LOWER AND UPPER BOUNDS FOR THE FIRST EIGENVALUE OF NONLOCAL DIFFUSION PROBLEMS IN THE WHOLE SPACE

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Abstract. We find lower and upper bounds for the first eigenvalue of a nonlocal diffusion operator of the form $T(u) = -\int_{\mathbb{R}^d} K(x,y)(u(y) - u(x))\,dy$. Here we consider a kernel $K(x,y) = \psi(y-a(x)) + \psi(x-a(y))$ where ψ is a bounded, nonnegative function supported in the unit ball and a means a diffeomorphism on $\mathbb{R}^d$. A simple example being a linear function $a(x) = Ax$. The upper and lower bounds that we obtain are given in terms of the Jacobian of a and the integral of ψ. Indeed, in the linear case $a(x) = Ax$ we obtain an explicit expression for the first eigenvalue in the whole $\mathbb{R}^d$ and it is positive when the the determinant of the matrix A is different from one. As an application of our results, we observe that, when the first eigenvalue is positive, there is an exponential decay for the solutions to the associated evolution problem. As a tool to obtain the result, we also study the behaviour of the principal eigenvalue of the nonlocal Dirichlet problem in the ball B_R and prove that it converges to the first eigenvalue in the whole space as $R \to \infty$.

1. Introduction

Nonlocal problems have been recently widely used to model diffusion processes. When $u(x,t)$ is interpreted as the density of a single population at the point x at time t and $J(x-y)$ is the probability of "jumping" from location y to location x, the convolution $(J * u)(x) = \int_{\mathbb{R}^d} J(y-x)u(y,t)\,dy$ is the rate at which individuals arrive to position x from all other positions, while $-\int_{\mathbb{R}^d} J(y-x)u(x,t)\,dy$ is the rate at which they leave position x to reach any other position. If in addition no external source is present, we obtain that u is a solution to the following evolution problem

$$u_t(x,t) = \int_{\mathbb{R}^d} J(y-x)(u(y,t) - u(x,t))\,dy. \tag{1.1}$$

This equation is understood to hold in a bounded domain, this is, for $x \in \Omega$ and has to be complemented with a "boundary" condition. For example, $u = 0$ in $\mathbb{R}^d \setminus \Omega$ which means that the habitat Ω is surrounded by a hostile environment (see [16] and [15] for a general nonlocal vector calculus). Problem (1.1) and its stationary version have been considered recently in connection with real applications (for example to peridynamics, a recent model for elasticity), we quote for instance [1], [11], [12], [5], [6], [13], [14], [8], [7], [24], [25], [26] and the recent book [3]. See also [21] for the appearance of convective terms, [2] for a problem with nonlinear nonlocal diffusion and [9], [10] for other features in related nonlocal problems.

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On the other hand, it is well known that eigenvalue problems are a fundamental tool to deal with local problems. In particular, the so-called principal eigenvalue of the Laplacian with Dirichlet boundary conditions,

$$\begin{cases} -\Delta v(x) = \sigma v(x), & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

plays an important role, since it gives the exponential decay of solutions to the associated parabolic problem, $u_t = \Delta u$ with $u|_{\partial\Omega} = 0$. The properties of the principal eigenvalue of (1.2) are well-known, see [19].

For the nonlocal problem, in [18] the authors consider the "Dirichlet" eigenvalue problem for a nonlocal operator in a smooth bounded domain Ω, that is,

$$\begin{cases} (J * u)(x) - u(x) = -\lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^d \setminus \Omega. \end{cases} \tag{1.3}$$

They show that the first eigenvalue has associated a positive eigenfunction and that the eigenvalue goes to zero as the domain is expanded, i.e., $\lambda_1(k\Omega) \to 0$ as $k \to \infty$. In addition, it is proved in [7] that solutions to (1.1) in the whole $\mathbb{R}^d$ decay in the L^2-norm as $t^{-d/4}$. Therefore the first eigenvalue in the whole space $\mathbb{R}^d$ is zero for the convolution case. When we face a convolution one of the main tools is the use of the Fourier transform, see [7].

For more general kernels, in [22] energy methods where applied to obtain decay estimates for solutions to nonlocal evolution equations whose kernel is not given by a convolution, that is, equations of the form

$$u_t(x,t) = \int_{\mathbb{R}^d} K(x,y)(u(y,t) - u(x,t))\, dy \tag{1.4}$$

with $K(x,y)$ a symmetric nonnegative kernel. The obtained decay estimates are of polinomial type, more precisely, $\|u(\cdot,t)\|_{L^2(\mathbb{R}^d)} \le Ct^{-d/4}$. We remark that this decay bound need not be optimal, in fact, in [22] there is a particular example of a kernel K that give exponential decay in $L^2(\mathbb{R})$. The exponential decay of solutions suggests that the associated first eigenvalue is positive.

Our main goal in the present work is to study properties of the principal eigenvalue of nonlocal diffusion operators when the associated kernel is not of convolution type. Some preliminary properties are already known, as existence, uniqueness and a variational characterization. To go further, we need to assume some structure for the kernel. Let us consider a function ψ nonnegative, bounded and supported in the unit ball in $\mathbb{R}^d$. We associate with this function a kernel of the form

$$K(x,y) = \psi(y - a(x)) + \psi(x - a(y)) \tag{1.5}$$

where $a(x)$ is a diffeomorphism on $\mathbb{R}^d$. Note that K is symmetric and that the convolution type kernels also take the form (1.5) (just put $a(x) = x$). For this kernels let us look for the first eigenvalue of the associated nonlocal operator, that is,

$$-\int_{\mathbb{R}^d} K(x,y)(u(y) - u(x))\, dy = \lambda_1 u(x). \tag{1.6}$$

Some known results (that we state in the next section for completeness) read as follows: For any bounded domain Ω there exists a principal eigenvalue $\lambda_1(\Omega)$ of problem (1.6) with $u \equiv 0$ in $\mathbb{R}^d \setminus \Omega$. The corresponding non-negative eigenfunction $\phi_1(x)$ is strictly positive in Ω. Moreover, the first eigenvalue is given by

$$\lambda_1(\Omega) = \inf_{u \in L^2(\Omega)} \frac{\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(\tilde{u}(x) - \tilde{u}(y))^2 dxdy}{\displaystyle\int_\Omega u^2(x)dx}. \tag{1.7}$$

Here we have denoted by $\tilde{u}$ the extension by zero of u,

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^d \setminus \Omega. \end{cases}$$

We will use this notation trough the whole paper. When we deal with the whole space we have

$$\lambda_1(\mathbb{R}^d) = \inf_{u \in L^2(\mathbb{R}^d)} \frac{\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(u(x) - u(y))^2 dxdy}{\displaystyle\int_{\mathbb{R}^d} u^2(x)dx}. \tag{1.8}$$

The main results of this paper are the following:

Theorem 1.1. *Let Ω be a bounded domain with $0 \in \Omega$ and consider its dilations by a real factor R, $R\Omega = \{Rx \,:\, x \in \Omega\}$. Then*

$$\lambda_1(\mathbb{R}^d) = \lim_{R \to \infty} \lambda_1(R\Omega). \tag{1.9}$$

Now, we state our result concerning lower bounds for the first eigenvalue.

Theorem 1.2. *Assume that the kernel is given by (1.5) and that the Jacobian of a^{-1}, $J_{a^{-1}}$, verifies*

$$\sup_{x \in \mathbb{R}^d} |J_{a^{-1}}(x)| = M < 1 \qquad \text{or} \qquad \inf_{x \in \mathbb{R}^d} |J_{a^{-1}}(x)| = m > 1.$$

Then

$$\lambda_1(\mathbb{R}^d) \geq 2(1 - M^{1/2})^2 \left(\int_{\mathbb{R}^d} \psi(x)dx \right),$$

in the first case and

$$\lambda_1(\mathbb{R}^d) \geq 2(m^{1/2} - 1)^2 \left(\int_{\mathbb{R}^d} \psi(x)dx \right),$$

in the second case.

Concerning upper bounds we have the following less general result.

Theorem 1.3. *Let a be a diffeomorphism homogeneous of degree one, that is, $a(Rx) = Ra(x)$. Assume that the kernel is given by (1.5). Then*

$$\lambda_1(\mathbb{R}^d) \leq 2 \left(\int_{\mathbb{R}^d} \psi(x)dx \right) \inf_{\|\phi\|_{L^2(B_1)}=1} \int_{\mathbb{R}^d} \big(\phi(x) - \phi(a(x))\big)^2 dx, \tag{1.10}$$

where the infimum is taken over all functions ϕ supported in the unit ball of $\mathbb{R}^d$.

Remark 1.1. *Since we can consider $\phi \geq 0$, we get*

$$\begin{aligned}\int_{\mathbb{R}^d} \big(\phi(x) - \phi(a(x))\big)^2 dx &= \int_{\mathbb{R}^d} \phi^2(x)dx + \int_{\mathbb{R}^d} \phi^2(a(x))dx - 2\int_{\mathbb{R}^d} \phi(x)\phi(a(x))dx \\ &\leq \int_{\mathbb{R}^d} \phi^2(x)dx + \int_{\mathbb{R}^d} \phi^2(a(x))dx.\end{aligned}$$

Hence, from (1.10) *we immediately obtain the following bound*

$$\lambda_1(\mathbb{R}^d) \leq 2\big(1 + \sup_{x\in\mathbb{R}^d} |J_{a^{-1}}(x)|\big) \left(\int_{\mathbb{R}^d} \psi(x)dx\right). \tag{1.11}$$

For invertible linear maps a on $\mathbb{R}^d$ we obtain the following sharp result.

Theorem 1.4. *Let K be given by* (1.5) *with an invertible linear map $a(x) = Ax$. Then*

$$\lambda_1(\mathbb{R}^d) = \lim_{R\to\infty} \lambda_1(B_R) = 2(1 - |\det(A)|^{-1/2})^2 \left(\int_{\mathbb{R}^d} \psi(x)dx\right). \tag{1.12}$$

Remark 1.2. *Note that for a linear function a the bound* (1.11) *is not sharp. However, Theorems 1.2 and 1.3 provide lower and upper bounds for $\lambda_1(\mathbb{R}^d)$ when $M < 1$ or $m > 1$ that depend linearly on $\int \psi$ in terms of the jacobian of the diffeomorphism a^{-1}.*

As an immediate application of our results, we observe that, when the first eigenvalue is positive, we have exponential decay for the solutions to the associated evolution problem in $\mathbb{R}^d$. In fact, let us consider,

$$u_t(x,t) = \int_{\mathbb{R}^d} K(x,y)(u(y,t) - u(x,t))\, dy,$$

with an initial condition $u(x,0) = u_0(x) \in L^2(\mathbb{R}^d)$. Multiply by $u(x,t)$ and integrate to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} u^2(x,t)\, dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(u(y,t) - u(x,t))u(x,t)\, dy\, dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(u(y,t) - u(x,t))^2\, dy\, dx \\ &\leq -\frac{1}{2}\lambda_1 \int_{\mathbb{R}^d} u^2(x,t)\, dx.\end{aligned}$$

Thus, an exponential decay of u in L^2-norm follows

$$\int_{\mathbb{R}^d} u^2(x,t)\, dx \leq \left(\int_{\mathbb{R}^d} u^2(x,0)\, dx\right) \cdot e^{-\lambda_1 t}.$$

The paper is organized as follows: in Section 2 we collect some preliminary results and prove Theorem 1.1; while in Section 3 we collect the proofs of the lower and upper bounds for the first eigenvalue; we prove Theorem 1.2, Theorem 1.3 and Theorem 1.4.

2. Properties of the first eigenvalue. Proof of Theorem 1.1

First, let us state some known properties of the first eigenvalue of our nonlocal operator.

Theorem 2.1. *For any bounded domain Ω there exists a principal eigenvalue $\lambda_1(\Omega)$ of problem* (1.6)*, i.e. the corresponding non-negative eigenfunction $\phi_1(x)$ is strictly positive in Ω.*

Proof. It follows from [23]. $\square$

Theorem 2.2. *The first eigenvalue of problem* (1.6) *satisfies*

$$\lambda_1(\Omega) = \inf_{u \in L^2(\Omega)} \frac{\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(\tilde{u}(x) - \tilde{u}(y))^2 \, dx \, dy}{\displaystyle\int_{\Omega} u^2(x) \, dx}. \tag{2.1}$$

Proof. See [18]. $\square$

Now, to simplify the presentation, we prove Theorem 1.1 in the special case of balls B_R that are centered at the origin with radius R (we will use this notation in the rest of the paper) and next we deduce from this fact the general case, Ω a bounded domain.

Lemma 2.1. *Let $\lambda_1(\mathbb{R}^d)$ be defined by* (1.8)*. Then*

$$\lambda_1(\mathbb{R}^d) = \lim_{R\to\infty} \lambda_1(B_R). \tag{2.2}$$

Proof. First of all, observe that for any $R_1 \le R_2$ we have $B_{R_1} \subset B_{R_2}$ and then

$$\lambda_1(B_{R_1}) \ge \lambda_1(B_{R_2}) > 0.$$

Then we deduce that there exists the limit

$$\lim_{R\to\infty} \lambda_1(B_R) \ge 0.$$

Step I. Let us choose $u \in L^2(B_R)$. By the definition of $\lambda_1(\mathbb{R}^d)$ we get

$$\frac{\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(\tilde{u}(x) - \tilde{u}(y))^2 dxdy}{\displaystyle\int_{B_R} u^2(x)dx} = \frac{\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(\tilde{u}(x) - \tilde{u}(y))^2 dxdy}{\displaystyle\int_{\mathbb{R}^d} \tilde{u}^2(x)dx} \ge \lambda_1(\mathbb{R}^d).$$

Taking the infimum in the right hand side over all functions $u \in L^2(B_R)$ we obtain that for any $R > 0$

$$\lambda_1(B_R) \ge \lambda_1(\mathbb{R}^d). \tag{2.3}$$

Step II. Let be $\varepsilon > 0$. Then there exists $u_\varepsilon \in L^2(\mathbb{R})$ such that

$$\lambda_1(\mathbb{R}^d) + \varepsilon \ge \frac{\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(u_\varepsilon(x) - u_\varepsilon(y))^2 dxdy}{\displaystyle\int_{\mathbb{R}^d} u_\varepsilon^2(x)dx}. \tag{2.4}$$

We choose $u_{\varepsilon,R}$ defined by

$$u_{\varepsilon,R}(x) = u_\varepsilon(x)\chi_{B_R}(x).$$

We claim that

$$\int_{B_R} u_{\varepsilon,R}^2(x)dx \to \int_{\mathbb{R}^d} u_\varepsilon^2(x)dx \tag{2.5}$$

and

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(u_{\varepsilon,R}(x)-u_{\varepsilon,R}(y))^2\,dx\,dy \to \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(u_\varepsilon(x)-u_\varepsilon(y))^2\,dx\,dy. \tag{2.6}$$

Assume these claims for the momment; using that $u_{\varepsilon,R}$ vanishes outside the ball B_R and the definition of $\lambda_1(B_R)$ we get

$$\frac{\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(u_{\varepsilon,R}(x)-u_{\varepsilon,R}(y))^2dxdy}{\displaystyle\int_{B_R} u_{\varepsilon,R}^2(x)dx} \ge \lambda_1(B_R).$$

Using claims (2.5) and (2.6) and taking $R\to\infty$ we obtain

$$\frac{\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(u_{\varepsilon}(x)-u_{\varepsilon}(y))^2dxdy}{\displaystyle\int_{\mathbb{R}^d} u_{\varepsilon}^2(x)dx} \ge \lim_{R\to\infty}\lambda_1(B_R).$$

By (2.4), for any $\varepsilon>0$, we have $\lambda_1(\mathbb{R}^d)+\varepsilon \ge \lim_{R\to\infty}\lambda_1(B_R)$. Thus

$$\lambda_1(\mathbb{R}^d) \ge \lim_{R\to\infty}\lambda_1(B_R).$$

Using now (2.3) the proof of (2.2) is finished.

It remains to prove claims (2.5) and (2.6). The first claim follows from Lebesgue's dominated convergence theorem, since $|u_{\varepsilon,R}|\le |u_\varepsilon|\in L^2(\mathbb{R}^d)$. For the second one we have that

$$u_{\varepsilon,R}(x)-u_{\varepsilon,R}(y)\to u_\varepsilon(x)-u_\varepsilon(y), \quad \text{as} \quad R\to\infty$$

and

$$K(x,y)|u_{\varepsilon,R}(x)-u_{\varepsilon,R}(y)|^2 \le 2K(x,y)(u_{\varepsilon,R}^2(x)+u_{\varepsilon,R}^2(y)) \le 2K(x,y)(u_\varepsilon^2(x)+u_\varepsilon^2(y)). \tag{2.7}$$

We show that under the assumptions on K the right hand side in (2.7) belongs to $L^1(\mathbb{R}^d\times\mathbb{R}^d)$:

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(u_\varepsilon^2(x)+u_\varepsilon^2(y))dxdy &= 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)u_\varepsilon^2(x)dxdy\\ &= 2\int_{\mathbb{R}^d} u_\varepsilon^2(x)dx\int_{\mathbb{R}^d} K(x,y)dy \le 2\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}K(x,y)dy\int_{\mathbb{R}^d} u_\varepsilon^2(x)dx\\ &\le 2\int_{\mathbb{R}^d}\psi(x)(1+|J_{a^{-1}}(x)|)dx\int_{\mathbb{R}^d} u_\varepsilon^2(x)d \le C\int_{\mathbb{R}^d} u_\varepsilon^2(x)dx.\end{aligned}$$

Applying now Lebesgue's convergence theorem we obtain (2.6). □

When we consider dilations of a domain Ω with $0\in\Omega$ we get the same limit. This provides a proof of Theorem 1.1.

Proof of Theorem 1.1. Let us consider $B_{r_1} \subset \Omega \subset B_{r_2}$ then

$$\lambda_1(RB_{r_1}) \geq \lambda_1(R\Omega) \geq \lambda_1(RB_{r_2}),$$

and we just observe that

$$\lim_{R\to\infty} \lambda_1(RB_{r_1}) = \lim_{R\to\infty} \lambda_1(RB_{r_1}) = \lambda_1(\mathbb{R}^d).$$

This ends the proof. $\square$

3. Proofs of lower and upper bounds for the first eigenvalue

In this section we obtain estimates on $\lambda_1(\mathbb{R}^d)$ defined by (1.8). First we prove Theorem 1.2.

Proof of Theorem 1.2. First of all, let us perform the following computations: let θ be a positive constant which will be fixed latter. Using the elementary inequality

$$(b-c)^2 = b^2 + c^2 - 2bc \geq b^2 + c^2 - \theta b^2 - \frac{1}{\theta}c^2 = (1-\theta)(b^2 - \frac{c^2}{\theta})$$

we get

$$\begin{aligned}
\iint_{\mathbb{R}^{2d}} \psi(y-a(x))(u(x)-u(y))^2 dxdy &\geq (1-\theta) \iint_{\mathbb{R}^{2d}} \psi(y-a(x))\Big(u^2(x) - \frac{u^2(y)}{\theta}\Big) dxdy \\
&= (1-\theta)\Big(\int_{\mathbb{R}^d} u^2(x)dx \int_{\mathbb{R}^d} \psi(y)dy - \frac{1}{\theta}\int_{\mathbb{R}^d} u^2(y) \int_{\mathbb{R}^d} \psi(y-a(x))dxdy\Big) \\
&= (1-\theta) \int_{\mathbb{R}^d} u^2(x)\Big(\int_{\mathbb{R}^d} \psi(y)dy - \frac{1}{\theta}\int_{\mathbb{R}^d} \psi(x-a(y))dy\Big)dx \\
&= (1-\theta) \int_{\mathbb{R}} u^2(x)\Big(\int_{\mathbb{R}^d} \psi(y)dy - \frac{1}{\theta}\int_{\mathbb{R}^d} \psi(x-y)||J_{a^{-1}}(y)||dy\Big)dx \\
&= \frac{1-\theta}{\theta} \int_{\mathbb{R}^d} \psi(y)dy \int_{\mathbb{R}^d} u^2(x)\Big(\theta - \frac{(\psi * |J_{a^{-1}}|)(x)}{\int_{\mathbb{R}^d} \psi(y)dy}\Big)dx.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{2}\iint_{\mathbb{R}^{2d}} K(x,y)(u(x)-u(y))^2\, dx\, dy \\
&\quad \geq \begin{cases} \dfrac{1-\theta}{\theta}\left(\displaystyle\int_{\mathbb{R}^d} \psi(y)dy\right)\displaystyle\int_{\mathbb{R}^d} u^2(x)\Big(\theta - \dfrac{\sup_{x\in\mathbb{R}^d} \psi * |J_{a^{-1}}|}{\int_{\mathbb{R}^d} \psi(y)dy}\Big)dx, & \theta < 1, \\[2ex] \dfrac{1-\theta}{\theta}\left(\displaystyle\int_{\mathbb{R}^d} \psi(y)\, dy\right)\displaystyle\int_{\mathbb{R}^d} u^2(x)\Big(\theta - \dfrac{\inf \psi * |J_{a^{-1}}|}{\int_{\mathbb{R}^d} \psi(y)\, dy}\Big)\, dx, & \theta > 1. \end{cases} \\
&\quad \geq \begin{cases} \dfrac{1-\theta}{\theta}\left(\displaystyle\int_{\mathbb{R}^d} \psi(y)dy\right)\displaystyle\int_{\mathbb{R}^d} u^2(x)\Big(\theta - M\Big)dx, & \theta < 1, \\[2ex] \dfrac{\theta-1}{\theta}\left(\displaystyle\int_{\mathbb{R}^d} \psi(y)\, dy\right)\displaystyle\int_{\mathbb{R}^d} u^2(x)\Big(m - \theta\Big)\, dx, & \theta > 1. \end{cases}
\end{aligned}$$

In the first case we choose $\theta = M^{1/2}$. In the second case we get $\theta = m^{1/2}$.

Therefore, the statement holds from the definition of $\lambda_1(\mathbb{R}^d)$. □

In the following we deal with upper bounds for the first eigenvalue.

First, let us state a lemma with an upper bound for $\lambda_1(B_R)$ in terms of the radius of the ball, R, and the function ψ. Note that here we are assuming that a is $1-$homogeneous.

Lemma 3.1. *Let $K(x,y) = \psi(y - a(x)) + \psi(x - a(y))$ with an $1-$homogeneous map a. For every $\delta > 0$ there exists a constant $C(\delta)$ such that the following*

$$\lambda_1(B_R) \le (2+\delta) \int_{\mathbb{R}^d} \psi(z)dz \int_{\mathbb{R}^d} \Big(\phi(x) - \phi(a(x))\Big)^2 dx \\ + \frac{C(\delta)}{R^2} \int_{\mathbb{R}^d} \psi(z)|z|^2 dz \int_{\mathbb{R}^d} |\nabla \phi(x)|^2 dx \sup_{y \in B_{1+1/R}} |J_{a^{-1}}(y)|$$

holds for any function ϕ supported in the unit ball with $\|\phi\|_{L^2(B_1)} = 1$ and all $R > 0$.

Proof. Let ϕ be a smooth function supported in the unit ball with $\int_{B_1} \phi^2(x)dx = 1$. Taking as a test function $\phi_R(x) = \phi(x/R)$ in the variational characterization (1.7), we obtain

$$\lambda_1(B_R) \le \frac{\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(\phi_R(x) - \phi_R(y))^2 dxdy}{\displaystyle\int_{B_R} \phi_R^2(x)dx} = \frac{1}{R^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)\Big(\phi(\frac{x}{R}) - \phi(\frac{y}{R})\Big)^2 dxdy \\ = R^d \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(Rx,Ry)\Big(\phi(x) - \phi(y)\Big)^2 dxdy.$$

Using that $K(x,y) = \psi(y-a(x)) + \psi(x-a(y))$ and that the right hand side in the last term is symmetric we get

$$\begin{aligned}
\lambda_1(B_R) &\le 2R^d \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(Ry - a(Rx))\Big(\phi(x) - \phi(y)\Big)^2 dxdy \\
&= 2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(z)\Big(\phi(x) - \phi(\frac{z + a(Rx)}{R})\Big)^2 dxdz \\
&\le (2+\delta) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(z)\Big(\phi(x) - \phi(\frac{a(Rx)}{R})\Big)^2 dxdz \\
&\quad + C(\delta) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(z)\Big(\phi(\frac{a(Rx)}{R}) - \phi(\frac{z + a(Rx)}{R}\Big)^2 dxdz \\
&\le (2+\delta) \int_{\mathbb{R}^d} \psi(z)dz \int_{\mathbb{R}^d} \Big(\phi(x) - \phi(a(x))\Big)^2 dx \\
&\quad + \frac{C(\delta)}{R^2} \int_{|z|\le 1} \psi(z) \int_{\mathbb{R}^d} \Big(\int_0^1 \nabla\phi(a(x) + s\frac{z}{R}) \cdot z ds\Big)^2 dxdz.
\end{aligned}$$

Observe that we have

$$
\begin{aligned}
\int_{|z|\le 1} \psi(z) \int_{\mathbb{R}^d} \Big(\int_0^1 \nabla\phi(a(x) + s\frac{z}{R}) \cdot z ds\Big)^2 dxdz \\
&\le \int_{\mathbb{R}^d} \psi(z)|z|^2 \int_{\mathbb{R}^d} \int_0^1 |\nabla\phi\big(a(x) + \frac{sz}{R}\big)|^2 dsdxdz \\
&\le \int_{\mathbb{R}^d} \psi(z)|z|^2 \int_0^1 \int_{\mathbb{R}^d} |\nabla\phi\big(x + \frac{sz}{R}\big)|^2 |J_{a^{-1}}(x)| dxdsdz \\
&\le \int_{\mathbb{R}^d} \psi(z)|z|^2 \int_0^1 \int_{|x|\le 1} |\nabla\phi\big(x\big)|^2 |J_{a^{-1}}(x - \frac{sz}{R})| dxdsdz \\
&\le \int_{\mathbb{R}^d} \psi(z)|z|^2 dz \int_{\mathbb{R}^d} |\nabla\phi(x)|^2 dx \sup_{y\in B_{1+1/R}} |J_{a^{-1}}(y)|.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\lambda_1(B_R) \le (2+\delta) \int_{\mathbb{R}^d} \psi(z)dz \int_{\mathbb{R}^d} \Big(\phi(x) - \phi(a(x))\Big)^2 dx \\
+ \frac{C(\delta)}{R^2} \int_{\mathbb{R}^d} \psi(z)|z|^2 dz \int_{\mathbb{R}^d} |\nabla\phi(x)|^2 dx \sup_{y\in B_{1+1/R}} |J_{a^{-1}}(y)|,
\end{aligned}
$$

as we wanted to show. □

Now we are ready to prove our general upper bound.

Proof of Theorem 1.3. Let us fix $\delta > 0$ and a function ϕ supported in the unit ball with $\|\phi\|_{L^2(\mathbb{R}^d)} = 1$. We apply Lemma 3.1 and let $R \to \infty$. Then

$$
\lambda_1(\mathbb{R}^d) \le (2+\delta) \int_{\mathbb{R}^d} \psi(z)dz \int_{\mathbb{R}^d} \Big(\phi(x) - \phi(a(x))\Big)^2 dx
$$

Letting $\delta \to 0$ we obtain the desired result. □

Now, we deal with the case in which a is an invertible linear map on $\mathbb{R}^d$ of the form $a(x) = Ax$. To clarify the presentation we first treat the case of a diagonal matrix A. We then extend the result to the case of a general matrix. The proof in the first case is simpler while the proof of the general case is more involved and requires different techniques.

Lemma 3.2. *Let $a(x) = Ax$ be an invertible linear map that in addition is assumed to be diagonal, that is, $a(x) = (\alpha_1 x_1, \dots, \alpha_d x_d)^T$ with $\alpha_i \in \mathbb{R}$. Then, if we consider functions $\phi \in L^2(\mathbb{R}^d)$ supported in the unit ball, we have*

$$
\inf_{\|\phi\|_{L^2(B_1)}=1} \int_{\mathbb{R}^d} \Big(\phi(x) - \phi(a(x))\Big)^2 dx = (1 - |\det(A)|^{-1/2})^2. \tag{3.1}
$$

Proof. For any function ϕ as in the statement we have

$$\begin{aligned}\int_{\mathbb{R}^d} \Big(\phi(x)-\phi(a(x))\Big)^2 &= 1+|\det(A)|^{-1} - 2\int_{\mathbb{R}^d} \phi(x)\phi(a(x)) \\ &\geq 1+|\det(A)|^{-1} - 2\left(\int_{\mathbb{R}^d}\phi^2(x)dx\right)^{1/2}\left(\int_{\mathbb{R}^d}\phi^2(a(x))dx\right)^{1/2} \\ &= 1+|\det(A)|^{-1} - 2|\det(A)|^{-1/2} = (1-|\det(A)|^{-1/2})^2.\end{aligned}$$

In order to prove (3.1) we need to show the existence of a sequence of functions ϕ as in the statement such that

$$\frac{\displaystyle\int_{\mathbb{R}^d} \phi(x)\phi(a(x))dx}{|\det(A)|^{-1/2}\displaystyle\int_{\mathbb{R}^d}\phi^2(x)dx} \to 1.$$

Choosing ϕ of the form (we use a standard separation of variables here)

$$\phi(x) = \prod_{i=1}^d \phi_i(x_i)\chi_{B_\varepsilon}(x), \quad x=(x_1,\dots,x_d),$$

with ε small enough such that ϕ to be supported in the unit ball we reduce the problem to the one dimensional case: $a(x)=\alpha x$ and construct a sequence of functions ϕ_σ supported in $[-\varepsilon,\varepsilon]$ such that

$$\frac{\displaystyle\int_{\mathbb{R}} \phi_\sigma(x)\phi_\sigma(a(x)))dx}{\alpha^{-1/2}\displaystyle\int_{\mathbb{R}}\phi_\sigma^2(x)dx} \to 1.$$

We choose

$$\phi_\sigma(x) = \frac{1}{|x|^\sigma}\chi_{(0,\varepsilon)}(x), \qquad \text{with } \sigma<1/2.$$

Then

$$\int_{\mathbb{R}}\phi_\sigma^2(x)dx = \int_0^\varepsilon \frac{1}{|x|^{2\sigma}} = \frac{\varepsilon^{1-2\sigma}}{1-2\sigma}$$

and

$$\begin{aligned}\int_{\mathbb{R}}\phi_\sigma(x)\phi_\sigma(a(x))dx &= \int_0^{\min\{\varepsilon,\varepsilon/\alpha\}} \frac{1}{|x|^\sigma}\frac{1}{|\alpha x|^\sigma} \\ &= \alpha^{-\sigma}\int_0^{\min\{\varepsilon,\varepsilon/\alpha\}}\frac{1}{|x|^{2\sigma}} = \frac{\alpha^{-\sigma}\min\{\varepsilon,\varepsilon/\alpha\}^{1-2\sigma}}{1-2\sigma}.\end{aligned}$$

Thus

$$\frac{\displaystyle\int_{\mathbb{R}} \phi_\sigma(x)\phi_\sigma(a(x)))dx}{\alpha^{-1/2}\displaystyle\int_{\mathbb{R}}\phi_\sigma^2(x)dx} = \frac{\alpha^{-\sigma}\min\{\varepsilon,\varepsilon/\alpha\}^{1-2\sigma}}{\alpha^{-1/2}\varepsilon^{1-2\sigma}} \to 1, \qquad \text{as } \sigma\to 1/2.$$

This ends the proof. □

We proceed now to prove our result concerning linear functions a when A is diagonal.

Theorem 3.1. *Let $a(x) = Ax$ be an invertible linear map that in addition is assumed to be diagonal, that is, $a(x) = (\alpha_1 x_1, \dots, \alpha_d x_d)^T$ with $\alpha_i \in \mathbb{R}$. Then*

$$\lambda_1(\mathbb{R}^d) = \lim_{R\to\infty} \lambda_1(B_R) = 2(1 - |\det(A)|^{-1/2})^2 \int_{\mathbb{R}^d} \psi(z)dz.$$

Proof. Using the results of Theorem 1.3 and Lemma 3.2 (here we are using that A is diagonal) we obtain that

$$\lim_{R\to\infty} \lambda_1(B_R) \le 2(1 - |\det(A)|^{-1/2})^2 \int_{\mathbb{R}^d} \psi(z)dz.$$

On the other hand Theorem 1.2 gives us that

$$\lim_{R\to\infty} \lambda_1(B_R) = \lambda_1(\mathbb{R}^d) \ge 2(1 - |\det(A)|^{-1/2})^2 \int_{\mathbb{R}^d} \psi(z)dz.$$

Thus we conclude that

$$\lim_{R\to\infty} \lambda_1(B_R) = 2(1 - |\det(A)|^{-1/2})^2 \int_{\mathbb{R}^d} \psi(z)dz.$$

and the proof is finished. □

Now our task is to extend the result, using different arguments to a general lineal invertible map $a(x) = Ax$. In this case we use the Jordan decomposition of A.

Recall that a linear map $a : \mathbb{R}^d \to \mathbb{R}^d$, $a(x) = Ax$ is called expansive if the absolute value of the (complex) eigenvalues of A are bigger than one.

Lemma 3.3. *Let $a : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear map. If a or a^{-1} is expansive then for functions $\phi \in L^2(\mathbb{R}^d)$ supported in the unit ball with $\|\phi\|_{L^2(B_1)} = 1$ the following holds:*

$$\sup_\phi \int_{\mathbb{R}^d} \phi(x)\phi(a(x))dx = |\det(A)|^{-1/2}.$$

Moreover, the supremum is not attained.

Proof. First, given ϕ as in the statement, we observe that

$$\int_{\mathbb{R}^d} \phi(x)\phi(a(x))dx \le \left(\int_{\mathbb{R}^d} \phi^2(x)dx\right)^{1/2} \left(\int_{\mathbb{R}^d} \phi^2(a(x))dx\right)^{1/2}. \tag{3.2}$$

Hence

$$\sup_\phi \int_{\mathbb{R}^d} \phi(x)\phi(a(x))dx \le |\det(A)|^{-1/2}.$$

Observe that in (3.2) we cannot have equality since in this case $\phi(a(x)) = \mu\phi(x)$ $a.e$ for some constant μ. Since a is expansive this implies that ϕ should vanish identically.

Now we want to obtain the reverse inequality. Let us assume that a is expansive. So, there exists $B \subset \mathbb{R}^d$ a ball with center the origin such that $a^{-j}(B) \subset B_1$, $\forall j \in \{0, 1, \dots, \}$. Take the following sets

$$F = \bigcup_{j=0}^{\infty} a^{-j}(B), \qquad E_l = a^{-l}(F) \setminus a^{-l-1}(F), \quad \text{for } l \in \{0, 1, \dots\}$$

and

$$E = \bigcup_{j=0} E_j.$$

Observe that given $l \in \{0, 1, \dots\}$ we have $\mid E_l \mid_d > 0$. Here and in what follows we denote by $|\cdot|_d$ the Lebesgue measure of a set in $\mathbb{R}^d$.

Since $|\det\ A| > 1$, then

$$|a^{-l}(F)|_d = |a(a^{-l-1}(F))|_d = |\det(a)||a^{-l-1}(F)|_d > |a^{-l-1}(F)|_d.$$

Next, let us observe that

$$E_j \cap E_l = \emptyset \quad \text{if } j, l \in \{0, 1, \dots\} \text{ and } j \neq l. \tag{3.3}$$

Also, since $F \supset a^{-1}(F) \supset a^{-2}(F) \supset \dots$ we have

$$|E_j|_d = |a^{-j}(F)|_d - |a^{-j-1}(F)|_d = |\det(A)|^{-j}(|F|_d - |a^{-1}(F)|_d) = |\det(A)|^{-j}|E_0|_d.$$

For any $0 < \sigma < |\det(A)|^{1/2}$ we now choose

$$\phi_\sigma(x) = \sum_{j=0}^{\infty} \sigma^j \chi_{E_j}(x).$$

These functions are supported in the unit ball and belong to $L^2(\mathbb{R}^d)$, in fact,

$$\|\phi_\sigma\|^2_{L^2(\mathbb{R}^d)} = \sum_{j=0}^{\infty} \sigma^{2j}|E_j|_d = |E_0|_d \sum_{j=0}^{\infty} \sigma^{2j}|\det(A)|^{-j} (< \infty).$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}} \phi_\sigma(x)\phi_\sigma(a(x))dx &= \sum_{j=1}^{\infty} \sigma^{j-1}\sigma^j|E_j|_d = \sum_{j=1}^{\infty} \sigma^{j-1}\sigma^j|\det(A)|^{-j}|E_0|_d \\
&= \sigma|\det(A)|^{-1}|E_0|_d \sum_{j=1}^{\infty} \sigma^{2(j-1)}|\det(A)|^{-j+1} \\
&= \sigma|\det(A)|^{-1}\|\phi_\sigma\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Thus

$$\frac{\displaystyle\int_{\mathbb{R}} \phi(x)\phi(a(x)))dx}{|\det(A)|^{-1/2} \displaystyle\int_{\mathbb{R}} \phi^2(x)dx} = \sigma|\det(A)|^{-1/2} \to 1, \quad \text{as} \quad \sigma \to (|\det(A)|^{1/2})^-,$$

which proves Lemma 3.3 in the case of an expansive function.

Assume now that a^{-1} is expansive. Let ϕ as in the statement, then after the change of variable $a(x) = y$ we have

$$\int_{\mathbb{R}^d} \phi(x)\phi(a(x))dx = |\det(A)|^{-1} \int_{\mathbb{R}^d} \phi(x)\phi(a^{-1}(x))dx.$$

Hence, the proof finishes using the previous expansive case. $\square$

Lemma 3.4. *Let $a : \mathbb{R}^d \to \mathbb{R}^d$, $a(x) = Ax$ be such that A is diagonalizable with all of its complex eigenvalues having the absolute value equal to one. For functions $\phi \in L^2(\mathbb{R}^d)$ supported in the unit ball with $\|\phi\|_{L^2(B_1)} = 1$ the following holds*

$$\sup_{\phi} \int_{\mathbb{R}^d} \phi(x)\phi(a(x))dx = \max_{\phi} \int_{\mathbb{R}^d} \phi(x)\phi(a(x))dx = 1.$$

Proof. Take $\phi = |B_1|_d^{-1/2}\chi_{B_1}$ where χ_{B_1} is the characteristic function of the ball with center the origin and radius 1. Since $\phi(a(x)) = \chi_{B_1}(x)$, then the assertion follows. □

Lemma 3.5. *Let $a : \mathbb{R}^d \to \mathbb{R}^d$, $a(x) = Ax$ be an invertible linear map such that the corresponding matrix associated to the canonical basis is given by*

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & & 1 \\ & & & \lambda \end{pmatrix}, \tag{3.4}$$

or

$$\widetilde{J}_k(\theta) = \begin{pmatrix} \mathbf{M} & \mathbf{I} & & \\ & \ddots & & \mathbf{I} \\ & & & \mathbf{M} \end{pmatrix}, \tag{3.5}$$

where $\lambda \in \{\pm 1\}$, $\theta \in \mathbb{R}$, $\mathbf{M} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, if we consider functions $\phi \in L^2(\mathbb{R}^d)$ supported in the unit ball with $\|\phi\|_{L^2(B_1)} = 1$, we get

$$\sup_{\phi} \int_{\mathbb{R}^d} \phi(x)\phi(a(x))dx = 1.$$

Proof. **Case I.** Assume that the corresponding matrix of the linear map a associated to the canonical basis is given by (3.4). Given $j \in \mathbb{N}$, $a^j(\mathbf{p})^t = (\lambda^j + j\lambda^{j-1}, \lambda^j, 0, \dots, 0)^t$ where $\mathbf{p} = (1, 1, 0, \dots, 0) \in \mathbb{R}^d$. Observe that $a^j(\mathbf{p}) \neq a^l(\mathbf{p})$ if $l, j \in \mathbb{N}$ and $j \neq l$. Indeed $\|a^j(\mathbf{p}) - a^l(\mathbf{p})\| \geq 1$ if $j \neq l$. Thus, $a^j(B_{1/4}(\mathbf{p})) \cap a^l(B_{1/4}(\mathbf{p})) = \emptyset$ if $j \neq l$, where $B_{1/4}(\mathbf{p})$ is the ball with center the point $\mathbf{p}$ and radius $1/4$.

Given $k \in \mathbb{N}$, $k \geq 5$, set the function

$$\phi_k(x) = \sum_{j=0}^{k} \chi_{a^j(2^{-k}B_{1/4}(\mathbf{p}))}(x).$$

Observe that the function ϕ_k is supported in the unit ball. If x is in the support of ϕ_k then there exists $j \in \{0, 1, \dots, k\}$ such that $|x - a^j(\mathbf{p})| \leq 2^{-k-2}$ and we have

$$\begin{aligned} |x| &\leq |x - a^j(\mathbf{p})| + |a^j(\mathbf{p})| \\ &\leq 2^{-k-2} + 2^{-k}((1+k)^2+1)^{1/2} \leq 2^{-k-2} + 2^{-k}3k \leq 2^{-k+1}3k < 1. \end{aligned}$$

Further,

$$\|\phi_k\|^2_{L^2(\mathbb{R}^d)} = \sum_{j=0}^{k} |a^j(2^{-k}B_{1/4}(\mathbf{p}))|_d = 2^{-kd}(k+1)|B_{1/4}(\mathbf{p})|_d.$$

On the other hand,

$$\int_{\mathbb{R}} \phi_k(x)\phi_k(a(x))dx \quad = \quad \sum_{j=1}^{k} |a^j(2^{-k}B_{1/4}(\mathbf{p}))|_d = 2^{-kd}(k)|B_{1/4}(\mathbf{p})|_d$$

Thus

$$\frac{\displaystyle\int_{\mathbb{R}} \phi_k(x)\phi_k(a(x)))dx}{|\det(A)|^{-1/2}\displaystyle\int_{\mathbb{R}} \phi_k^2(x)dx} = \frac{k}{k+1} \to 1, \quad k \to \infty.$$

Having in mind that $|\det(A)| = 1$, that ϕ satisfies the hypotheses in the statement and using Hölder's inequality we obtain that

$$\sup_{\phi} \int_{\mathbb{R}^d} \phi(x)\phi(a(x))dx \leq 1 = |\det(A)|^{-1/2}.$$

Hence, the conclussion follows.

Case II. Assume that the corresponding matrix of A in the canonical basis is given by (3.5). For any $j \in \mathbb{N}$ we set

$$a^j(\mathbf{q}) = \begin{pmatrix} \cos(j\theta) + \sin(j\theta) + (j-1)\cos((j-1)\theta) + (j-1)\sin((j-1)\theta) \\ \cos(j\theta) - \sin(j\theta) + (j-1)\cos((j-1)\theta) - (j-1)\sin((j-1)\theta) \\ \cos(j\theta) + \sin(j\theta) \\ \cos(j\theta) - \sin(j\theta) \\ 0 \\ \cdots \\ 0 \end{pmatrix}. \tag{3.6}$$

Observe that for $\mathbf{q} = (1, 1, 1, 1, 0 \ldots 0)$, we have $a^j(\mathbf{q}) \neq \mathbf{q}$ if $j \in \{0, \ldots, k\}$, where k is a no negative integer number. So $a^j(\mathbf{q}) \neq a^l(\mathbf{q})$ if $l, j \in \{0, \ldots, k\}$ and $j \neq l$. Thus, by continuity of the linear map a, there exists $B \subset \mathbb{R}^d$ a ball with the center at the point $\mathbf{q}$ and radius less or equal to 1 such that $a^j(B) \cap a^l(B) = \emptyset$ if $j, l \in \{0, 1, \ldots, k\}$, $j \neq l$.

Given $k \in \mathbb{N}$, $k \geq 7$, we set the function

$$\phi_k(x) = \sum_{j=0}^{k} \chi_{a^j(2^{-k}B)}(x).$$

Observe that the function ϕ_k is supported in the unit ball. If x is in the support of ϕ_k then there exists $j \in \{0, 1, \ldots, k\}$ such that $|x - a^j(\mathbf{q})| \leq 2{-}k$ and we have

$$\begin{aligned} |x| &\leq |x - a^j(\mathbf{q})| + |a^j(\mathbf{q})| \\ &\leq 2^{-k} + 2^{-k}(2(2 + 2(j-1))^2 + 2^3)^{1/2} \leq 2^{-k} + 2^{-k}(2(2 + 2(k-1))^2 + 2\ 2^2)^{1/2} \\ &\leq 2^{-k} + 2^{-k}(2(4(k-1))^2 + 2(k-1)^2)^{1/2} \\ &\leq 2^{-k} + 2^{-k}(2^6(k-1)^2)^{1/2} = 2^{-k} + 2^{-k+3}(k-1) \leq 2^{-k+4}(k-1) < 1. \end{aligned}$$

Further,

$$\|\phi_k\|^2_{L^2(\mathbb{R}^d)} = \sum_{j=0}^{k} |a^j(2^{-k}B)|_d = 2^{-kd}(k+1)|B|_d.$$

On the other hand,

$$\int_{\mathbb{R}} \phi_k(x)\phi_k(a(x))dx \quad = \quad \sum_{j=1}^{k} |a^j(2^{-k}B)|_d = 2^{-kd}k|B|_d$$

Thus

$$\frac{\displaystyle\int_{\mathbb{R}} \phi_k(x)\phi_k(a(x)))dx}{|\det(A)|^{-1/2}\displaystyle\int_{\mathbb{R}} \phi^2(x)dx} = \frac{k}{k+1} \to 1, \quad k \to \infty.$$

Now, we observe, as we did before, that

$$\sup_{\phi} \int_{\mathbb{R}^d} \phi(x)\phi(a(x))dx \leq 1.$$

Hence, the conclusion follows. □

Now we are ready to proceed with the proof of our main result concerning linear maps a.

Proof of Theorem 1.4. According to Theorem 1.2,

$$\lim_{R\to\infty} \lambda_1(B_R) = \lambda_1(\mathbb{R}^d) \geq 2(1 - |\det(A)|^{-1/2})^2 \int_{\mathbb{R}^d} \psi(z)dz.$$

So, if we prove

$$\lim_{R\to\infty} \lambda_1(B_R) = \lambda_1(\mathbb{R}^d) \leq 2(1 - |\det(A)|^{-1/2})^2 \int_{\mathbb{R}^d} \psi(z)dz, \tag{3.7}$$

the proof is finished. Let us see that (3.7) holds.

Using Jordan's decomposition, there exist C and J two $d \times d$ invertible matrices with real entries such that $A = CJC^{-1}$. Note that J is defined by Jordan blocks, i.e,

$$J = \begin{pmatrix} J_1(\lambda_1) & & & & & \\ & \ddots & & & & \\ & & J_r(\lambda_r) & & & \\ & & & J_{r+1}(\alpha_1, \beta_1) & & \\ & & & & \ddots & \\ & & & & & J_{r+s}(\alpha_s, \beta_s) \end{pmatrix}, \tag{3.8}$$

with

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & \\ & \ddots & 1 \\ & & \lambda \end{pmatrix}, \quad k = 1, \ldots, r, \tag{3.9}$$

or

$$J_k(\alpha,\beta) = \begin{pmatrix} \mathbf{M} & \mathbf{I} & & \\ & \ddots & & \mathbf{I} \\ & & & \mathbf{M} \end{pmatrix}, \quad k = r+1,\dots,r+s. \tag{3.10}$$

Here λ, α and β are real numbers, $\mathbf{M} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Given a $d_k \times d_k$ Jordan block J_k as in (3.9) or (3.10), then either J_k or J_k^{-1} is expansive, or the corresponding eigenvalue has absolute value equal to 1. Then by Lemma 3.3 or Lemma 3.5 there exists $\{\phi_j^{(k)}\}_{j=1}^{\infty} \in L^2(\mathbb{R}^{d_k})$, $\|\phi_j\|_{L^2(\mathbb{R}^{d_k})} = 1$, a sequence of functions supported in the unit ball of $\mathbb{R}^{d_k}$ such that

$$\lim_{j\to\infty} \int_{\mathbb{R}^{d_k}} \phi_j^{(k)}(x)\phi_j^{(k)}(J_k(x))dx = |\det\ J_k|^{-1/2}. \tag{3.11}$$

For $j \in \mathbb{N}$, we choose

$$\varphi_j(x_1^{(1)},\dots,x_{d_1}^{(1)},\dots\dots,x_1^{(r+s)},\dots,x_{d_1}^{(r+s)}) = \prod_{k=1}^{r+s} \phi_j^{(k)}(x_1^{(k)},\dots,x_{d_k}^{(k)})$$

and

$$\Phi_j(x) = (r+s)^{-d/4}\|C^{-1}\|^{-1/2}|\det C|^{-1/2}\varphi_j((r+s)^{-1/2}\|C^{-1}\|^{-1}C^{-1}x),$$

where $\|C^{-1}\|$ denotes the norm of C^{-1} as operator on $\mathbb{R}^d$. Observe that Φ_j is supported in B_1 and $\|\Phi_j\|_{L^2(\mathbb{R}^d)} = 1$ After the change of variable $\|C^{-1}\|^{-1}C^{-1}x = y$, we have

$$\begin{aligned} &\lim_{j\to\infty} \int_{\mathbb{R}^d} \Phi_j(x)\Phi_j(a(x))dx \\ &= (r+s)^{-d/2}\|C^{-1}\|^{-1}|\det C|^{-1} \lim_{j\to\infty} \int_{\mathbb{R}^d} \varphi_j((r+s)^{-1/2}\|C^{-1}\|^{-1}C^{-1}x) \\ &\qquad\qquad \times\varphi_j((r+s)^{-1/2}\|C^{-1}\|^{-1}C^{-1}CJC^{-1}(x))dx \\ &= \lim_{j\to\infty} \int_{\mathbb{R}^d} \varphi_j(y)\varphi_j(Jy)dy = \prod_{k=1}^{r+s} \lim_{j\to\infty} \int_{\mathbb{R}^{d_k}} \phi_j^{(k)}(x)\phi_j^{(k)}(J_k(x))dx \\ &= \prod_{k=1}^{r+s} |\det\ J_k(\lambda_k)|^{-1/2} = |\det(A)|^{-1/2}. \end{aligned} \tag{3.12}$$

Again, using Holder's inequality, we obtain, for any function ϕ as in the statement,

$$\int_{\mathbb{R}^d} \phi(x)\phi(a(x))dx \le |\det(A)|^{-1/2}. \tag{3.13}$$

Therefore, we have

$$\int_{\mathbb{R}^d} \Big(\phi(x) - \phi(a(x))\Big)^2 = 1 + |\det(A)|^{-1} - 2\int_{\mathbb{R}^d} \phi(x)\phi(a(x)),$$

then by (3.12) and (3.13),

$$\inf_{\|\phi\|_{L^2(B_1)}=1} \int_{\mathbb{R}^d} \Big(\phi(x) - \phi(a(x))\Big)^2 = (1 - |\det(A)|^{-1/2})^2.$$

Hence, using the results contained in Lemma 3.1 the proof is finished. □

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