BLOW-UP RESULTS AND LOCALIZATION OF BLOW-UP POINTS IN AN N-DIMENSIONAL SMOOTH DOMAIN

D. F. Rial and J. D. Rossi

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales,
Universidad de Buenos Aires, (1428) Buenos Aires, Argentina

Abstract. We prove that every positive solution of the heat equation with the boundary condition $\frac{\partial u}{\partial \eta} = f(u)$ blows-up in L^∞ norm provided that $\int^{+\infty} \frac{1}{f} < +\infty$ and the blow-up time verifies

$$T \leq \frac{1}{\mid \partial\Omega \mid} \int_\Omega \Big(\int_{u_0(x)}^{+\infty} \frac{1}{f(s)} ds \Big) dx$$

Also we prove for f convex that the blow-up points are localized at the boundary of a ball or at the boundary of a smooth domain if u_0 verifies $\Delta u_0 \geq c > 0$.

I. Introduction.

In this paper we study the behavior of positive solutions of the following problem;

$$(1.1) \qquad \begin{cases} u_t = \Delta u & \text{in } \Omega \times (0,T) \\ \frac{\partial u}{\partial \eta} = f(u) & \text{in } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

Where Ω is a bounded domain in $\ ^n$ with smooth boundary $\partial\Omega$, f is C^2 increasing and positive in $\ _+$, u_0 is $C^{2+\alpha}(\overline{\Omega})$, positive and verifies $\frac{\partial u_0}{\partial \eta} = f(u_0)$.

Under these hypothesis, existence and uniqueness of a classical solution up to some time T was proved in [3].

For problem (1.1) it is known that for each f the existence of global solutions only depends in the behavior of f at infinity. This problem was first studied by H. A. Levine and L. E. Payne in [2]. W. Walter ([4]) proved that if f is convex a necessary and sufficient condition for global existence is $\int^{+\infty} \frac{1}{ff'} = +\infty$ (for every positive initial data u_0). In 1991, J. Lopez Gomez, V. Marquez and N. Wolanski showed that if $1/f$ is locally in L^1 at ∞ (i.e., $\int^\infty \frac{1}{f}$ converges) then blow-up of positive solutions necessarily occurs at a finite time (at least for domains in $\ ^2$, see [3]).

Supported by Universidad de Buenos Aires under grant EX117 . J.D. Rossi is a fellow of CONICET.

If the solutions are nonglobal (this means that the maximal interval of existence is finite, say $(0,T)$) we have

$$\limsup_{t\nearrow T} \|u(x,t)\|_{L^\infty(\Omega)} = +\infty$$

and we say that the solution $u(x,t)$ blows-up at time T.

In Section II we give a blow-up result for every f such that

$$\int^{+\infty} \frac{1}{f} < +\infty \tag{1.2}$$

without any convexity hypothesis. And we obtain for the blow-up time an explicit bound in terms of the initial data

$$T \leq \frac{1}{\mid \partial\Omega \mid} \int_\Omega \Big(\int_{u_0(x)}^{+\infty} \frac{1}{f(s)} ds\Big) dx$$

We prove :

Theorem 1. *Let $u(x,t)$ be a classical solution of (1.1) if (1.2) holds then*

$$T \leq \frac{1}{\mid \partial\Omega \mid} \int_\Omega \Big(\int_{u_0(x)}^{+\infty} \frac{1}{f(s)} ds\Big) dx$$

In Section III and IV we localize the blow-up set. In our problem in which the nonlinear absorption term appears at the boundary it seems reasonable to expect that blow-up occurs only at the boundary .

This was proved to be the case in a ball for a solution that is radially symmetric and in a smooth 2-dimensional domain ([3]); also in a general domain if $\Delta u_0 \geq c > 0$ and $f(s) = s^p, p > 1$ ([1]).

In this paper we generalize the previous results and prove the localization result for the ball with the aditional restriction of f convex or for a general domain Ω with $\Delta u_0 \geq c > 0$ and f convex.

We state our results :

Theorem 2. *Let $\Omega = B(0,1)$ and $u(x,t)$ a solution of (1.1) with f as in theorem 1 with $f(0) = 0$ and convex. Then, given a subdomain $\Omega' \subset\subset \Omega$ there exists a finite constant $K = K(u_0, \Omega') > 0$ such that*

$$\sup_{0<t<T} \|u(x,t)\|_{L^\infty(\Omega')} < K$$

Theorem 3. *In a general smooth domain Ω with the aditional hypothesis of $\Delta u_0 \geq c > 0$ the same conclusion of theorem 2 remains valid .*

II. Blow-up results.

Proof of theorem 1.

Let

$$m(t) = \int_\Omega \left(\int_{u(x,t)}^{+\infty} \frac{1}{f(s)} ds \right) dx$$

We observe that $m(t)$ is well defined for every $t \in (0, T)$ where T is the maximal time of existence for $u(x,t)$ (T finite or not). Also $m(t)$ is positive.

We claim that there exists a constant $k > 0$ such that

$$m'(t) \le -k < 0$$

In order to prove this claim we compute $m'(t)$,

$$m'(t) = -\int_\Omega \frac{u_t}{f(u)}$$

Using that $u(x,t)$ is a solution of the heat equation we get,

$$m'(t) = -\int_\Omega \frac{\Delta u}{f(u)}$$

or equivalentely,

$$m'(t) = -\int_\Omega \mathrm{div}\left(\frac{\nabla u}{f(u)}\right) - \int_\Omega \frac{f'(u)\|\nabla u\|^2}{f(u)^2}$$

Since f is increasing the second integral is nonnegative and by Gauss Theorem we get,

$$m'(t) \le -\int_{\partial\Omega} \frac{1}{f(u)} \frac{\partial u}{\partial \eta} = - \mid \partial\Omega \mid$$

and we have prove the claim.

We finish the proof of theorem 1 by making the following remark. As $m(t)$ is nonnegative, t must be less than or equal to $\frac{m(0)}{|\partial\Omega|}$ and therefore,

$$T \le \frac{1}{\mid \partial\Omega \mid} \int_\Omega \left(\int_{u_0(x)}^{+\infty} \frac{1}{f(s)} ds \right) dx$$

Corollary 2.1. *If f is only continuous, nondecreasing, positive and such that (1.2) holds then every positive solution of (1.1) blows-up in finite time.*

Proof. We can consider an auxiliary function $\overline{f}$ under the same hypothesis of theorem 1 and an initial data $\overline{u}_0$ compatible with this $\overline{f}$ and less than or equal to u_0. With these functions $\overline{f}$ and $\overline{u}_0$ we obtain a solution $\overline{u}$ with finite time blow-up T. Then using a comparison result (Prop. 1.2 of [3]) we obtain $u(x,t) \ge \overline{u}(x,t)$ and hence u cannot be global.

Remark 2.1. There are valid analogous blow-up results for

$$(2.1)\qquad \begin{cases} u_t = \Delta(\phi(u)) & \text{in } \Omega\times(0,T) \\ \frac{\partial(\phi(u))}{\partial\eta} = f(u) & \text{in } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

$$(\text{with } \phi'(u) > 0)$$

and for

$$(2.2)\qquad \begin{cases} u_t = div(A\cdot\nabla u) & \text{in } \Omega\times(0,T) \\ < A\cdot\nabla u;\eta >= f(u) & \text{in } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

$$(\text{with } A \text{ uniformly elliptic})$$

For problem (2.1) see [5] and [6].

III. Localization results for $\Omega = B(0,1)$.

Note . From now on we assume that f is convex.

First we state the following lemmas where $u(x,t)$ is a solution of (1.1) with $\Omega = B(0,1)$ (observe that u is not necessarily radial).

Lemma 3.1. *Let*

$$w(r,t) = \frac{1}{\mid S^{n-1}\mid}\int_{S^{n-1}} u(r\xi,t)\ d\sigma(\xi)$$

Then w satisfies

$$(3.1)\qquad \begin{cases} w_t = w_{rr} + \frac{n-1}{r}w_r \\ w_r \geq f(w) \qquad at \qquad r=1 \end{cases}$$

Proof. We compute w_t using (1.1).

$$(3.2)\qquad w_t(r,t) = \frac{1}{\mid S^{n-1}\mid}\int_{S^{n-1}} \Delta u(r\xi,t)\ d\sigma(\xi)$$

On the other hand we have,

$$w_r(r,t) = \frac{1}{\mid S^{n-1}\mid r^{n-1}}\int_{S^{n-1}} \nabla u(r\xi,t)\xi r^{n-1}\ d\sigma(\xi)$$

and by Gauss Theorem we get,

$$w_r(r,t) = \frac{1}{\mid S_r\mid}\int_{B_r} \Delta u(x,t)dx$$

passing to spherical coordinates,

$$(3.3)\qquad w_r(r,t) = \frac{1}{\mid S^{n-1}\mid r^{n-1}}\int_0^r \Big(\int_{S^{n-1}} \Delta u(\rho\xi,t)\rho^{n-1}\ d\sigma(\xi)\Big)\ d\rho$$

Now we compute w_{rr},

$$w_{rr}(r,t) = \frac{1}{\mid S^{n-1} \mid r^{n-1}} \int_{S^{n-1}} \Delta u(r\xi,t) r^{n-1} \; d\sigma(\xi) \tag{3.4}$$

$$+\frac{1-n}{\mid S^{n-1} \mid r^{n}} \int_0^r (\int_{S^{n-1}} \Delta u(\rho\xi,t)\rho^{n-1} \; d\sigma(\xi)) \; d\rho$$

From (3.2), (3.3) and (3.4) we get the first statement of (3.1).

And from (3.3) with $r = 1$

$$w_r(1,t) = \frac{1}{\mid S^{n-1} \mid} \int_{S^{n-1}} \frac{\partial u}{\partial \eta}(\xi,t) \; d\sigma(\xi)$$

using (1.1),

$$w_r(1,t) = \frac{1}{\mid S^{n-1} \mid} \int_{S^{n-1}} f(u(\xi,t)) \; d\sigma(\xi)$$

and now we can use the fact that f is convex and obtain (via Jensen's inequality),

$$w_r(1,t) \geq f(w(1,t))$$

Lemma 3.2. *Let $x \in B(0,1)$ and $v(r,t) = \frac{1}{|B_r|} \int_0^r (\int_{S^{n-1}} u(x+\rho\xi,t)\rho^{n-1} \; d\sigma(\xi)) \; d\rho$*

Then v satisfies

$$v_t = v_{rr} + \frac{(n+1)}{r} v_r \qquad (0 < r < 1 - \|x\|) \tag{3.5}$$

Proof. First we compute v_t

$$v_t(r,t) = \frac{1}{\mid B \mid r^n} \int_0^r (\int_{S^{n-1}} \Delta u(x+\rho\xi,t)\rho^{n-1} \; d\sigma(\xi)) \; d\rho$$

Now we compute v_r

$$v_r(r,t) = \frac{1}{\mid B \mid r} \int_{S^{n-1}} u(x+r\xi,t) \; d\sigma(\xi) - \tag{3.6}$$

$$-\frac{n}{\mid B \mid r^{n+1}} \int_0^r (\int_{S^{n-1}} u(x+\rho\xi,t)\rho^{n-1} \; d\sigma(\xi)) \; d\rho$$

And v_{rr},

$$v_{rr}(r,t) = \frac{1}{\mid B \mid r} \int_{S^{n-1}} \nabla u(x+r\xi,t)\xi \; d\sigma(\xi) - \frac{1}{\mid B \mid r^2} \int_{S^{n-1}} u(x+r\xi,t) \; d\sigma(\xi) -$$

$$-\frac{n}{\mid B \mid r^2} \int_{S^{n-1}} u(x+r\xi,t) \; d\sigma(\xi) + \frac{n(n+1)}{\mid B \mid r^{n+2}} \int_0^r (\int_{S^{n-1}} u(x+\rho\xi,t)\rho^{n-1} \; d\sigma(\xi)) \; d\rho$$

using the divergence theorem for the first term we obtain,

$$v_{rr}(r,t) = \frac{1}{\mid B \mid r^n} \int_{B_r} \Delta u(x+y,t) dy -$$

$$-\frac{(n+1)}{r} \left\{ \frac{1}{\mid B \mid r} \int_{S^{n-1}} u(x+r\xi,t) d\sigma(\xi) - \frac{n}{\mid B \mid r^{n+1}} \int_0^r (\int_{S^{n-1}} u(x+\rho\xi,t)\rho^{n-1} d\sigma(\xi)) d\rho \right\}$$

from this we obtain (3.5) taking spherical coordinates and using (3.6).

Lemma 3.3. *If v is as in Lemma 3.2 then,*

$$u(x,t) \leq \max\left\{ \|u_0\|_{L^\infty(\Omega)}, \sup_t\{v(r,t)\} \right\}$$

for every $r \in (0, 1 - \|x\|)$.

Proof. By lemma 3.2 we may consider $v(r,t)$ as a radial solution of the heat equation in $^{n+2}$ and then the maximun principle implies,

$$v(0,t) \leq \max\left\{ \sup_{0<\rho<r}\{v(\rho,0)\}, \sup_t\{v(r,t)\} \right\} \qquad \text{with } 0<r<1-\|x\|$$

Now, by the definition of $v(r,t)$, $v(0,t) = u(x,t)$ and $\sup_{0<\rho<r} v(\rho,0) \leq \|u_0\|_{L^\infty(B)}$ and the conclusion follows.

With these lemmas we can prove theorem 2

Proof of theorem 2. By lemma 3.3 it is enough to control $\sup_t\{v(r,t)\}$

It holds,

$$v(r,t) \leq \frac{|\, S^{n-1} \,|}{|\, B \,|\, r^n} \int_0^{r+\|x\|} w(\rho,t)\rho^{n-1}\, d\rho$$

which leads us to control $w(\rho,t)$ with $0 \leq \rho \leq r + \|x\| < 1$.

Since w satisfies (3.1), we may use Proposition 2.2 of [3] which states,

Proposition 2.2 ([3]). *Let $f \in C^2$ convex and positive in $_+$ with $f(0)=0$. Let w be a classical solution of (3.1), $r + \|x\| < 1$ and*

$$G(s) = \int_s^{+\infty} \frac{1}{f}, \qquad H = G^{-1}$$

Then there exists a smooth function $g : [0,1] \mapsto {}_+$ and a constant $\lambda \geq 0$ such that $g(1) > 0$ and

$$w(\rho,t) \leq H(\int_\rho^1 g(s)ds) + \lambda(1-\rho) \leq H(\int_{r+\|x\|}^1 g(s)ds) + \lambda$$

And we finally obtain,

$$\sup_t\{v(r,t)\} \leq c\left\{ H(\int_{r+\|x\|}^1 g(s)ds) + \lambda \right\} \frac{|\, S^{n-1} \,|}{|\, B \,|} (\frac{r+\|x\|}{r})^n$$

IV. Localization results for smooth Ω.

We want to follow the same ideas as in Section III but we have to sort some technical facts.

First we introduce some notation and a well known fact.

Note 1. Let Ω be a bounded domain with smooth boundary $\partial\Omega$. We denote $\Omega_r = \{x \in \Omega; dist(x,\partial\Omega) > r\}$ and $\partial\Omega_r = \{x \in \Omega; dist(x,\partial\Omega) = r\}$ and we use the following contruction. We define

$$\Phi(\xi, r) = \xi - r\eta(\xi)$$

Where $\eta(\xi)$ is the exterior normal vector at $\xi \in \partial\Omega$.

$$\Phi : \partial\Omega \times (0, R) \mapsto \Omega \quad \overline{\Omega}_R$$

We recall that Φ is a difeomorphism if R is small enough.

We also use the fact that there exists a function $\mu(r, \xi)$ such that,

$$\int_{\partial\Omega_r} u \, d\sigma = \int_{\partial\Omega} u(\Phi(r, \xi))\mu(r, \xi) \, d\sigma(\xi)$$

and

$$\int_{\Omega \quad \overline{\Omega}_r} u(x)dx = \int_0^r \int_{\partial\Omega} u(\Phi(\rho, \xi)\mu(\rho, \xi) \, d\sigma(\xi) \, d\rho$$

(if Ω is a ball $\mu(r, \xi) = r^{n-1}$ as in Section III).

Note 2. If u is a solution of (1.1) with $\Delta u_0 \geq c > 0$ then $\Delta u \geq c > 0$ (Prop 1.3 of [3]) and also $\Delta f(u) \geq 0$.

Lemma 4.1. *There exists a constant $C = C(\Omega)$ such that, for every $u(x)$ positive with $\Delta u \geq 0$,*

$$\int_\Omega u \, dx \leq C \int_{\partial\Omega} u \, d\sigma$$

in particular $\int_{\Omega_r} f(u) \, dx \leq C \int_{\partial\Omega_r} f(u) \, d\sigma$.

Proof. In this Proof C denotes a constant that depends only on Ω. For $0 \leq r \leq r_0 < R$ we define

$$v(r) = \int_{\partial\Omega_r} u \, d\sigma = \int_{\partial\Omega} u(\Phi(r, \xi))\mu(r, \xi) \, d\sigma(\xi)$$

We compute $v'(r)$

$$v'(r) = -\int_{\partial\Omega_r} \frac{\partial u}{\partial \eta} \, d\sigma + \int_{\partial\Omega} u\mu_r \, d\sigma = -\int_{\Omega_r} \Delta u \, dx + \int_{\partial\Omega} u\mu_r \, d\sigma$$

$$\leq \int_{\partial\Omega} u\{\frac{\mu_r}{\mu}\}\mu \, d\sigma \leq Av(r)$$

Where $A = \max_{[0,r_0]\times\partial\Omega} \frac{|\mu_r(r,\xi)|}{\mu(r,\xi)}$. By Gronwall's Lemma,

$$v(r) \leq Cv(0)$$

And then

$$\int_{\Omega \quad \Omega_{r_0}} u \, dx = \int_0^{r_0} \int_{\partial\Omega_\rho} u \, d\sigma \, d\rho \leq Cv(0)$$

Now we observe that, if $x \in \partial\Omega_{r_0/2}$ then $B = B(x, r_0/4) \subset \Omega \quad \Omega_{r_0}$ and then

$$u(x) \leq \frac{1}{|B|} \int_B u \leq C \int_{\Omega \quad \Omega_{r_0}} u \leq Cv(0)$$

We obtain

$$\max_{\partial\Omega_{r_0/2}} u(x) \leq Cv(0)$$

and then

$$\int_\Omega u\ dx \le \int_{\Omega\ \ \Omega_{r_0}} u\ dx + \int_{\Omega_{r_0/2}} u\ dx \le Cv(0) + \mid \Omega \mid \max_{\partial\Omega_{r_0/2}} u(x) \le Cv(0) = C\int_{\partial\Omega} u\ d\sigma$$

The last statement is a consequence of $\Delta f(u) \ge 0$ and the fact that we may replace Ω by Ω_r for $0 \le r \le r_0/2$ by changing a little bit the constant C.

As in Section III we need to prove some previous lemmas before getting to the proof of Theorem 3.

Lemma 4.2. *Let $u(x,t)$ be a positive solution of $u_t = \Delta u$.*
Then $v(r,t) = \int_{\Omega_r} u(x,t)dx$ satisfies,

$$v_t(r,t) \ge v_{rr}(r,t) + Av_r(r,t) \tag{4.1}$$

$$\textit{where } A = \max_{[0,r_0]\times\partial\Omega} \frac{\mid \mu_r(r,\xi) \mid}{\mu(r,\xi)}$$

<u>*Proof*</u>. We compute v_t,

$$v_t(r,t) = \int_{\Omega_r} u_t(x,t)dx = \int_{\Omega_r} \Delta u(x,t)dx \tag{4.2}$$

Now we observe that,

$$v(r,t) = \int_{\Omega_r} u(x,t)dx = \int_\Omega u(x,t)dx - \int_{\Omega\ \ \Omega_r} u(x,t)dx$$
$$= \int_\Omega u(x,t)dx - \int_0^r \int_{\partial\Omega} u(\Phi(\rho,\xi),t)\mu(\rho,\xi)\ d\sigma(\xi)\ d\rho$$

We compute v_r,

$$v_r(r,t) = -\int_{\partial\Omega} u(\Phi(r,\xi),t)\mu(r,\xi)\ d\sigma(\xi) \tag{4.3}$$

and v_{rr},

$$v_{rr}(r,t) = \int_{\partial\Omega} < \nabla u(\Phi(r,\xi),t); \eta(\xi) > \mu(r,\xi)\ d\sigma(\xi) - \int_{\partial\Omega} u(\Phi(r,\xi),t)\mu_r(r,\xi)\ d\sigma(\xi)$$
$$= \int_{\partial\Omega} \frac{\partial u}{\partial\eta}(\Phi(r,\xi),t)\mu(r,\xi)\ d\sigma(\xi) - \int_{\partial\Omega} u(\Phi(r,\xi),t)\mu(r,\xi)\{\frac{\mu_r}{\mu}\}\ d\sigma(\xi)$$
$$= \int_{\Omega_r} \Delta u(x,t)dx - \int_{\partial\Omega} u(\Phi(r,\xi),t)\mu(r,\xi)\{\frac{\mu_r}{\mu}\}\ d\sigma(\xi)$$

using (4.2),

$$= v_t(r,t) - \int_{\partial\Omega} u(\Phi(r,\xi),t)\mu(r,\xi)\{\frac{\mu_r}{\mu}\}\ d\sigma(\xi)$$

which implies by using (4.3) and the expression for A,

$$v_t(r,t) = v_{rr}(r,t) + \int_{\partial\Omega} u(\Phi(r,\xi),t)\mu(r,\xi)\{\frac{\mu_r}{\mu}\}\ d\sigma(\xi)$$
$$\ge v_{rr}(r,t) + Av_r(r,t)$$

Corolary 4.1. *Let u as in Lemma 4.2, $a = \frac{A}{2}$ and $b = a^2 - Aa$. Then*

$$\overline{z}(r,t) = \exp(ar - bt) \int_{\partial\Omega_r} \frac{\partial u}{\partial \eta}\, d\sigma$$

satisfies,

$$\overline{z}_t \geq \overline{z}_{rr}$$

Proof. We observe that $\overline{v}(r,t) = \int_{\partial\Omega_r} \frac{\partial u}{\partial \eta}\, d\sigma = \int_{\Omega_r} \Delta u(x,t)dx$. And now we apply Lemma 4.2 to Δu which is a solution of the heat equation. We obtain,

$$\overline{v}_t \geq \overline{v}_{rr} + A\overline{v}_r$$

Now we compute,

$$\overline{z}_t = \exp(ar - bt)[\overline{v}_t - b\overline{v}]$$

$$\overline{z}_r = \exp(ar - bt)[\overline{v}_r + a\overline{v}]$$

$$\overline{z}_{rr} = \exp(ar - bt)[\overline{v}_{rr} + 2a\overline{v}_r + a^2\overline{v}]$$

Then we obtain,

$$\overline{z}_t \geq \overline{z}_{rr} + (A - 2a)\overline{z}_r + (-b + a^2 - Aa)\overline{z}$$

Using our choice of a and b,

$$\overline{z}_t \geq \overline{z}_{rr}$$

Remark 4.1.

$$\frac{\partial}{\partial r}(e^{-Ar} \int_{\partial\Omega_r} u\, d\sigma) \leq -e^{-Ar} \int_{\partial\Omega_r} \frac{\partial u}{\partial \eta}\, d\sigma$$

Proof.

$$\frac{\partial}{\partial r}(e^{-Ar} \int_{\partial\Omega_r} u\, d\sigma) = -Ae^{-Ar} \int_{\partial\Omega_r} u\, d\sigma + e^{-Ar} \frac{\partial}{\partial r}(\int_{\partial\Omega_r} u\, d\sigma)$$

$$= -Ae^{-Ar}(\int_{\partial\Omega_r} u\, d\sigma) + e^{-Ar} \frac{\partial}{\partial r}(\int_{\partial\Omega} u(\Phi(r,\xi))\mu(r,\xi)\, d\sigma(\xi)$$

$$= e^{-Ar}\left\{-A\int_{\partial\Omega_r} u d\sigma + \int_{\partial\Omega} u(\Phi(r,\xi))\frac{\mu_r}{\mu}\mu d\sigma(\xi) - \int_{\partial\Omega} < \nabla u(\Phi(r,\xi)); \eta(\xi) > \mu(r,\xi) d\sigma(\xi)\right\}$$

$$\leq -e^{-Ar} \int_{\partial\Omega_r} \frac{\partial u}{\partial \eta}\, d\sigma$$

Lemma 4.3. *Let $w(r,t) = \int_{\partial\Omega_r} f(u)\, d\sigma$. Then w satisfies,*

$$w_t \leq w_{rr} + Bw_r + Cw \qquad \textit{and} \tag{4.4}$$

$$0 \leq \int_{\partial\Omega_r} \Delta(f(u))dx \leq Aw - w_r \tag{4.5}$$

Proof. We begin with (4.5).

$$w_r(r,t)=\frac{\partial}{\partial r}\left\{\int_{\partial\Omega} f(u(\Phi(r,\xi),t))\mu(r,\xi)\ d\sigma(\xi)\right\}$$

$$=-\left\{\int_{\partial\Omega} <\nabla(f(u));\eta(\xi)>(\Phi(r,\xi),t)\mu(r,\xi)\ d\sigma(\xi)\right\}+$$

$$\left\{\int_{\partial\Omega} f(u(\Phi(r,\xi),t))\frac{\mu_r}{\mu}(r,\xi)\mu(r,\xi)\ d\sigma(\xi)\right\}$$

It follows that,

$$w_r(r,t)\le-\int_{\partial\Omega_r}\frac{\partial(f(u))}{\partial\eta}\ d\sigma+A\int_{\partial\Omega_r} f(u)\ d\sigma$$

and using the divergence theorem we obtain the second inequality (the first one is trivial because f is convex and $\Delta u\ge 0$).

$$w_r(r,t)\le-\int_{\Omega_r}\Delta(f(u))dx+Aw$$

In order to prove (4.4) we rewrite w_r as,

$$w_r(r,t)=-\int_{\Omega}\Delta(f(u))dx+\int_0^r\int_{\partial\Omega}\Delta(f(u))(\Phi(\rho,\xi),t)\mu(\rho,\xi)\ d\sigma(\xi)\ d\rho+$$

$$+\int_{\partial\Omega} f(u(\Phi(r,\xi),t))\mu_r(r,\xi)\ d\sigma(\xi)$$

And we compute w_{rr},

$$w_{rr}=\int_{\partial\Omega}\Delta(f(u))(\Phi(r,\xi),t)\mu(r,\xi)\ d\sigma(\xi)-$$

$$\int_{\partial\Omega} <\nabla(f(u));\eta(\xi)>(\Phi(r,\xi),t)\frac{\mu_r}{\mu}\mu(r,\xi)\ d\sigma(\xi)+$$

$$+\int_{\partial\Omega} f(u(\Phi(r,\xi),t))\frac{\mu_{rr}}{\mu}\mu(r,\xi)\ d\sigma(\xi)$$

Now let $\overline{\mu}$ be a $C^2(\overline{\Omega})$ function that verifies $\overline{\mu}(\Phi(r,\xi))=\frac{\mu_r}{\mu}(r,\xi)$ for $(r,\xi)\in[0,r_0]\times\partial\Omega$. With this $\overline{\mu}$ we can compute, using the conclusion of Lemma 4.1 ($\Delta f(u)\ge 0$),

$$\int_{\partial\Omega_r}\frac{\partial f(u)}{\partial\eta}\overline{\mu}\ d\sigma=\int_{\Omega_r}\Delta(f(u))\overline{\mu}dx-\int_{\Omega_r} f(u)\Delta\overline{\mu}dx+\int_{\partial\Omega_r} f(u)\frac{\partial\overline{\mu}}{\partial\eta}\ d\sigma$$

$$\Big|\int_{\partial\Omega_r}\frac{\partial f(u)}{\partial\eta}\overline{\mu}\ d\sigma\Big|\le C_1\int_{\Omega_r}\Delta(f(u))+C_2\int_{\partial\Omega_r} f(u)\ d\sigma$$

$$\Big|\int_{\partial\Omega_r}\frac{\partial f(u)}{\partial\eta}\overline{\mu}\ d\sigma\Big|\le C_1(Aw-w_r)+C_2w$$

It follows that,

$$w_{rr}(r,t)\ge\int_{\partial\Omega_r}\Delta(f(u))\ d\sigma-Bw_r-Cw\ge\int_{\partial\Omega_r} f'(u)\Delta u\ d\sigma-Bw_r-Cw$$

$$w_t(r,t)=\int_{\partial\Omega_r} f'(u)\Delta u\ d\sigma$$

And we finally obtain,

$$w_t\le w_{rr}+Bw_r+Cw$$

Corolary 4.2. *Let $\beta < -\max\{A, \frac{|B|}{2}\}$ and $\gamma = C + \beta^2 - B\beta$*
Then $z(r,t) = \exp(\beta r - \gamma t) w(r,t)$ satisfies

$$z_r < 0 \tag{4.6}$$

$$z_t \leq z_{rr} \tag{4.7}$$

Proof.

$$z_r = \exp(\beta r - \gamma t)[\beta w + w_r] = \exp(\beta r - \gamma t)[(A + \beta) w + (w_r - Aw)]$$

And each term is negative so we have (4.6). Next we compute

$$z_t = \exp(\beta r - \gamma t)[w_t - \gamma w]$$

$$z_r = \exp(\beta r - \gamma t)[w_r + \beta w]$$

$$z_{rr} = \exp(\beta r - \gamma t)[w_{rr} + 2\beta w_r + \beta^2 w]$$

Then, by Lemma 4.3, we have,

$$z_t \leq z_{rr} + (B - 2\beta) z_r + (-\gamma + C + \beta^2 - B\beta) z$$

And then, by our choice of β and γ, we obtain (4.7).

Proposition 4.1. *There exists $g \in C^2([0, r_0])$, nonnegative, decreasing and convex with $g(r_0) = 0$ and $g(0) < L = \sup_{0 \leq t \leq T} \frac{e^{-bt}}{e^{-\gamma t}}$ such that,*

$$\int_{\partial \Omega_r} \frac{\partial u}{\partial \eta}\, d\sigma \geq C g(r) \int_{\partial \Omega_r} f(u)\, d\sigma \tag{4.8}$$

Proof. We use a comparison argument here.

First we choose a C^2, nonnegative, decreasing and convex function g such that $\overline{z}(r,0) \geq g(r) z(r,0)$ with $g(r_0) = 0$ and $g(0) < L$. Next we observe that $g(r) z(r,t)$ is a subsolution of the heat equation (we use Corolary 4.2 and the properties of g) and, by Corolary 4.1, $\overline{z}$ is a supersolution. To finish the comparison argument we observe that at $r = 0$ we have $\overline{z}(0,t) \geq g(0) z(0,t)$ since $g(0) < L$, and at $r = r_0$, $\int_{\partial \Omega_{r_0}} \frac{\partial u}{\partial \eta}\, d\sigma \geq 0$. We conclude that $\overline{z} \geq g(r) z$.

We finish the proof of the Proposition by recalling the definitions of $\overline{z}$ and z.

$$\exp(ar - bt) \int_{\partial \Omega_r} \frac{\partial u}{\partial \eta}\, d\sigma = \overline{z} \geq g(r) z = g(r) \exp(\beta r - \gamma t) \int_{\partial \Omega_r} f(u)\, d\sigma$$

We choose

$$C = \inf_{[0,R] \times [0,T]} \frac{\exp(\beta r - \gamma t)}{\exp(ar - bt)}$$

Proof of theorem 3. From Remark 4.1 we have

$$\frac{\partial}{\partial r}\left(e^{-Ar} \int_{\partial \Omega_r} u\, d\sigma\right) \leq -e^{-Ar} \int_{\partial \Omega_r} \frac{\partial u}{\partial \eta}\, d\sigma$$

And then using Proposition 4.1, Jensen's inequality and the fact that f is increasing

$$\frac{\partial}{\partial r}(e^{-Ar}\int_{\partial\Omega_r} u\ d\sigma) \leq -Cg(r)(e^{-Ar})f(\int_{\partial\Omega_r} u\ d\sigma) \leq -Cg(r)f(e^{-Ar}\int_{\partial\Omega_r} u\ d\sigma)$$

Thus we obtain, with G as in the proof of Theorem 2

$$G(e^{-Ar}\int_{\partial\Omega_r} u\ d\sigma) \geq C\int_0^r g(s)ds$$

Therefore,

$$\int_{\partial\Omega_r} u\ d\sigma \leq e^{Ar}H(C\int_0^r g(s)ds)$$

Now we observe that since u verifies $\Delta u > 0$, (with $x \in \Omega_{r_0} \quad \Omega_{r_1}$ and $r_0/4 < \delta < r_0/2$)

$$u(x,t) \leq \frac{1}{\mid B(x,\delta)\mid}\int_{B(x,\delta)} u \leq \frac{1}{\mid B(x,\delta)\mid}\int_{r_0/2}^{r_1+r_0/2}\int_{\partial\Omega_r} u\ d\sigma$$

And the last term is bounded by an expresion that only depends on r_0 and r_1, so we can control u in $\Omega_{r_0} \quad \Omega_{r_1}$ and then in Ω_{r_0} using again that $\Delta u \geq 0$.

REFERENCES

[1] Bei Hu and Hong-Ming Yin, *The Profile Near Blow-up Time for the Solution of the Heat Equation with a Nonlinear Boundary Condition*, Trans. Amer. Math. Soc **346 (1)** (1994), 117–135.

[2] H. A. Levine and L. E. Payne, *Nonexistence Theorems for the heat equation with non-linear boundary conditions and for the porous medium equation backward in time*, Jour. Diff. Eq. **16** (1974), 319–334.

[3] J. Lopez Gomez , V. Marquez and N. Wolanski, *Blow-up Results and Localization of Blow-up Points for the Heat Equation with a Nonlinear Boundary Condition*, Jour. Diff. Eq. **92(2)** (1991), 384–401.

[4] W. Walter, *On Existence and Nonexistence in the Large of Solutions of Parabolic Differential Equations with a Nonlinear Boundary Condition*, SIAM J. Math. Anal. **6(1)** (1975), 85–90.

[5] Mingxin Wang and Yonghui Wu, *Global existence and blow-up problems for quasilinear parabolic equations with nonlinear boundary conditions*, SIAM J. Math. Anal. **24(6)** (1993), 1515–1521.

[6] N. Wolanski, *Global behavior of positive solutions to nonlinear diffusion problems with nonlinear absorption through the boundary*, SIAM J. Math. Anal. **24(2)** (1993), 317–326.