GLOBAL EXISTENCE AND NONEXISTENCE FOR A PARABOLIC SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

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We find both necessary and sufficient conditions on the nonnegative increasing functions f and g so that positive solutions of

$$\begin{array}{lll} u_t = \Delta u + h(u,v) & v_t = \Delta v + r(u,v) & \text{in } \Omega \times (0,T) \\ \frac{\partial}{\partial \nu} u = f(v) & \frac{\partial}{\partial \nu} v = g(u) & \text{on } \partial\Omega \times (0,T) \end{array}$$

exist globally in time. We assume throughout that h and r are nonnegative, smooth and $\frac{h(u,v)}{u}$, $\frac{r(u,v)}{v}$ are globally bounded.

I. Introduction.

In this paper we study the global existence or nonexistence of positive solutions to the system,

$$\begin{cases} u_t = \Delta u + h(u,v) & v_t = \Delta v + r(u,v) & \text{in } \Omega \times (0,T) \\ \frac{\partial}{\partial \nu} u = f(v) & \frac{\partial}{\partial \nu} v = g(u) & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & v(x,0) = v_0(x) & \text{in } \Omega \end{cases} \tag{1.1}$$

Existence and uniqueness of a solution up to some time $T \leq +\infty$ for f' and g' Lipschitz continuous, h and r smooth has been proved in [1] and [2]. We assume in addition that f, g, h and r are nonnegative and $\frac{h(u,v)}{u}$, $\frac{r(u,v)}{v}$ are uniformly bounded in $R \times R$. If $T = +\infty$ we say that (u,v) is global. In [1] the author proves that if $T < +\infty$ either u or v should blow up in L^∞. In our case it is easy to see and we prove it here that in the latter case both functions should blow up in L^∞ norm at time T. Also we show that we can relax the regularity condition to allow f' and g' to be locally Lipschitz continuous in $R \quad 0$.

These results are well known for a single equation. In this case it is known that for each f the existence of global solutions is independent both of the domain Ω

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and the initial data; it only depends on how fast $f(s)$ and $f'(s)$ grow as s goes to infinity.

We find the same situation for system (1.1). In this case the existence of global solutions depends on the growth of certain functions derived from f and g. Precisely, assume that for each $s_0 > 0$, there exist such that,

$$\frac{g'(s)}{g^2(s)} \le \frac{f'(F^{-1}(G(s)))}{f^2(F^{-1}(G(s)))} \qquad \text{for } s > s_0 \tag{1.2}$$

Also,

$$\text{let } p(z) := \frac{f'(F^{-1}(G(z)))g(z)}{f(F^{-1}(G(z)))} \text{ be monotone (incresing or decreasing)} \tag{1.3}$$

and,

$$\int^{\infty} \frac{ds}{f(F^{-1}(G(s)))} = \int^{\infty} \frac{ds}{g(s)f'(F^{-1}(G(s)))} = +\infty$$

then any positive solution of (1.1) exists globally. Conversely, if one of the integrals is finite and,

$$\frac{g'(s)}{g^2(s)} \ge {}^{-1}\frac{f'(F^{-1}(G(s)))}{f^2(F^{-1}(G(s)))} \qquad \text{for } s > s_0 \tag{1.2'}$$

every positive solution blows up in finite time.

Here F and G are positive primitives of f and g respectively with $F(0) = G(0) = 0$.

Conditions (1.2) and (1.2') hold in many situations. For instance, if $F(s) = G(s^\beta)$ for some $\beta > 1$, (1.2) is true. For F and G as above, if $\frac{g(s)}{s} \le Kg'(s)$ for some positive constant K both members are seen to be equivalent. On the other hand if $\beta \le 1$ (1.2') holds.

As an application of this general result we get for instance,

(a) Let $f(v) = (p+1)v^p$, $g(u) = (q+1)u^q$ in (1.1) then if $pq \le 1$ every positive solution is global whereas if $pq > 1$ every positive solution blows up in finite time.

(b) Let $f(v) = (1+v)\log(1+v)$ and $g(u) = (\frac{p+1}{2})^2(1+u)^p\log(1+u)$ in (1.1) then if $p < 1$ every positive solution is global but if $p \ge 1$ every positive solution blows up in finite time.

We would like to point out at this time that for $u_0 = v_0$, $f = g$ and $h = r$ we have $u(x,t) = v(x,t)$ for every x, t. Thus u is a solution of the problem

$$u_t = \Delta u + h(u,u) \qquad ; \qquad \frac{\partial}{\partial \nu} u = f(u)$$

It is known (see [13]) that in this case if f is convex u blows up in finite time if and only if

$$\int^{\infty} \frac{ds}{f(s)f'(s)} < \infty$$

So that case (b) with $p = 1$ should not be in the global existence range.

(1.1) is a simple example of a weakly coupled system. Questions on global solvability for weakly coupled systems have been addressed recently by several authors in the case of semilinear systems in R^n or in bounded domains with Dirichlet boundary conditions (see for instance [3], [4], [5], [9], [12]). On the other hand case (a) above has been generalized in [10] to a system of m heat equations with boundary conditions,

$$\frac{\partial}{\partial \nu} u_i = u^{p_i} \qquad \text{for } i = 1, \cdots, m$$

where $u = (u_1, \cdots, u_m)$ is the solution and for each i, $p_i = (p_{i1}, \cdots, p_{im})$ with $p_{ij} \geq 0$.

An interesting question for problem (1.1) is the localization of the blow up set and the rate at which solutions blow up. For a single equation the localization question has been answerd in [7] for $f(s) = s^p$ and in [8] and [11] for f convex with $\int^{\infty} \frac{ds}{f(s)} < \infty$ in the N-dimensional ball and in general smooth domains respectively. They showed that the blow up set is contained in $\partial\Omega$. For the rate of blow up for the single equation we refer to [6], [7]. We leave the treatment of these problems for a future paper.

The paper is organized as follows. Based on the results of Amann we show in Section II that for f' and g' locally Lipschitz continuous in R and $u_0,\ v_0$ nonnegative or locally Lipschitz continuous in R 0 and $u_0,\ v_0$ nonnegative in Ω and strictly positive on $\partial\Omega$ the solutions can be continued as long as they remain bounded and that both functions should blow up at the same time if the solution does not exist globally. Also we show that u and v become strictly positive for $t > 0$ in $\overline{\Omega}$ if they are nonnegative initially. In Section III we prove our results on global existence and blow up of solutions and in Section IV we apply these results to several examples.

For the sake of simplicity we will state and prove our results for $h = r = 0$ although it will be clear from the proofs that all the arguments work for h and r nonnegative and smooth satisfying,

$$\begin{aligned} &\frac{h(u,v)}{|u|} \leq C \qquad \text{in } R \times R \qquad \text{and} \\ &\frac{r(u,v)}{|v|} \leq C \qquad \text{in } R \times R \end{aligned} \tag{1.4}$$

so we will assume throughout that (1.4) holds.

The same techniques may be applied to find necessary and sufficient conditions for global solvability without the restriction (1.4). Clearly in this case global existence will depend also on the growth of h and r at infinity.

II. Local Existence and Positivity.

Theorem 2.0 ([1]). *Let $u_0,\ v_0 \in C^2(\overline{\Omega})$ and strictly positive in $\overline{\Omega}$. Let f, g be uniformly Lipschitz continuous in R together with their derivatives and strictly positive in $R \quad 0$. There exists $T > 0$ such that there exists a unique classical solution of (1.1) in $C(\overline{\Omega} \times [0, T))$. If T is the maximal time of existence and $T < +\infty$,*

$$||u||_{L^\infty(\Omega)} + ||v||_{L^\infty(\Omega)} \nearrow \infty \qquad \text{as } t \nearrow T$$

Remark. This theorem is a consequence of Theorem 3 in [1] since for $u_0,\ v_0 \geq \gamma$ we have $u(x, t),\ v(x, t) \geq \gamma$ for $t > 0$. In fact we state this minimum principle as Lemma 2.1.

Lemma 2.1. *Let u, v be continuous solutions to (1.1). Let f and g be strictly positive in $R \quad 0$. Then the minimun of u and v in $\Omega \times [0, T)$ is attained at time $t = 0$. Moreover if u_0 and v_0 are nonnegative and not both identically 0 (assuming a solution exists which is continuous in $\overline{\Omega}$ for $t > 0$) and $\tau > 0$ there exists $\gamma_\tau > 0$ such that $u(x, t) \geq \gamma_\tau$ and $v(x, t) \geq \gamma_\tau$ for $t \geq \tau$.*

Proof. Since u and v are supercaloric they cannot attain an interior minimum unless they are constant. On the other hand they cannot attain a minimum at the lateral boundary unless they are constant. In fact if for instance u attains a minimum at a lateral point and u is not constant $\frac{\partial}{\partial\nu}u < 0$ there. But $\frac{\partial}{\partial\nu}u = f(v) \geq 0$. Therefore both functions attain their minima at time $t = 0$.

Now let $\tau > 0$.

Then $\min_{t\geq\tau} u(x, t) \geq \min u(x, \tau)$ and $\min_{t\geq\tau} v(x, t) \geq \min v(x, \tau)$. Let us see that there exists $\gamma_\tau > 0$ such that $u(x, \tau) \geq \gamma_\tau$ and $v(x, \tau) \geq \gamma_\tau$. In fact it is enough to prove that both functions are strictly positive in $\overline{\Omega} \times (0, T)$. We already know that they are nonnegative since they are nonnegative initially. If u_0 is not identically 0, u should be strictly positive for $t > 0$. In particular $\frac{\partial}{\partial\nu}v > 0$ therefore v is not identically zero even if it is initially. So that both u and v become strictly positive in $\overline{\Omega}$ for $t > 0$.

We will also need a comparison principle for sub and supersolutions to (1.1)

Definition. A pair of continuous functions in $\overline{\Omega} \times [0, T)$ $(\overline{u}, \overline{v})$ is a classical supersolution to problem (1.1) if $u \in C^{2,1}(\Omega \times (0, T))$, $v \in C^{2,1}(\Omega \times (0, T))$ and,

$$\begin{cases} \overline{u}_t \geq \Delta\overline{u} & \overline{v}_t \geq \Delta\overline{v} & \text{in } \Omega \times (0, T) \\ \frac{\partial}{\partial\nu}\overline{u} \geq f(\overline{v}) & \frac{\partial}{\partial\nu}\overline{v} \geq g(\overline{u}) & \text{on } \partial\Omega \times (0, T) \end{cases}$$

Analogously, $(\underline{u}, \underline{v})$ is a subsolution if it satisfies the opossite inequalities.

Lemma 2.2. *Let f and g be locally Lipschitz continuous in $R \quad \{0\}$ and nondecreasing in R_+, if $(\underline{u}, \underline{v})$ is a nonnegative classical subsolution to (1.1) and $(\overline{u}, \overline{v})$ is a nonnegative classical supersolution with $\underline{u}(x, 0) < \overline{u}(x, 0)$ and $\underline{v}(x, 0) < \overline{v}(x.0)$ in $\overline{\Omega}$ the same relation holds for $t > 0$.*

Proof. Let $(w_1, w_2) = (\overline{u} - \underline{u}, \overline{v} - \underline{v})$. Then w_i $i = 1, 2$ are supersolutions of the heat equation that satisfy on the lateral boundary for $t \leq T < \min(T(\underline{u}_0, \underline{v}_0), T(\overline{u}_0, \overline{v}_0))$,

$$\frac{\partial}{\partial \nu} w_1 \geq A(x, t) \qquad ; \qquad \frac{\partial}{\partial \nu} w_2 \geq B(x, t)$$

with $A = f(\overline{v}) - f(\underline{v})$ and $B = g(\overline{u}) - g(\underline{u})$.

Let $\delta > 0$ such that $w_i(x, 0) \geq \delta$ in Ω. Then $w_i > \frac{\delta}{2}$ for $t > 0$. In fact assume it is not true and let $t_0 > 0$ be such that both functions remain larger than $\frac{\delta}{2}$ for $t < t_0$ but there exists x_0 such that either $w_1(x_0, t_0) = \frac{\delta}{2}$ or $w_2(x_0, t_0) = \frac{\delta}{2}$. Let this happen for w_1. Since w_1 is supercaloric $x_0 \notin \Omega$. Also x_0 cannot belong to $\partial\Omega$ since in this case we must have $\frac{\partial}{\partial \nu} w_1(x_0, t_0) < 0$ but on the other hand $A(x_0, t_0)$ is nonnegative there, a contradiction.

Corollary 2.1. *Let u_0 , $v_0 \in C^2(\overline{\Omega})$ and nonnegative in $\overline{\Omega}$. Let f and g be strictly positive in $R \quad 0$, nondecreasing in R_+ with f' , g' locally Lipschitz in R. There exists T positive and a solution of (1.1) in $\Omega \times (0, T)$. Moreover there exists a unique maximal solution (u, v). Let $T(u_0, v_0)$ be the time of existence of the maximal solution. If $T(u_0, v_0) < \infty$,*

$$\lim_{t \nearrow T(u_0, v_0)} ||u(\cdot, t)||_{L^\infty(\Omega)} = +\infty$$
$$\lim_{t \nearrow T(u_0, v_0)} ||v(\cdot, t)||_{L^\infty(\Omega)} = +\infty$$

Proof. Let us first assume that u_0 and v_0 are strictly positive.

Let $M \geq 2(||u_0||_{L^\infty(\Omega)} + ||v_0||_{L^\infty(\Omega)})$. Let f_M, g_M be strictly positive in $R \quad 0$ and such that they coincide with f and g for $|s| < M$ and f'_M, g'_M are uniformly Lipschitz continuous. Let (u_M, v_M) be the solution of (1.1) with f and g replaced by f_M and g_M. Since u_M and v_M are continuous in $\overline{\Omega} \times [0, \infty)$ there exists $T > 0$ such that $||u_M||_{L^\infty(\Omega\times(0,T))} + ||v_M||_{L^\infty(\Omega\times(0,T))} \leq M$.Therefore (u_M, v_M) is a solution of problem (1.1) with f and g in $\Omega \times (0, T)$.

Uniquenes of the maximal solution is a consequence of the results in [1].

By Theorem 2.0 and a construction similar to the one above but with $M \geq 2(||u||_{L^\infty(\Omega\times(0,T))} + ||v||_{L^\infty(\Omega\times(0,T))})$ it is clear that if u and v remain bounded in $\Omega \times (0, T)$ they may be extended beyond $t = T$. Thus if $T(u_0, v_0) < \infty$ either u or v should become unbounded.

Let us assume that v remains bounded. Then u is a solution of

$$\begin{array}{ll} u_t = \Delta u & \text{in } \Omega \times (0, T) \\ \frac{\partial}{\partial \nu} u = H(x, t) & \text{on } \partial\Omega \times (0, T) \end{array}$$

with H and $u(x, 0)$ bounded. It is well known that in this case u stays bounded which is a contradiction if $T = T(u_0, v_0)$. So that both functions should blow up at the same time.

Now let u_0 and v_0 be nonnegative. Let (u_n, v_n) be the solution to,

$$\begin{cases} u_t = \Delta u & v_t = \Delta v & \text{in } \Omega \times (0, T_n) \\ \frac{\partial}{\partial\nu} u = f(v) & \frac{\partial}{\partial\nu} v = g(u) & \text{on } \partial\Omega \times (0, T_n) \\ u(x,0) = u_0(x) + \frac{1}{n} & v(x,0) = v_0(x) + \frac{1}{n} & \text{in } \Omega \end{cases} \tag{2.1}$$

¿From the comparison principle (Lemma 2.2) it follows that the sequences u_n and v_n are decreasing, in particular $T_n \geq T_1$ for $n > 1$. Since $u_n \geq 0$, $v_n \geq 0$ there exists $u = \lim_{n \nearrow \infty} u_n$, $v = \lim_{n \nearrow \infty} v_n$ in $\Omega \times (0, T)$ for $T = T_1$.

Let $G(x, t; y, \tau)$ be the Green's function of the heat equation with Neumann boundary conditions. Then

$$u_n(x,t) = \int_\Omega (u_0(y) + \frac{1}{n}) G(x,t;y,0)\, dy + \int_0^t \int_{\partial\Omega} f(v_n(y,\tau)) G(x,t;y,\tau)\, dS_y\, d\tau$$

Passing to the limit we see that u satisfies a similar integral equation.

Let us see that both u and v are continuous in $\overline{\Omega} \times [0, T_1)$. It is enough to see that the convergence is uniform in $\overline{\Omega} \times [0, T_1 - \varepsilon]$ for each $\varepsilon > 0$. In fact, let $u_{nm} = u_n - u_m$, $v_{nm} = v_n - v_m$ and $w_{nm} = u_{nm} + v_{nm}$. Then,

$$\begin{array}{ll} \frac{\partial}{\partial t} w_{nm} = \Delta w_{nm} & \text{in } \Omega \times (0, T_1) \\ \frac{\partial}{\partial\nu} w_{nm} = f(v_n) - f(v_m) + g(u_n) - g(u_m) & \text{on } \partial\Omega \times (0, T_1) \\ w_{nm}(x, 0) = \frac{2}{n} - \frac{2}{m} & \text{in } \Omega \end{array}$$

Since for $t \leq T_1 - \varepsilon$, $u_n \leq u_1 \leq C$, $v_n \leq v_1 \leq C$ and f and g are locally Lipschitz,

$$0 \leq f(v_n) - f(v_m) \leq L v_{nm}$$
$$0 \leq g(u_n) - g(u_m) \leq L u_{nm}$$

therefore, $0 \leq \frac{\partial}{\partial\nu} w_{nm} \leq L w_{nm}$ so that,

$$w_{nm} \leq C(L, T_1, \varepsilon)(\frac{2}{n} - \frac{2}{m}) \tag{2.2}$$

for $t \leq T_1 - \varepsilon$. Since u_{nm} and v_{nm} are nonnegative (2.2) also holds for these functions. Thus u_n and v_m are uniformly convergent for $t \leq T_1 - \varepsilon$ and u and v are continuous in $\overline{\Omega} \times [0, T_1)$.

Thus u is a solution of,

$$\begin{array}{ll} u_t = \Delta u & \text{in } \Omega \times (0, T) \\ \frac{\partial}{\partial\nu} u = f(v) & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{array}$$

Analogously for v. So that (1.1) has a continuous solution in $\overline{\Omega} \times [0, T)$.

¿From Lemma 2.1 we see that if u_0 and v_0 are not both identically 0 they become strictly positive for $t > 0$. Therefore the results we have proved for strictly positive initial data hold in this case.

Corollary 2.2. *Let f' and g' be locally Lipschitz continuous in $R \quad \{0\}$. Let f and g be strictly positive in $R \quad 0$, continuous in R and nondecreasing in R_+. Let u_0 and v_0 be strictly positive on $\partial\Omega$ and in $C^2(\overline{\Omega})$. There exists a unique maximal classical solution to (1.1) which is continuous in $\overline{\Omega} \times [0, T)$. Let T^* be its time of existence. If $T^* < \infty$,*

$$\lim_{t \nearrow T^*} ||u(\cdot, t)||_{L^\infty(\Omega)} = +\infty$$
$$\lim_{t \nearrow T^*} ||v(\cdot, t)||_{L^\infty(\Omega)} = +\infty$$

Proof. In fact, let γ be such that $u_0 \geq \gamma$ and $v_0 \geq \gamma$ on $\partial\Omega$. Let f_γ and g_γ strictly positive in R_+, nondecreasing in R and equal to f and g for $|s| \geq \frac{\gamma}{2}$ be such that their derivatives are locally Lipschitz continuous in R. Let (u_γ, v_γ) be the solution to (1.1) with $f,\ g$ replaced by $f_\gamma,\ g_\gamma$. Since u_γ and v_γ are continuous up to $t = 0$ there exists $T > 0$ such that $u_\gamma \geq \frac{\gamma}{2}$ and $v_\gamma \geq \frac{\gamma}{2}$ on $\partial\Omega$ for $0 \leq t \leq T$. So that (u_γ, v_γ) is a solution to (1.1) with f and g for $t < T$.

Let $(\overline{u}, \overline{v})$ be a continuous solution. Since for some $\overline{T} > 0$ we have $\overline{u} \geq \frac{\gamma}{2}$, $\overline{v} \geq \frac{\gamma}{2}$ on $\partial\Omega$ for $0 \leq t \leq \overline{T}$, $(\overline{u}, \overline{v})$ is a continuous classical solution with f_γ and g_γ in $\Omega \times (0, \overline{T})$. By uniqueness $\overline{u} = u_\gamma$, $\overline{v} = v_\gamma$ in $\Omega \times (0, \min(T, \overline{T}))$. On the other hand there exists $\delta > 0$ such that $\overline{u} = u_\gamma \geq \delta$, $\overline{v} = v_\gamma \geq \delta$ in $\overline{\Omega}$ for $0 < \tau \leq t < \min(T, \overline{T})$. Therefore $(\overline{u}, \overline{v})$ and (u_γ, v_γ) are solutions of (1.1) with f and g replaced by f_δ and g_δ smooth at the origin and equal to f and g for $|s| \geq \min(\frac{\gamma}{2}, \delta)$. So that $u_\gamma = \overline{u}$ and $v_\gamma = \overline{v}$ as long as they exist.

¿From the arguments above we see that we may apply the results of Corollary 2.1 to deduce that if the maximal time of existence is finite, both u and v blow up in L^∞ norm at this time.

Remark. From now on we will always assume that u_0 and v_0 are nonnegative. If f and g are smooth at the origin there will be no further assumptions. If not we will assume that they are strictly positive on $\partial\Omega$. Also we will always assume that f and g are strictly positive in $R \quad 0$, nonnegative in R and nondecreasing in R_+.

III. Global Existence and Blow up Results.

Following the ideas in [13] we will construct global supersolutions or blowing up subsolutions.

Theorem 3.1. *Let f and g strictly positive in $R \quad 0$, nonnegative in R with f' and g' locally Lipschitz continuous in $R \quad \{0\}$ and nonnegative in R_+. Let F and G be such that they vanish at 0 and $F' = f$ and $G' = g$. Assume that Ω is smooth and there exist positive constants and s_0 such that (1.2) holds. If also,*

$$\int^\infty \frac{ds}{f(F^{-1}(G(s) + c))} = \int^\infty \frac{ds}{g(s) f'(F^{-1}(G(s) + c))} = +\infty$$

and the monotonicity condition (1.3) holds, every positive solution is global.

Proof. We will construct supersolutions that are initially larger than any positive constant. From Remark 3.1 it is clear that this is all we need in order to prove our result.

Let $\overline{u}(x,t) = \psi(a(x)+b(t))$, $\overline{v}(x,t) = \alpha(a(x)+b(t))$. Let us take as (ψ,α) a solution of,

$$\begin{aligned} \psi' &= f(\alpha) \\ \alpha' &= g(\psi) \end{aligned} \tag{3.1}$$

so that the boundary conditions are satisfied if we take as $a(x)$ a $C^2(\overline{\Omega})$ function such that $\frac{\partial}{\partial\nu}a = 1$ on $\partial\Omega$. Let $L \geq \Delta a$ in Ω. Then $(\overline{u},\overline{v})$ is a supersolution of (1.1) if,

$$\begin{aligned} b'(t) &\geq |\nabla a|^2 \frac{\psi''(s)}{\psi'(s)} + L \\ b'(t) &\geq |\nabla a|^2 \frac{\alpha''(s)}{\alpha'(s)} + L \end{aligned} \tag{3.2}$$

for $s = a(x)+b(t)$.

Let us multiply the first equation in (3.1) by $g(\psi)$ and the second by $f(\alpha)$ and substract. An integration gives,

$$F(\alpha(s)) - G(\psi(s)) = c \tag{3.3}$$

for some arbitrary constant c. Let us take as initial condition (ψ_0,α_0) a pair of arbitrarily large constants such that $F(\alpha_0) = G(\psi_0)$ so that $c = 0$ in (3.3).

On the other hand, since f and g are positive for $s \neq 0$, F and G are strictly increasing therefore (3.3) is equivalent to,

$$\alpha(s) = F^{-1}(G(\psi(s)))$$

and (3.1) becomes,

$$\psi' = f(F^{-1}(G(\psi)))$$

Since by hypothesis,

$$\int^{\infty} \frac{ds}{f(F^{-1}(G(s)))} = +\infty$$

$\psi(s)$ exists globally for any positive initial datum. Therefore any solution of (3.1) is global.

Let us observe that $\lim_{s\nearrow\infty}\psi(s) = \infty$ and the same is true for α. In fact, both ψ and α are strictly increasing thus $\alpha(s) > \alpha(0) > 0$. Therefore, $\psi'(s) \geq f(\alpha(0)) > 0$ and $\lim_{s\nearrow\infty}\psi(s) = \infty$. And analogously for α.

Thus in order to get a global supersolution initially larger than any positive constant we only have to find a globally defined function $b(t)$ that verifies (3.2) with $a(x)$ as above and $b(0)$ arbitrarily large.

On the other hand,

$$\frac{\psi''(s)}{\psi'(s)} = \frac{f'(\alpha(s))g(\psi(s))}{f(\alpha(s))} = \frac{f'(F^{-1}(G(\psi(s))))g(\psi(s))}{f(F^{-1}(G(\psi(s))))} = p(\psi(s)) \tag{3.4}$$

and

$$\frac{\alpha''(s)}{\alpha'(s)} = \frac{g'(\psi(s))f(F^{-1}(G(\psi(s))))}{g(\psi(s))}$$

From the monotonicity condition (1.3) and condition (1.2) we see that in order to verify (3.2) b only needs to satisfy,

$$b'(t) \geq kp(\psi(N + b(t)) + L \tag{3.5}$$

where $N = 0$ if $p(z)$ is decreasing and $N = \max_\Omega a(x)$ if it is increasing.
Now,

$$\int^\infty \frac{ds}{p(\psi(s))} = \int^\infty \frac{dz}{g(z)f'(F^{-1}(G(z)))} = \infty$$

Therefore there is a global solution to (3.5) for any initial datum b_0.

Conversely,

Theorem 3.2. *Let f, g and Ω as in Theorem 3.1. Let us assume again that the monotonicity condition (1.3) holds. Let (1.2') be true. If either*

$$\int^\infty \frac{dz}{f(F^{-1}(G(z)))} < \infty \tag{3.6}$$

or,

$$\int^\infty \frac{dz}{g(z)f'(F^{-1}(G(z)))} < \infty \tag{3.7}$$

every positive solution to (1.1) blows up in finite time.

Proof. Let (ψ, α) as in Theorem (3.1). If (3.6) happens both ψ and α blow up in finite time for any initial datum. If not they exist globally.

Let $a(x) = \varepsilon|x - x_0|^2$ where $x_0 \notin \overline{\Omega}$. Let $\underline{u}(x,t) = \psi(a(x) + b(t))$ and $\underline{v}(x,t) = \alpha(a(x) + b(t))$. If ε is small the boundary condition,

$$\frac{\partial}{\partial\nu}\underline{u} \leq f(\underline{v})$$
$$\frac{\partial}{\partial\nu}\underline{v} \leq g(\underline{u})$$

holds. (Also ε, $b(0)$, $\psi(0)$ and $\alpha(0)$ should be so small as to have $\underline{u}(x,\tau) < \gamma_\tau$ and $\underline{v}(x,\tau) < \gamma_\tau$). So that in order to have a subsolution to (1.1) it is enough to verify,

$$b'(t) \leq 2n\varepsilon + (2n\varepsilon\, d)^2 \frac{\psi''}{\psi'}(a(x) + b(t))$$
$$b'(t) \leq 2n\varepsilon + (2n\varepsilon\, d)^2 \frac{\alpha''}{\alpha'}(a(x) + b(t))$$

where $d = dist(x_0, \Omega)$.

Conditions (1.2') and (1.3) and identity (3.4) show that it is enough to have,

$$\begin{aligned} b' &= 2n\varepsilon \qquad \text{for } t \le t_0 \text{ (where } t_0 \text{ is such that } \psi(2n\varepsilon t_0) \ge s_0) \\ b' &= (2n\varepsilon\, d)^2 \quad {}^{-1}p(\psi(N+b)) \qquad \text{if not} \end{aligned} \tag{3.8}$$

with $N = 0$ if p is increasing and $N = \max_\Omega a(x)$ if it is decreasing.

If condition (3.6) is not true, condition (3.7) is. Therefore any solution of (3.8) blows up in finite time and there is nothing else to prove since $\psi \nearrow \infty$ and $\alpha \nearrow \infty$. (Remember that as soon as $t \ge \tau > 0$, $u(x,t) \ge \gamma_\tau > 0$, $v(x,t) \ge \gamma_\tau$).

If (3.6) holds instead of taking b as a solution to (3.8) we take it as a solution to $b' = 2n\varepsilon$ so that it becomes larger than any positive constant in finite time and therefore $\underline{u} \nearrow \infty$ and $\underline{v} \nearrow \infty$ in finite time.

The Theorem is proved.

IV. Examples.

In this Section we will apply the results of Section III to several examples.

Let $F' = f$, $G' = g$ with $F(0) = G(0) = 0$. Let us assume that,

$$F(z) = G(z^\beta) \tag{4.1}$$

then, $f(z) = \beta z^{\beta-1} g(z^\beta)$ and $f'(z) = \beta(\beta-1)z^{\beta-2}g(z^\beta) + \beta^2 z^{2(\beta-1)}g'(z^\beta)$. Let (ψ, α) be a positive solution of (3.1). Then,

$$\begin{aligned} \frac{\psi''}{\psi'} &= (\beta-1)\frac{g(\alpha^\beta)}{\alpha} + \beta\alpha^{\beta-1}g'(\alpha^\beta) \\ \frac{\alpha''}{\alpha'} &= \beta\alpha^{\beta-1}g'(\alpha^\beta) \end{aligned}$$

so that if $\beta \ge 1$,

$$\frac{\psi''}{\psi'} \ge \frac{\alpha''}{\alpha'} \tag{4.2}$$

which is equivalent to condition (1.2) with $\ = 1$ since (4.2) holds for any initial datum.

On the other hand, if $\beta \le 1$ (1.2') is true.

Also we would like to remark that when $\beta \ge 1$ and $\frac{g(z)}{z} \le Kg'(z)$ both (1.2) and (1.2') hold.

In this Section we will analyze two examples:

- Case (a) $f(z) = (p+1)z^p$, $g(z) = (q+1)z^q$ with $p,\ q > 0$
- Case (b) $f(z) = (1+z)\log(1+z)$, $g(z) = (\frac{p+1}{2})^2(1+z)^p\log(1+z)$

In Case(a) we have $\beta = \frac{(p+1)}{(q+1)}$ and $\frac{g(z)}{z} = (q+1)z^{q-1}$, $g'(z) = (q+1)qz^{q-1}$, so that both (1.2) and (1.2') hold for any $p \geq q > 0$. (By the symmetry of the problem we can always assume that this is the case. In fact all the other conditions are verified independently of the order relation between p and q).

In Case(b) we first make a change of variables so that we actually consider $f(z) = z \log z$ and $g(z) = (\frac{p+1}{2})^2 z^p \log z$ (and consequently we ask the initial data to be larger than or equal to 1 everywhere in Ω. We have $\frac{1}{\beta} = \frac{(p+1)}{2}$ so that $\beta \geq 1$ if and only if $p \leq 1$. So that (1.2) holds for $p \leq 1$ and (1.2') for $p \geq 1$.

Also in case (a) the monotonicity condition (1.3) holds. In fact, $p(z) = p(q+1)$ $z^{\frac{pq-1}{p+1}}$.

On the other hand in case (b) $p(z) = \frac{(p+1)}{2} z^{\frac{p-1}{2}} \left[\frac{(p+1)}{2} \log z + 1 \right]$. So that the monotonicity condition holds if $p \geq 1$ but if $p < 1$, $p(z)$ increases as long as $z \leq z_0$ and then it decreases. We will see that a small modification of the arguments in Section III will handle this case.

Theorem 4.1. *Let $p \geq q > 0$ and $f(z) = (p+1)z^p$, $g(z) = (q+1)z^q$. If $pq \leq 1$ every positive solution to (1.1) is global whereas if $pq > 1$ every positive solution blows up in finite time.*

Proof. We only have to analyze the convergence of the integrals,

$$\int^{\infty} \frac{ds}{f(F^{-1}(G(s)))} = \int^{\infty} \frac{ds}{(p+1)z^{p\frac{q+1}{p+1}}} \tag{4.3}$$

and,

$$\int^{\infty} \frac{ds}{g(s)f'(F^{-1}(G(s)))} = \int^{\infty} \frac{ds}{p(p+1)(q+1)z^{\frac{2pq+p-1}{p+1}}} \tag{4.4}$$

so that both (4.3) and (4.4) are finite if and only if $pq > 1$.

Theorem 4.2. *Let $f(z) = z \log z$ and $g(z) = (\frac{p+1}{2})^2 z^p \log z$ with $p > 0$ and $u_0 \geq 1$, $v_0 \geq 1$ in (1.1). If $p < 1$ every solution is global whereas if $p \geq 1$ every solution blows up in finite time.*

Proof. If $p \geq 1$ the function $p(z)$ of condition (1.3) is monotone increasing therefore we only have to analyze the convergence of the integrals.

$$\int^{\infty} \frac{ds}{f(F^{-1}(G(s)))} = \frac{2}{p+1} \int^{\infty} \frac{ds}{s^{\frac{p+1}{2}} \log s} = \frac{2}{p+1} \int^{\infty} \frac{e^{\frac{1-p}{2}z}}{z}\, dz \tag{4.5}$$

and,

$$\begin{aligned} \int^{\infty} \frac{ds}{g(s)f'(F^{-1}(G(s)))} &= (\frac{2}{p+1})^2 \int^{\infty} \frac{ds}{s^p \log s(\frac{p+1}{2}\log s + 1)} = \\ &= (\frac{2}{p+1})^2 \int^{\infty} \frac{e^{(1-p)z}}{z(\frac{p+1}{2}z + 1)}\, dz \end{aligned} \tag{4.6}$$

If $p > 1$ both integrals are convergent. If $p = 1$ (4.5) is divergent but (4.6) convergent. Thus if $p \geq 1$ every solution blows up in finite time.

Now let $p < 1$ in this case $p(z)$ is increasing for $z \leq z_0$ and decreasing thereafter. We prove that every solution is global. In fact we only need to construct global supersolutions that are initially large so that we may assume that $\psi(b(0)) > z_0$ and take b as the solution of,

$$b' = kp(\psi(N+b)) + L$$

since $p(z)$ is decreasing in the range where $\psi(N+b)$ moves. Thus since for $p < 1$ both integrals in (4.5) and (4.6) diverge and (1.2) holds, every solution is global.

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