}$ in $C^{2,\eta}(\Omega_+ \cup \partial\Omega_-)$ as $K \to \infty$. Furthermore, the same arguments yield

$$\lim_{m\to\infty} u_{+,m} = u_+ \ , \tag{4.13}$$

in $C^{2,\eta}(\Omega_+ \cup \partial\Omega_-)$ with u_+ the minimal solution to

$$\begin{cases} \Delta u = e^{q_+(x)u} & x \in \Omega_+ \\ u = \infty & x \in \partial\Omega_+. \end{cases}$$

Finally, define

$$\overline{u}_m(x) = \begin{cases} u_{-,m}(x) & x \in \overline{\Omega}_- \\ u_{+,m}(x) & x \in \Omega_+. \end{cases}$$

To check that $\overline{u}_m$ defines a weak supersolution we first notice that $\frac{\partial u_{-,m}}{\partial \nu_-} > 0$ where ν_- stands for the outer unit normal to Ω_- at $\partial\Omega_-$. To conclude that $\overline{u}_m$ is a supersolution it is only needed to ensure that

$$\frac{\partial u_{+,m}}{\partial \nu_+} > 0 \tag{4.14}$$

on $\partial\Omega_-$ where now $\nu_+ = -\nu_-$ is the outer unit normal to Ω_+ at the component $\partial\Omega_-$ of its boundary. Thus, to show (4.14) choose a small strip $\mathcal{U}_\delta = \{x \in \Omega_+ : \mathrm{dist}(x, \partial\Omega_-) < \delta\}$, set $\Gamma_\delta = \{x \in \Omega_+ : \mathrm{dist}(x, \partial\Omega_-) = \delta\} \subset \partial\mathcal{U}_\delta$ and notice that, in virtue of (4.13), $u_{+,m}$ keeps finite on Γ_δ as $m \to \infty$. Hence, for m conveniently large $u_{+,m}$ takes its maximum on $\partial\Omega_-$ and so (4.14) follows from Hopf's maximum principle.

In order to construct a suitable subsolution $\underline{u}$ to be compared with $\overline{u}_m$ we choose R slightly bigger than $\mathrm{diam}(\Omega)/2$ and $x_0 \in \Omega$ so that $\Omega \subset B(x_0, R)$ and use again the function $\phi(x) = \frac{1}{2N}(|x - x_0|^2 - R^2)$. It can be checked that $\underline{u} = A\phi$ defines a subsolution to $\Delta u = e^{q_+(x)u}$ in Ω_+ for every $A \geq 1$. In virtue of the maximum principle we have in addition

$$\underline{u}(x) < u_+(x) \qquad x \in \Omega_+,$$

since an exact solution can be placed between $\underline{u}$ and u_+. On the other hand, $\underline{u}$ defines a subsolution to $\Delta u = e^{-q_-(x)u}$ in Ω_- if, for $A > 1$ fixed,

$$\sup_{\Omega_-} q_- \leq \left(\frac{1}{-\inf_{\Omega_-} \phi} \right) \frac{1}{A} \log A.$$

Observe that a positive R satisfying $\Omega \subset B(x_0, R)$ and coefficient $1 < A < e$ can be found if relation (1.6) holds. Finally, since $\phi < 0$ in $\overline{\Omega}$ then we also get $\underline{u} \leq \overline{u}_m$.

To conclude the proof of i) in Theorem 8 we obtain the minimal solution to (1.1) from $\underline{u}$ and $\overline{u}_m$. Thus, we construct an increasing sequence $u_n \in C^{2,\eta}(\overline{\Omega})$ solving

$$\begin{cases} \Delta u = e^{q(x)u} & x \in \Omega, \\ u = n & \partial\Omega \end{cases} \tag{4.15}$$

and satisfying $u_n < \overline{u}_m$ for every n. Then, according to the arguments in Section 2, $u = \lim u_n = \sup u_n$ defines the minimal solution to (1.1). Proceeding inductively, to find u_{n+1} from u_n, we regard $\underline{u}_n := u_n$ as a subsolution. To construct the corresponding supersolution to $(4.15)_{n+1}$ we choose $\varepsilon_0 > 0$ such that $\overline{u}_m > n+1$ in $\Omega_{\varepsilon_0} = \{x \in \Omega : d(x) < \varepsilon_0\}$. For $0 < \varepsilon < \varepsilon_0$ we obtain a solution $u = \tilde{u}_{n+1}$ to the problem

$$\begin{cases} \Delta u = e^{q(x)u} & x \in \Omega^\varepsilon, \\ u = n+1 & \partial\Omega^\varepsilon \end{cases}$$

in $\Omega^\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ by regarding the restrictions of $\underline{u}_n = u_n$, $\overline{u} = \overline{u}_m$ to Ω^ε as sub and supersolutions, respectively. Thus, $\overline{u} = \tilde{u}_{n+1}$ if $x \in \Omega^\varepsilon$ and $\overline{u} = n+1$ for $x \in \Omega \backslash \Omega^\varepsilon$ defines the searched supersolution. Since $\overline{u} \le \overline{u}_m$ we obtain a solution u_{n+1} to $(4.15)_{n+1}$ satisfying $u_n \le u_{n+1} \le \overline{u}_m$, as desired.

Let us show now part ii) and observe that

$$L_\Omega = \inf_{k \in S_{N-1}} \sup_{x,y \in \Omega} |k(x-y)| \,.$$

We proceed to obtain a suitable positive subsolution $\underline{u}$ which can be compared to $\overline{u}_m$ for large m. From the expression for L_Ω it is possible to find $y \in \overline{\Omega}$ and $k \in S_{N-1} = \partial B(0,1)$ such that

$$\Omega \subset \{x \in \mathbb{R}^N : |k(x-y)| \le L \},$$

while a number $L > L_\Omega$ can be chosen so that

$$\bar{q} L^2 < 2\pi^2, \tag{4.16}$$

where $\bar{q} = \sup_\Omega q = \sup_{\Omega_+} q^+$. Now we set

$$\underline{u}(x) = w(t),$$

where $t = k(x-y) \in (-\frac{L}{2}, \frac{L}{2})$ solves

$$\begin{cases} w'' = e^{\bar{q}} w, & 0 < t < L/2, \\ w'(0) = 0, \\ w(L/2) = \infty. \end{cases}$$

A direct computation shows that

$$w(t) = \frac{2}{\bar{q}} \log\left(\frac{2\omega^*}{\sqrt{\bar{q}} L}\right) + \frac{1}{\bar{q}} z\left(\frac{2\omega^*}{L} t\right),$$

where $\omega^* = \pi/\sqrt{2}$ and $z = z(t)$ is the solution to:

$$\begin{cases} z'' = e^z \\ z(0) = z'(0) = 0. \end{cases}$$

In particular, for each $x \in \overline{\Omega}$

$$\underline{u}(x) \ge w(0) = \frac{2}{\bar{q}} \log\left(\frac{2\omega^*}{\sqrt{\bar{q}} L}\right) > 0$$

provided condition (4.16) holds.

On the other hand, the positivity of $\underline{u}$ implies that it defines a subsolution to $\Delta u = e^{q(x)u}$ in Ω which is finite in $\overline{\Omega}$. This implies

$$\underline{u}(x) < u_+(x) \qquad x \in \overline{\Omega}_+,$$

which in turn says that $\underline{u}(x) < \overline{u}_m(x)$ for $x \in \Omega$ if m is large. As shown in part i) one obtains a positive solution u to (1.1) lying in the interval $[\underline{u}, \overline{u}_m]$. □

Proof of Corollary 9. It is an immediate consequence of inequality (1.7) and the fact that $\sup q_\lambda = \sup q$ while $L_{\Omega_\lambda} = \lambda L_\Omega$ for all $\lambda > 0$. □

Remark 4. Theorem 8 and Corollary 9 also hold –without changes in their proofs– under slightly more general conditions on q. For instance, the connectedness of Ω_- is not necessary provided it remains being a smooth open subset of Ω (possessing finitely many components). On the other hand, q may also exhibit a null domain Ω_0 provided $\overline{\Omega}_0 \subset \Omega$ and $\partial\Omega_0 \cap \partial\Omega_- = \emptyset$.

5. A reference example

In this section, we analyze a class of prospective examples to estimate the strength of the conclusions in Theorem 8, regarding the sizes of $q_\pm$ and Ω, and suggesting some multiplicity results.

We will be concerned with the analysis of the one-dimensional problem:

$$\begin{cases} u'' = e^{q(x)u} & -L < x < L, \\ u(\pm L) = \infty, \end{cases} \tag{5.17}$$

where q designates the two-parametric family of symmetric coefficients

$$q(x) = -q_0\chi_{I_0}(|x|) + q_1\chi_{I_1}(|x|),$$

q_0, q_1 positive parameters, I_0, I_1 the intervals $[0, a]$, $(a, a + b]$, respectively, $a + b = L$ (χ_I standing for the characteristic function of I). Solutions to (5.17) will be understood in a weak sense, i. e. $u \in C^1(-L, L) \cap C^2\{(-L, L) \setminus \{-a, a\}\}$ and pointwise solving (5.17) with the sole exception of $x = \pm a$.

Notice that every solution u to (5.17) is strictly convex in $(-L, L)$ and thus possesses a unique critical point x_0 where it achieves the minimum, $u_0 = u(x_0)$. It is easily checked that u is symmetric if and only if $x_0 = 0$. On the other hand, and as a general rule, symmetry of solutions if often obtained as a consequence of a uniqueness result. However, it will be seen below that our problem (5.17) may exhibit multiple solutions. Nevertheless, all such solutions must be symmetric.

Lemma 11. *Every solution $u \in C^1(-L, L) \cap C^2\{(-L, L) \setminus \{-a, a\}\}$ to (5.17) is symmetric and therefore defines a solution to the boundary value problem*

$$\begin{cases} u'' = e^{q(x)u} & 0 \le x < L \\ u(0) = u_0, \quad u'(0) = 0 \\ u(L) = \infty, \end{cases} \tag{5.18}$$

where $u_0 = \min u$.

We will show the symmetry assertion in the course of the next discussion of the main features of (5.17).

To analyze the existence of solutions to (5.17) observe that any solution u to (5.18) separately solves

$$\begin{cases} v'' = e^{-q_0 v} & 0 \le x < a, \\ v(0) = u_0, \quad v'(0) = 0, \end{cases} \tag{5.19}$$

in $[0,a]$ and

$$
\begin{cases} w'' = e^{q_1 w} & a \le x < L = a+b, \\ w(a) = u_1, \quad w'(a) = u_1', & \\ w(L) = \infty, & \end{cases} \tag{5.20}
$$

in $[a,L)$, where $u_1 = v(a), u_1' = v'(a)$ in (5.20). Conversely, if u_1, u_1' are regarded as parameters in (5.20) and u_0 is another parameter in (5.19), v solves (5.19) in I_0, w is a solution of (5.20) in I_1, then u defined as v in I_0 and as w in I_1 constitutes a solution to (5.18) (i. e, to (5.17)) if and only if the relations

$$
v(a) = w(a) \qquad v'(a) = w'(a), \tag{5.21}
$$

hold.

In order to discuss equations (5.21) we first characterize separately the solvability of problems (5.19) and (5.20) in the form,

$$
v(a) = f(v'(a)), \qquad w(a) = g(w'(a)), \tag{5.22}
$$

respectively, with functions f, g which depend on the parameters a, q_0 and b, q_1, respectively. Once f and g have been found, (5.21) simply reduces to

$$
f(u_1') = g(u_1'). \tag{5.23}
$$

After solving (5.23) a solution to (5.17) is obtained by inserting u_1' and the common value $u_1 = f(u_1') = g(u_1')$ in (5.19), (5.20) (in the case of (5.19), initial conditions u_1, u_1' are taken on $x = a$).

Accordingly, we begin with the analysis of (5.18). In order to obtain the function f in (5.22) we compute the solution $v(x)$ to (5.19) keeping u_0 as a parameter. Direct integration shows that:

$$
v(x) = u_0 + \frac{2}{q_0} \log\cosh\left(\sqrt{\frac{q_0}{2}} e^{-q_0 u_0/2} x\right).
$$

Setting $u_1 = v(a), u_1' = v'(a)$ we find

$$
\begin{aligned} u_1 &= \frac{2}{q_0} \log\left(a\sqrt{\frac{q_0}{2}} \frac{\cosh\xi}{\xi}\right) \\ u_1' &= \frac{2}{a q_0} \xi \tanh\xi, \end{aligned} \tag{5.24}
$$

where $\xi = a\sqrt{\frac{q_0}{2}} e^{-q_0 u_0/2}$. Since u_1' is increasing in $\xi \ge 0$ then (5.24) defines

$$
u_1 = f(u_1', a, q_0) := \frac{2}{q_0} \log\left(a\sqrt{\frac{q_0}{2}} \frac{\cosh H(a q_0 u_1'/2)}{H(a q_0 u_1'/2)}\right),
$$

where $H(t)$ is defined through $H(t)\tanh H(t) = t$. The next features of f, relevant in our discussion, are listed without proof.

i) f is smooth in $u_1' > 0$ and

$$
f(u_1') = -\frac{1}{q_0} \log\left(\frac{u_1'}{a}\right) + o(1) \qquad u_1' \to 0,
$$

while

$$
f(u_1') = a u_1' + o(u_1') \qquad u_1' \to \infty.
$$

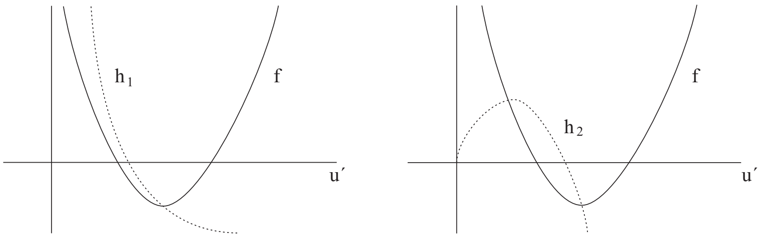

FIGURE 1. Curves of minima of f: $u = h_1(u')$ and $u = h_2(u')$

ii) f has an absolute minimum at $u_1' = u_m'$ with

$$u_m' = \frac{2}{aq_0}, \qquad u_m := f(u_m') = \frac{1}{q_0} \log \frac{a^2 q_0 \sinh^2 \theta}{2}, \tag{5.25}$$

$\theta > 1$ the positive root of $x \tanh x - 1 = 0$ ($\theta \cong 1.19968$). Moreover, f decreases when $u_1' \in (0, u_m')$ and it increases for $u_1' \in (u_m', \infty)$.

iii) f is increasing in a. Moreover, $f = \dfrac{1}{q} \log(a/u_1') + o(1)$ as $a \to 0$ and so $\lim_{a\to 0} f = -\infty$ uniformly in compacts of $\mathbb{R}^+$, while $f = au_1' + O(1)$ as $a \to \infty$ and $\lim_{a\to\infty} f = \infty$ uniformly in compacts of $\mathbb{R}^+$.

iv) $f = \dfrac{1}{q_0} \log(\dfrac{a}{u_1'}) + o(\dfrac{1}{q_0})$ as $q_0 \to 0$ and so $f \to \infty$ uniformly in compacts of $(0, a)$, $f \to -\infty$ uniformly in compacts of (a, ∞) as $q_0 \to 0$. In addition, $\lim_{q_0\to\infty} f(u_1', a, q_0) = au_1'$ uniformly in compacts of $\mathbb{R}^+$.

It is also convenient for our purposes to set $u_m = h_1(u_m')$, the curve (5.25) of minima of f when parameterized by a (q_0 fixed), $u_m = h_2(u_m')$ the corresponding curve when parameterized by q_0 and a is kept fixed (see Figure 1). From (5.25),

$$h_1(u_m') = \frac{1}{q_0} \log \left(\frac{2 \sinh^2 \theta}{q_0 {u_m'}^2} \right), \qquad h_2(u_m') = \frac{au_m'}{2} \log \left(\frac{a \sinh^2 \theta}{u_m'} \right).$$

Let us consider now problem (5.20). Since solutions to (5.18) are convex we restrict ourselves to study $u_1' > 0$. By setting the change:

$$u(t) = q_1 w(bt + a) + \log q_1 b^2 \qquad 0 \le t < 1,$$

(5.20) is reduced to

$$\begin{cases} u'' = e^u & 0 \le x < 1 \\ u(0) = q_1 u_1 + \log q_1 b^2, \quad u'(0) = bq_1 u_1' \\ u(1) = \infty. \end{cases} \tag{5.26}$$

Introducing the function

$$\omega(z, y) = \frac{1}{\sqrt{2}} \int_0^\infty \frac{ds}{\sqrt{(e^s - 1)e^z + \frac{y^2}{2}}}, \tag{5.27}$$

it can be checked that solving (5.26) amounts to solving the equation

$$\omega(z, y) = 1, \tag{5.28}$$

with $z = u(0), y = u'(0)$. The next statement summarizes the relevant features concerning (5.28).

Lemma 12. *Equation* (5.28) *is uniquely solvable in* $\mathbb{R} \times \overline{\mathbb{R}}^+$ *in the form*

$$z = g_1(y),$$

where g_1 *is a smooth decreasing function in* $y \geq 0$ *such that* $g_1(0) = 2\log\omega^*$ ($\omega^* = \frac{\pi}{\sqrt{2}}$), $g_1'(y) < 0$ *for* $y > 0$ *and*

$$\lim_{y\to\infty} \frac{g_1(y)}{y} = -1, \qquad \lim_{y\to\infty} g_1'(y) = -1. \tag{5.29}$$

Proof. The function $\omega(z, y)$ is separately decreasing in each variable and for each fixed $y \geq 0$ we find $\lim_{z\to\infty} \omega(z, y) = 0$, $\lim_{z\to-\infty} \omega(z, y) = \infty$. This provides existence and uniqueness of a solution for every $y \geq 0$. Notice also that since $\omega(z, 0) = \omega^* e^{-z/2}$ the value $g_1(0)$ is readily obtained. The smoothness of g_1 follows from the implicit function theorem.

In order to show the first limit in (5.29) observe that it can be directly shown that $g_1 \to -\infty$ as $y \to \infty$. Computing the integral in (5.27) we obtain

$$\sqrt{y^2 - 2e^{g_1}} = 2\log\left(\frac{y + \sqrt{y^2 - 2e^{g_1}}}{\sqrt{2}}\right) - g_1,$$

for large y. Thus,

$$\frac{g_1}{y} = \frac{2}{y}\log\left(\frac{y + \sqrt{y^2 - 2e^{g_1}}}{\sqrt{2}}\right) - \frac{\sqrt{y^2 - 2e^{g_1}}}{y}.$$

As the first term in the right-hand side is $o(1)$ as $y \to \infty$ the desired limit follows. The second limit can be easily obtained by differentiating the last equality. □

Proof of Lemma 11. Let us show now the symmetry assertion and so let u be any solution to (5.17) achieving the minimum at $x_0 \in (-L, L)$. If u is not symmetric we can assume that $x_0 > 0$ and consider first the case $0 < x_0 < a$. By setting $\zeta = u(a), \zeta' = u'(a)$ we get by integration

$$b = \sqrt{q_1}\,\omega(q_1\zeta, \sqrt{q_1}\zeta') = \sqrt{\frac{q_1}{2}} \int_0^\infty \frac{dt}{\sqrt{(e^t - 1)e^{q_1\zeta} + q_1\frac{\zeta'^2}{2}}}.$$

Reasoning by symmetry we see that $u(a') = u(a)$, $u'(a') = u'(a)$ where $a' = 2x_0 - a$ satisfies $a' > -a$ and thus $\zeta_1 := u(-a) > \zeta$, $\zeta_1' := -u'(-a) > \zeta$. This implies that

$$\omega_1 := \sqrt{q_1}\,\omega(q_1\zeta_1, \sqrt{q_1}\zeta_1') < \sqrt{q_1}\,\omega(q_1\zeta, \sqrt{q_1}\zeta') = b.$$

However, it follows from this equality that $u(-a - \omega_1) = \infty$ with $-a - \omega_1 > -L$ which is not possible. The impossibility of $x_0 \in [a, a + b)$ is shown with a similar argument. Therefore x_0 must be zero, that is, u is symmetric. □

In view of the previous analysis we finally conclude that problem (5.20) admits a solution –in the regime $u_1' \geq 0$– if and only if

$$u_1 = g(u_1', b, q_1) := \frac{1}{q_1} g_1(bq_1u_1') - \frac{1}{q_1}\log q_1 b^2.$$

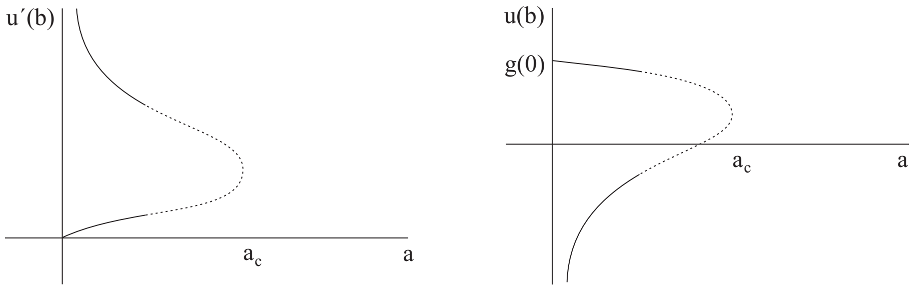

FIGURE 2. Bifurcation diagram for (5.17) with respect a representing solutions by $u'(b)$ (left) and $u(b)$ (right). Continuous drawing in curves means exact number of solutions.

Some relevant properties of g are next listed.

i) The function g is decreasing with respect to u_1', and it verifies $g(0) = \dfrac{1}{q_1}\log(\omega^{*2}/q_1 b^2)$ and $g(u_1') \sim -bu_1'$ as $u_1' \to \infty$.

ii) g is decreasing with respect to b, $\lim_{b\to 0} g = \infty$ uniformly in compacts of $[0,\infty)$ while $\lim_{b\to\infty} g = -\infty$ uniformly in $\overline{\mathbb{R}}^+$.

iii) $\lim_{q_1\to 0} g = \infty$ uniformly in compacts of $[0,\infty)$, and $g(u_1', b, q_1) \to -bu_1'$ uniformly in $\overline{\mathbb{R}}^+$ when $q_1 \to \infty$.

We proceed now to discuss the main features on existence, multiplicity and asymptotic profiles of solutions to (5.18) when the parameters a, q_0, b and q_1 are varied. The responses of our problem (5.17) to such variations will be separately analyzed in turn. To this objective we are using the notation $f(u')$, $g(u')$ instead of the more cumbersome $f(u', a, q_0)$, $g(u', b, q_1)$ but will keep in mind the hidden dependence on parameters. In most cases, a detailed analysis is omitted for the sake of simplicity.

Behavior with respect to a. For fixed q_0, b, q_1 and small a, equation (5.23) has exactly two solutions $u_{i,a}'$, $i = 1,2$, such that $u_{1,a}' \to 0$ and $u_{2,a}' \to \infty$ as $a \to 0$. On the contrary, no solutions exist if $a \gg 1$. This follows from the fact that the minimum function h_1 satisfies $h_1(u') > g(u')$ and (respectively, $h_1(u') < g(u')$ when u' is small (large). In addition, since f increases with a there exists $a_c > 0$ such that (5.23) has at least two solutions if $a < a_c$ and no solutions if $a > a_c$. See Figure 2 for a bifurcation diagram.

Corresponding to the values $u_1' = u_{i,a}', u_1 = g(u_1')$ in problems (5.19), (5.20), solutions $u_i(x,a)$, $i = 1,2$, to (5.18) exist with the following properties:

i) $u_1(x,a) \to u(x)$ in $C^2[0,L)$ as $a \to 0$ where $u(\cdot)$ is the solution to $u'' = e^{q_1 u}$ with $u(0) = g(0)$ and $u(L) = \infty$.

ii) $u_2(x,a) \to -\infty$ uniformly on compacts of $[0,L)$ as $a \to 0$.

To show the last assertion notice that for every $0 < x < b$ the relation

$$\sqrt{\frac{q_1}{2}}\int_0^{u_2(x)-\zeta} \frac{ds}{\sqrt{(e^{q_1 s}-1)e^{q_1\zeta} + q_1\frac{\zeta'^2}{2}}} = x$$

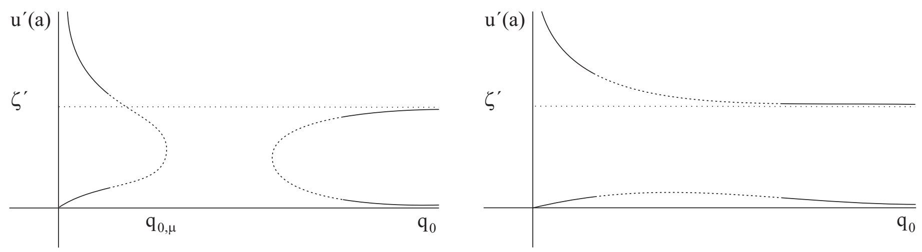

FIGURE 3. Bifurcation diagram for (5.17) plotting $u'(a)$ versus q_0. The figure on the left contains a generic configuration for the case $0 < \mu \leq \max h_2$ and the one on the right corresponds to the case $\mu \geq \mu^*$. As before, continuous drawing in curves means exact number of solutions.

holds true, where $u_2(x) = u_2(x,a)$, $\zeta' = u'_{2,a}$, $\zeta = g(\zeta)$ and $\zeta' \to \infty$, $\zeta \to -\infty$ as $a \to 0$. Now, for every $K \in \mathbb{R}$ it can be proved that

$$\lim_{\zeta' \to \infty} \int_{K-\zeta}^{\infty} \frac{ds}{\sqrt{(e^{q_1 s} - 1)e^{q_1 \zeta} + q_1 \frac{\zeta'^2}{2}}} = 0.$$

This says that $x_K \to b$ as $a \to 0$ where $x = x_K$ in defined through the equation $K = u_2(x,a)$. Thus $u_2(x,a) \to -\infty$ when $a \to 0$, as desired.

Behavior with respect to q_0. Suppose now a, b, q_1 are fixed. The minimum function h_2 becomes smaller than g for large u'. Hence, there exist two solutions $u'_{i,q_0,0}$, $i = 1, 2$, to (5.23) for q_0 lesser than certain amount. In addition $u'_{1,q_0,0} \to 0$, $u'_{2,q_0,0} \to \infty$ as $q_0 \to 0$.

However, for $q_0 = O(1)$ or large q_0 new features arise with respect to the preceding discussion. Fix $\mu = g(0) = g(0, b, q_1)$. Observe that $g(0) = \log(w^{*2}/bq_1^2)$ and so μ can achieve any real value. Two regimes are now possible.

a) $\mu \leq 0$. In this case $\min_{\mathbb{R}} f$ becomes larger than μ for $q_0 \geq q_{0,\mu}$ and thus (5.23) does not admit solutions for $q_0 \geq q_{0,\mu}$. Figure 3 shows a possible bifurcation diagram.
b) $\mu > 0$. Now, equation (5.23) possesses exactly two solutions $u'_{i,q_0,\infty}$, $i = 1, 2$, for $q_0 \gg 1$. Moreover $u'_{1,q_0,\infty} \to 0$ and $u'_{2,q_0,\infty} \to \zeta'$ as $q_0 \to \infty$ where $u' = \zeta'$ is defined by the equation $g(u') = au'$.

On the other hand, the case $\mu > 0$ allows two complementary regimes. A first one in which equation (5.23) supports at least two positive solutions for all $q_0 > 0$. This occurs, for instance, when $\mu \geq \mu^*$ where $\mu^* = \inf\{\mu > 0 : g(u') > h_2(u')$ for all $u' \geq 0\}$. A second regime in which solutions to (5.23) are only possible for $q_0 \ll 1$ or $q_0 \gg 1$ but they cease to exist in an intermediate interval $q_{0,1} \leq q_0 \leq q_{0,2}$. This is precisely the situation if, say, $0 < \mu \leq \max h_2$, $\max h_2 = a^2 \sinh\theta/2e$ (see Figures 3 and 4).

Regarding the limit profiles as $q_0 \to 0$ of the solutions $u = u_i(x, q_0)$ to (5.18) associated to the values $u'_{i,q_0,0}$, the next properties hold:

i) $u_1(x, q_0) \to u_0 + x^2/2$ in $C^2[0,a]$ with $u_0 = g(a) - a^2/2$; $u_1(x, q_0) \to u(x)$ in $C^2[a, L)$ where u is the solution to $u'' = e^{q_1 u}$ in $[a, L)$ subjected to the conditions $u(a) = g(a)$, $u(L) = \infty$.

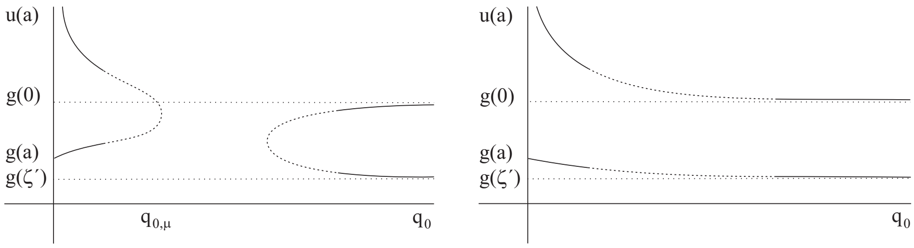

FIGURE 4. Bifurcation diagram for (5.17) now plotting $u(a)$ versus q_0 and again the cases $0 < \mu \le \max h_2$ (left) and $\mu \ge \mu^*$ (right).

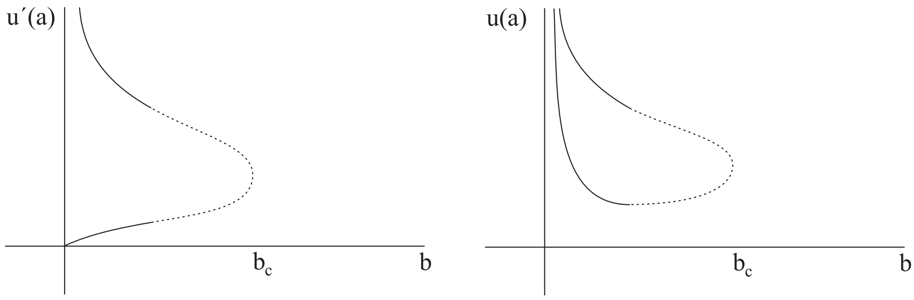

FIGURE 5. Bifurcation diagram for (5.17) plotting separately $u'(a)$ and $u(a)$ versus b (left and right, respectively).

ii) $u_2(x,a) \to -\infty$ uniformly on compacts of $[0,L)$ (see the discussion of ii) in the analysis of the parameter a).

As for the asymptotic behavior of the solutions $u = u_i(x,q_0)$ associated to the roots $u' = u'_{i,q_0,\infty}$ to (5.23), we obtain the following features:

i) $u_1(x,q_0) \to g(0)$ in $C^2[0,a]$, $u_1(x,q_0) \to u(x)$ in $C^2[a,L)$ as $q_0 \to \infty$, $u = u(x)$ solving $u'' = e^{q_1 u}$ in $a \le x < L$ and satisfying $u(a) = g(0)$, $u(L) = \infty$.

ii) $u_2(x,q_0) \to \zeta' x$ in $C[0,a]$, $u_2(x,q_0) \to u(x)$ in $C^2[a,L)$ as $q_0 \to \infty$ where now $u(x)$ solves $u'' = e^{q_1 u}$ in $a \le x < L$ and satisfies the boundary conditions $u(a) = g(\zeta')$, $u(L) = \infty$ (recall that ζ' is defined by the equality $g(\zeta') = a\zeta'$).

This completes the study of the parameter q_0.

Behavior with respect to b. As in the preceding cases, all parameters other than b will be now kept fixed. In particular f preserves its form while g varies in a decreasing way with respect to b. Thus, a positive b_c exists such that (5.23) possesses exactly two solutions $u'_{i,b}$ for $b \ll 1$, at least two solutions for $b < b_c$ and no solutions for $b > b_c$. In addition $u'_{1,b} \to 0$, $u'_{2,b} \to \infty$ as $b \to 0$ (Figure 5).

The limit profiles of the solutions $u = u_i(x,b)$ to (5.18) associated to the values $u'_{i,b}$, $i = 1,2$, are the following.

i) $u_1(x,b) \to \infty$ uniformly in $[0,L)$ as $b \to 0$.

ii) $u_2(x,b) \to \infty$ uniformly on compacts of $(0,L)$ while $\min u_2(\cdot,b) \to -\infty$ as $b \to 0$. More precisely:

$$u_2(x,b) = u_0 + \sqrt{\frac{2}{q_0}}e^{-q_0u_0/2}x - \frac{2}{q_0}\log 2 + o(1),$$

as $b \to 0$, $0 \le x \le a$, where $u_0 = -\frac{2}{q_0}\log\left(\sqrt{\frac{2}{q_0}}u'_{2,b}\right) + o(1)$. In particular $u_0 \to -\infty$ as $b \to 0$.

Behavior with respect to q_1. The behavior of problem (5.18) with respect to q_1 is quite similar the one observed with respect to the parameter q_0. For $q_1 \ll 1$ equation (5.23) has exactly two solutions $u'_{i,q_1,0}$ with $u'_{1,q_1,0} \to 0$ and $u'_{i,q_1,0} \to \infty$ as $q_1 \to 0$. The associated solutions $u_i(x,q_1)$, $i = 1,2$ to (5.18) satisfy:

i) $u_1(x,q_1) \to \infty$ uniformly in $[0,L)$ as $q_1 \to 0$.

ii) $\min u_2(\cdot,q_1) \to -\infty$ while $u_2(x,q_1) \to \infty$uniformly on compacts of $(0,L)$ as $q_1 \to \infty$. More precisely,

$$u_2(x,q_1) = u_0 + \sqrt{\frac{2}{q_0}}e^{-\frac{q_0u_0}{2}}x - \frac{2}{q_0}\log 2 + o(1) \qquad q_1 \to 0,$$

where $0 \le x \le a$ while $u_0 = \frac{2}{q_0}\log q_1 + o(\log q_1)$ as $q_1 \to 0$.

On the other hand, to describe the response of our problem when $q_1 \to \infty$ recall that $g(u',b,q_1) \to -bu'$ uniformly in $u' \ge 0$ (even in $C^1[0,\infty)$) as $q_1 \to \infty$. Thus, two options are possible. A first one when

$$f(u') > -bu'$$

for all $u' \ge 0$. Under such circumstance a $q_{1,c}$ exists such that (5.23) fails to have solutions for $q_1 > q_{1,c}$.

A second behavior occurs when $f(u') < -bu'$ at some $u' > 0$. In such case (5.23) admits at least two solutions for q_1 greater certain positive q_1^*. Moreover, assume for simplicity that the equation

$$f(u') = -bu' \tag{5.30}$$

possesses only two solutions $0 < \zeta'_1 < \zeta'_2$ (as is just the case provided either $a \ll 1$ or $q_0 \ll 1$). Then (5.23) admits exactly two solutions $u'_{i,q_1,\infty}$, $i = 1,2$, with $u'_{i,q_1,\infty} \to \zeta'_i$ as $q_1 \to \infty$. The associated solutions $u_i(x,q_1)$ to (5.18) satisfy the following properties:

a) For $i = 1,2$, $u_i(x,q_1) \to u_i(x)$ in $C^2[0,a]$ as $q_1 \to \infty$ where the $u_i(x)$ solve $u'' = e^{-q_0u}$ in $[0,a]$ together with the conditions $u(0) = u_{0,i}$, $u'(0) = 0$ where

$$u_{0,i} = -\frac{2}{q_0}\log\left(\frac{1}{a}\sqrt{\frac{2}{q_0}}H\left(\frac{aq_0}{2}\zeta'_i\right)\right) \qquad i = 1,2.$$

b) $u_i(x,q_1) \to -b\zeta'_i + \zeta'_i(x-a)$, $i = 1,2$, in $C^2[a,L)$ as $q_1 \to \infty$.

Finally, let us describe the general configuration of the set of solutions to (5.18) for the full range of values of q_1. To this proposal set

$$g_{\inf}(u') = \inf_{q_1>0} g(u',q_1).$$

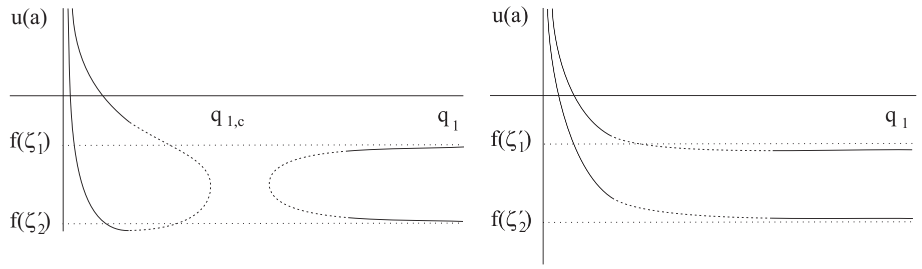

FIGURE 6. Bifurcation diagrams $u(a)$ versus q_1 for problem (5.17).

Notice that $g_{\inf}$ is well defined while $g_{\inf}(0) = -1/q_{\inf}$ with $q_{\inf} = e\omega^{*2}/b^2$. In particular,

$$g_{\inf}(u') \le g(u', q_1)_{q_1=q_{\inf}} \qquad u' \ge 0.$$

Two regimes are possible provided either of the next conditions is satisfied. Firstly, if $f(u') < g_{\inf}(u')$ holds for some $u' > 0$ then equation (5.23) admits at least two solutions for all $q_1 > 0$ (Figure 6 right). Alternatively if, for instance,

$$f(u') > g(u', q_1)_{q_1=q_{\inf}},$$

for all $u' \ge 0$ but $f(u') < -bu'$ at some $u' > 0$ (see the previous analysis of the case $q_1 \gg 1$ then two solutions to (5.23) exist for both $q_1 \ll 1$ and $q_1 \gg 1$ but solutions cease to exist in, at least, a finite interval $q_{1,1} \le q_1 \le q_{1,2}$, $q_{1,1} < q_{1,2}$ (Figure 6 left).

To summarize the conclusions driven from the analysis of problem (5.17) we see on one hand that existence of a solution if confirmed –according to Theorem 8– whenever either of the quantities q_0, q_1 and L (corresponding to q_-, q_+ and diam(Ω) or L_Ω in problem (1.1)) are small enough. We have also shown that existence might be lost if either of the quantities q_0, q_1, L becomes large enough.

Observe also that a second solution to (5.17) exists when one of the parameters q_0, q_1, a or b becomes small. In addition, a bifurcation from infinity appears in all cases.

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References

[1] C. Bandle, M. Marcus, *Sur les solutions maximales de problèmes elliptiques non linéaires: bornes isopérimetriques et comportement asymptotique*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 91–93.

[2] C. Bandle, M. Marcus, *'Large' solutions of semilinear elliptic equations: Existence, uniqueness and asymptotic behaviour*, J. Anal. Math. **58** (1992), 9–24.

[3] C. Bandle, M. Marcus, *On second order effects in the boundary behaviour of large solutions of semilinear elliptic problems*, Differential Integral Equations **11** (1) (1998), 23–34.

[4] L. Bieberbach, $\Delta u = e^u$ *und die automorphen Funktionen*, Math. Ann. **77** (1916), 173–212.

[5] M. G. Crandall, P. H. Rabinowitz, *Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems*, Arch. Rational Mech. Anal. **58** (3) (1975), 207–218.

[6] M. Chuaqui, C. Cortázar, M. Elgueta, C. Flores, J. García-Melián, R. Letelier, *On an elliptic problem with boundary blow-up and a singular weight: the radial case*, Proc. Roy. Soc. Edinburgh **133A** (2003), 1283–1297.

[7] M. Chuaqui, C. Cortázar, M. Elgueta, J. García-Melián, *Uniqueness and boundary behaviour of large solutions to elliptic problems with singular weights*, Comm. Pure Appl. Anal. **3** (2004), 653–662.

[8] M. Del Pino, R. Letelier, *The influence of domain geometry in boundary blow-up elliptic problems*, Nonlinear Anal. **48** (6) (2002), 897–904.

[9] G. Díaz, R. Letelier, *Explosive solutions of quasilinear elliptic equations: Existence and uniqueness*, Nonlinear Anal. **20** (1993), 97–125.

[10] Y. Du, Q. Huang, *Blow-up solutions for a class of semilinear elliptic and parabolic equations*, SIAM J. Math. Anal. **31** (1999), 1–18.

[11] S. Dumont, L. Dupaigne, O. Goubet, V. Rădulescu, *Back to the Keller-Osserman condition for boundary blow-up solutions*, Adv. Nonl. Stud. **7** (2007), 271–298.

[12] J. García-Melián, *Nondegeneracy and uniqueness for boundary blow-up elliptic problems*, J. Diff. Eqns. **223** (2006), 208–227.

[13] J. García-Melián, R. Letelier-Albornoz, J. Sabina de Lis, *Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up*, Proc. Amer. Math. Soc. **129** (12) (2001), 3593–3602.

[14] J. García-Melián, J. D. Rossi, J. Sabina de Lis, *Large solutions for the Laplacian with a power nonlinearity given by a variable exponent.* To appear in Ann. Inst. H. Poincaré (C) Analyse non linéaire.

[15] I. M. Gel'fand, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Transl. **29** (2) (1963), 295–381

[16] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer–Verlag, 1983.

[17] D. D. Joseph, T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. **49** (1972/73), 241–269.

[18] J. B. Keller, *On solutions of $\Delta u = f(u)$*, Comm. Pure Appl. Math. **10** (1957), 503–510.

[19] V. A. Kondrat'ev, V. A. Nikishkin, *Asymptotics, near the boundary, of a solution of a singular boundary value problem for a semilinear elliptic equation*, Differential Equations **26** (1990), 345–348.

[20] A. C. Lazer, P. J. McKenna, *On a problem of Bieberbach and Rademacher*, Nonlinear Anal. **21** (1993), 327–335.

[21] A. C. Lazer, P. J. Mckenna, *Asymptotic behaviour of solutions of boundary blow-up problems*, Differential Integral Equations **7** (1994), 1001–1019.

[22] C. Loewner, L. Nirenberg, *Partial differential equations invariant under conformal of projective transformations*, Contributions to Analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 1974, p. 245–272.

[23] J. López-Gómez, *Varying stoichometric exponents I: Classical steady states and metasolutions*, Adv. Nonl. Stud. **3** (2003), 327-354.

[24] M. Marcus, L. Véron, *Uniqueness and asymptotic behaviour of solutions with boundary blow-up for a class of nonlinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (2) (1997), 237–274.

[25] R. Osserman, *On the inequality $\Delta u \geq f(u)$*, Pacific J. Math. **7** (1957), 1641–1647.

[26] H. Rademacher, *Einige besondere Probleme partieller Differentialgleichungen*, in: P. Frank, R. von Mises (Eds.) Die Differential- und Integralgleichungen der Mechanik und Physik I, 2nd. ed., Rosenberg, New York (1943), 838–845.

[27] V. Rădulescu, *Singular phenomena in nonlinear elliptic problems: from boundary blow-up solutions to equations with singular nonlinearities*, in "Handbook of Differential Equations: Stationary Partial Differential Equations", Vol. **4** (Michel Chipot, Editor) (2007), 483–591.

[28] L. Véron, *Semilinear elliptic equations with uniform blowup on the boundary*, J. Anal. Math. **59** (1992), 231–250.

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