BLOW-UP VS. GLOBAL EXISTENCE FOR A SEMILINEAR REACTION-DIFFUSION SYSTEM IN A BOUNDED DOMAIN

by

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Dedicated to the memory of our teacher Prof. Julio E. Bouillet

I. Introduction and description of the results

In this papers we study the following semilinear parabolic system,

$$(1.1)\qquad \begin{cases} u_t = \Delta u + v^p e^u & \Omega \times (0,T) \\ v_t = \Delta v + u^q e^v & \Omega \times (0,T) \end{cases}$$

with Dirichlet boundary conditions,

$$(1.2)\qquad \begin{cases} u\big|_{\partial\Omega\times(0,T)} = 0 \\ v\big|_{\partial\Omega\times(0,T)} = 0 \end{cases}$$

and continuous, nonnegative initial data,

$$(1.3)\qquad \begin{cases} u(x,0) = u_0(x) \\ v(x,0) = v_0(x) \end{cases}$$

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where Ω is a smooth, bounded domain in ^{n}.

(1.1) constitutes a simple example of a semilinear reaction-diffusion system exhibiting nontrivial coupling in the unknowns. Local in time existence and uniqueness of solution of (1.1)-(1.3) is well known when $p, q \geq 1$ in which case the reaction term is a locally Lipschitz function of the unknowns. For $p, q > 0$ the existence result is obtained by a simple modification of an argument given in [2].

The main purpose of this paper is to investigate the global existence (resp. the finite time blow up) of the solutions.

More precisely, for a given solution $(u(x,t), v(x,t))$ we define $T^* = T^*(u,v)$ by,

$$T^* = \sup\{T > 0;\ (u,v) \text{ are bounded and satisfy (1.1)-(1.3) in } Q_T\}$$

We want to discuss the possibility of $T^* = +\infty$ (global existence) or $T^* < +\infty$ (blow-up) in terms of p, q, u_0, v_0 and Ω.

Note that if $T^* < +\infty$ one has

$$\limsup_{t \nearrow T^*} \|u(x,t)\|_{L^\infty(\Omega)} = +\infty \qquad \text{or}$$
$$\limsup_{t \nearrow T^*} \|v(x,t)\|_{L^\infty(\Omega)} = +\infty$$

since if not the solution could be continued beyond T^* (cf. [4]). We say that the solution under consideration blows-up in finite time.

In recent years there has been an increasing interest in blow-up problems for nonlinear systems.

In particular in [1] Escobedo and Herrero study the system

$$\text{(1.4)} \qquad \begin{cases} u_t = \Delta u + v^p \\ v_t = \Delta v + u^q \end{cases}$$

in $\Omega = {}^n$, and in [2] system (1.4) in Ω bounded with Dirichlet boundary conditions.

In [2] the authors show that for $p \cdot q \leq 1$ every solution of (1.4) is global and for $p \cdot q > 1$ there are solutions that blow-up and others that are global.

We find that (1.1)-(1.3) exhibites a rather different behavior when $p \cdot q \leq 1$. In fact, the exponential terms force blow up of solutions with large enough initial data, and this is true for any domain Ω and any $p, q > 0$. Moreover, in this case there are domains in which every solution blows up in finite time while in others there are both global and blowing up solutions. The blow up case is characterized by the "thickness" of the domain. In fact, we prove the following result (we recall here that we are only considering positive solutions),

THEOREM 1.1 *Let* $p \cdot q \leq 1$,

a) *For every* Ω *there exist blowing up solutions*

b) i) *If* Ω *is "thin" in one direction that is, if it is contained in a region between to parallel hyperplanes that are close enough to each other, global solutions do exist.*

ii) *If* Ω *is "thick" enough that is, if it contains a large enough ball, global solutions do not exist (every solution blows up)*

On the other hand, as for (1.4) if $p \cdot q > 1$ global solutions of (1.1)-(1.3) exist in any domain Ω. We prove the following result,

THEOREM 1.2 *Let* $p \cdot q > 1$. *For every* Ω *there exist both global and blowing up solutions.*

Since for $u_0 = v_0$ one has $u = v$ for $t > 0$, we get as a Corollary of these two theorems a similar result for the single equation (that had not been considered in the literature previously).

THEOREM 1.3 *Consider the problem,*

$$(1.5) \qquad \begin{cases} u_t = \Delta u + u^p e^u & \text{in } \Omega \times (0,T) \\ u\big|_{\partial\Omega} = 0 & \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) > 0 & \end{cases}$$

a) *If* $p > 1$ *or* $p \leq 1$ *and* Ω *"thin", there exist both global and blowing up solutions.*

b) *If $p \leq 1$ and Ω is "thick", every solution blows up in finite time.*

Let us observe that every solution of the associated ordinary system

$$\begin{cases} x' = y^p \cdot e^x \\ y' = x^q \cdot e^y \end{cases}$$

blows-up in finite time. The diffusion term plays a crucial role and allows the system to have global solutions for every domain if $p \cdot q > 1$, and for certain "thin" ones if $p \cdot q \leq 1$.

Another work related to ours is [3]. The authors characterize the existence of blowing-up or global solutions for

$$\begin{cases} u_t = \Delta u + u^{p_1} \cdot v^{q_1} \\ v_t = \Delta v + u^{p_2} \cdot v^{q_2} \end{cases} \tag{1.6}$$

in ^{n} or in cones in terms of p_1, p_2, q_1 and q_2. To our knowledge, no general study was available for (1.1)-(1.3) prior to this work.

The paper is organized as follows. In Section II we state the local existence and uniqueness results and we show that global solutions exist for $p \cdot q > 1$ in every domain Ω and for $p \cdot q \leq 1$ if Ω is "thin" in one direction. In Section III we prove that blow-up solutions exist in every Ω for any p, q positive and we show that for "thick" domains Ω every positive solution blows-up in finite time.

II. Global existence of solutions

By a nonnegative classical solution of (1.1)-(1.3) in some cylinder $Q_T = \Omega \times (0,T)$ $(T \leq +\infty)$ we will mean a pair of nonnegative $\mathbf{C}^{2,1}$ functions $(u(x,t), v(x,t))$ such that they satisfy (1.1), (1.2), (1.3) in Q_T.

Local existence and uniqueness of nonnegative solutions is well known if $p \geq 1$ and $q \geq 1$, since in this case the reaction term is locally Lipschitz continuous in the unknowns.(see for instance [5]).

For general $p, q > 0$, local existence follows in a standard way appoximating the source terms by locally Lippschitz continuous functions. In fact, let (u_n, v_n), $u_n \geq 1/n$, $v_n \geq 1/n$ be the maximal solution to

$$\begin{cases} u_t - \Delta u = f_n(v)e^u & \text{in } \Omega \times (0,T) \\ v_t - \Delta v = g_n(u)e^v & \\ u = v = 1/n & \text{in } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) + 1/n & v(x,0) = v_0(x) + 1/n \end{cases}$$

where $f_n(s) = s^p$ for $s \geq 1/n$, $g_n(s) = s^q$ for $s \geq 1/n$, f_n, g_n locally Lipschitz in .

It is easy to see that (u_n, v_n) are nonincreasing sequences. Since $u_n \geq 0$, $v_n \geq 0$ there exist $u = \lim_n u_n$, $v = \lim_n v_n$. And we see that (u, v) is a solution of (1.1)-(1.3) by passing to the limit in the "variation of constants formula". These idea applied to (1.4) can be found in [1] and [2].

Remark. Let (u, v) be a nonnegative solution to (1.1)-(1.3) then u and v are supercaloric functions. Therefore $u(x,t)$, $v(x,t)$ are strictly positive in Ω for $t > 0$.

In consequence when investigating the large time behavior of solutions of (1.1)-(1.3) we may assume without loss of generality that $u_0 > 0$, $v_0 > 0$ in Ω.

We now prove a comparison Lemma.

Lemma 2.1. *If $(\overline{u}, \overline{v})$ is a solution of*

$$\begin{cases} \overline{u}_t \geq \Delta\overline{u} + \overline{v}^p \cdot e^{\overline{u}} \\ \overline{v}_t \geq \Delta\overline{v} + \overline{u}^q \cdot e^{\overline{v}} \end{cases} \tag{2.1}$$

$$\begin{cases} \overline{u}(x,0) \geq u_0(x) + \delta \\ \overline{v}(x,0) \geq v_0(x) + \delta \end{cases} \tag{2.2}$$

$$\begin{cases} \overline{u}\big|_{\partial\Omega} \geq \delta \\ \overline{v}\big|_{\partial\Omega} \geq \delta \end{cases} \tag{2.3}$$

and if (u,v) is a solution of (1.1)-(1.3) with (u_0,v_0) as initial data, then

$$u(x,t) \le \overline{u}(x,t)$$
$$v(x,t) \le \overline{v}(x,t)$$

We call $(\overline{u},\overline{v})$ a supersolution of (1.1)-(1.3).

Proof. Let

$$t_0 = \sup\{t/(\overline{u}-u)(x,\tau) > \delta/2,\ (\overline{v}-v)(x,\tau) > \delta/2 \quad \text{in } \Omega\,,\ \text{for } 0<\tau<t\}$$

$t_0 > 0$. Let us assume that $t_0 < T^*(\overline{u},\overline{v})$. There exists $x_0 \in \Omega$ such that either $(\overline{u}-u)(x_0,t_0) = \delta/2$ or $(\overline{v}-v)(x_0,t_0) = \delta/2$. Let us assume that $(\overline{u}-u)(x_0,t_0) = \delta/2$.

We have

$$(\overline{u}-u)_t \ge \Delta(\overline{u}-u) + \overline{v}^p \cdot e^{\overline{u}} - v^p \cdot e^u \ge \Delta(\overline{u}-u) \quad \text{in } \Omega\times(0,t_0)$$

but on the parabolic boundary of $\Omega\times(0,t_0)$ $(\overline{u}-u)\ge\delta$ and the contradiction follows by an application of the minimum principle. □

We can now prove the following

LEMMA 2.2. *Let φ be such that $\Delta\varphi = -1$ in Ω, $\varphi \ge 0$, $\varphi\big|_{\partial\Omega} = 0$. If there exist $a,b>0$ such that for some $\delta>0$,*

$$\begin{cases} a \ge b^p(\delta+\varphi(x))^p \cdot e^{a(\delta+\varphi(x))} \\ b \ge a^q(\delta+\varphi(x))^q \cdot e^{b(\delta+\varphi(x))} \end{cases} \tag{2.4}$$

then there exists a global supersolution.

Proof. It suffices to take

$$\overline{u}(x,t) = a(\delta+\varphi(x))$$
$$\overline{v}(x,t) = b(\delta+\varphi(x))$$

then,

$$0 \ge -a + b^p(\delta+\varphi(x))^p \cdot e^{a(\delta+\varphi(x))}$$
$$0 \ge -b + a^q(\delta+\varphi(x))^q \cdot e^{b(\delta+\varphi(x))}$$

Thus inequalities (2.1) hold and the verification of the boundary conditions (2.2)-(2.3) is trivial. $\square$

LEMMA 2.3. a) *If $p \cdot q > 1$ there exists a global supersolution of (1.1)-(1.3) for every domain Ω.*

b) *If $p \cdot q \leq 1$ there exists a global supersolution of (1.1)-(1.3) if $\sup \varphi$ is sufficiently small, where φ is the function of Lemma 2.2. In particular, this is true if Ω is thin in one direction (that is, if Ω is located between two parallel hyperplanes that are close enough to each other).*

Proof. We set

$$K = \sup_{x \in \Omega} (\delta + \varphi(x))$$

and we have to find a, b that satisfy (2.4). We need

$$\begin{cases} a \geq b^p \cdot K^p \cdot e^{aK} \\ b \geq a^q \cdot K^q \cdot e^{bK} \end{cases} \tag{2.5}$$

If $p \cdot q > 1$ we choose $b^p = \dfrac{a}{K^p \cdot e^{aK}}$ and we have to verify that there exists $a > 0$ such that (in order to satisfy the second inequality),

$$\frac{1}{K^p e^{aK}} \geq a^{(p \cdot q - 1)} K^{pq} \cdot \exp\left(p \cdot \left(\frac{a}{e^{aK}}\right)^{\frac{1}{p}}\right) = h(a)$$

But we observe that $\lim\limits_{a \to 0+} h(a) = 0$ if $p \cdot q > 1$. So a) is true.

For b) we observe that for any $p, q > 0$ if K is small enough then (2.5) holds for some $a, b > 0$. In fact, if

$$\max\left\{K^p\, e^K,\, K^q\, e^K\right\} \leq 1$$

there exist $0 < a, b \leq 1$ such that

$$a \geq K^p\, e^K$$
$$b \geq K^q\, e^K$$

so that (2.5) holds.

Observe that K is small if and only if $\sup\limits_{\Omega} \varphi(x)$ and δ are small.

As an example if $\Omega = (0,\ell)$ is an interval, $\varphi(x) = \dfrac{x(\ell - x)}{2}$, and $\sup\limits_{x\in\Omega} \varphi(x) = \dfrac{\ell^2}{8}$.

We show that if the domain is thin enough in one direction conclusion b) holds. To this end we take a coordinate system such that

$$\Omega \subset\subset [0,\ell] \times \prod_{j=2}^{n} [0,k_j] = R$$

We take for $x_1 \in [0,\ell],\ \ y \in \prod\limits_{j=2}^{n} [0,k_j]$,

$$\psi(x_1,y) = \frac{1}{2} x_1(\ell - x_1)$$

Then $\Delta\psi = -1$ in R, $\ \psi\big|_{\partial R} \geq 0$.

Since $\psi > 0$ in $\overline{\Omega}$, by the comparison principle $\psi \geq \varphi$ in Ω. But

$$\|\psi\|_{L^\infty(R)} \leq \frac{\ell^2}{8}$$

so that this same estimate holds for φ.

Finally we choose ℓ and δ small so that K is small enough (we have $K \ \leq \delta + \dfrac{\ell^2}{8}$). $\square$

We can now conclude

THEOREM 2.1. a) *If $p \cdot q > 1$ then (1.1)-(1.3) has global solutions for small data (u_0,v_0) on an arbitrary Ω.*

b) *If $p \cdot q \leq 1$ and Ω is "thin" in one direction problem (1.1)-(1.3) has global solutions for small data (u_0,v_0).*

Proof. It suffices to combine Lemma 2.1 with Lemma 2.3 and take $u_0(x) \leq a\,\varphi(x),\ \ v_0(x) \leq b\,\varphi(x)$. $\square$

III. Blow-up results

We first state a well known result about a single equation. We prove it here in order to make this work self contained.

Lemma 3.1. *Let $w(x,t)$ be a solution of the following problem*

$$\begin{aligned} &w_t = \Delta w + \lambda \cdot e^w \\ &w\big|_{\partial B} = 0 \qquad\qquad \text{in } B \times (0,T) \\ &w_0 = w(x,0) > 0 \end{aligned}$$

then if $\lambda > \lambda_1$, w blows-up in finite time (λ_1 is the first eigenvalue of $-\Delta$ in B).

Proof. We use the "eigenfunction method". Let ϕ be the first eigenfunction of $-\Delta$ in B, ϕ satisfies $-\Delta\phi = \lambda_1\phi$, $\phi\big|_{\partial B} = 0$, $\phi > 0$ in Ω.

If we take

$$F(t) = \int_B w(x,t)\phi(x)dx$$

we observe that,

$$F'(t) = \int_B w_t \cdot \phi = \int_B \Delta w\phi + \lambda \int_B e^w \cdot \phi$$

By integration by parts and Jensen's inequality (we may assume $\int_B \phi = 1$)

$$F'(t) \geq \int_B w \cdot \Delta\phi + \lambda \cdot e^{\int_B w\phi} = -\lambda_1 F(t) + \lambda \cdot e^{F(t)}$$

and if $\lambda > \lambda_1$ then $F(t)$ cannot be global. $\square$

Theorem 3.1. *Let $p, q > 0$ and Ω arbitrary. Fix (u_0, v_0) with $u_0 > 0$, $v_0 > 0$ in Ω. There exists $\gamma_0 > 0$ such that, if $\gamma > \gamma_0$ then the solution of (1.1)-(1.3) with initial data $(u_0, \gamma \cdot v_0)$ blows-up in finite time.*

Proof. Let B be a ball such that $u_0\big|_B > \varepsilon > 0$ and $B \subset\subset \Omega$. (This ball exists because u_0 is positive and continuous.)

We take a λ, fixed, such that $\lambda > \lambda_1$ and w as in Lemma 3.1 with $0 < w(x,0) < \varepsilon$ in B. Then w blows-up in finite time, say T.

We consider now the following linear problem

$$\text{(3.1)} \qquad \begin{cases} \psi_t = \Delta\psi & \text{in } \Omega \times (0, +\infty) \\ \psi\big|_{\partial\Omega} = 0 & \text{in } \partial\Omega \times (0, +\infty) \\ \psi(x,0) = v_0(x), \end{cases}$$

and define $\alpha = \inf\limits_{\substack{x\in B\\ 0\le t\le T}} \psi$, α is positive since v_0 is positive in $\overline{B}$.

Let us assume that (u_γ, v_γ) exists and is bounded up to time T ((u_γ, v_γ) is the solution of (1.1)-(1.3) with initial data $(u_0, \gamma v_0)$).

Next we observe that $\gamma \cdot \psi(x,t)$ is a solution of (3.1) with initial data $\gamma \cdot v_0(x)$ and therefore $\gamma \cdot \psi(x,t) \le v_\gamma(x,t)$. (We use the comparison principle in Ω and the fact that v_γ is a supersolution of the heat equation.)

It then follows for every $\gamma > \gamma_0 = \dfrac{\lambda^{1/p}}{\alpha}$

$$\inf_{\substack{x\in B\\ 0\le t\le T}} v_\gamma(x,t) \ge \inf_{\substack{x\in B\\ 0\le t\le T}} \gamma \cdot \psi(x,t) \ge \gamma \cdot \alpha > \lambda^{1/p}$$

We use this fact to compare $u(x,t)$ with $w(x,t)$. We claim that $u(x,t) \ge w(x,t)$ in $B \times [0,T)$. In fact $u(x,t)$ satisfies

$$(3.2) \qquad \begin{cases} u_t = \Delta u + v^p \cdot e^u \ge \Delta u + \lambda \cdot e^u & B \times (0,T) \\ u\big|_{\partial B} > 0 & \partial B \times (0,T) \\ u(x,0) > w(x,0) & \text{in } B \end{cases}$$

We may apply the comparison principle again. Therefore u_γ cannot remain bounded up to time T. $\square$

We observe that γ depends not only of v_0 but also of u_0, p and $B \subset\subset \Omega$. On the other hand we are only using that v is supercaloric (this is, $\Delta v \le v_t$).

Remark. It is clear from the proof of Theorem 3.1 that if $u \nearrow \infty$ or $v \nearrow \infty$ uniformly in a ball B completly contained in Ω, (u,v) should blow up in finite time (it cannot be global).

Finally we prove a blow-up result for every initial data (u_0, v_0) in "thick" domains when $p \cdot q \le 1$ where for us a domain is thick if it contains a large enough ball.

First we take B_1 the unit ball in ^{n} and make the following construction.

Take a function $\phi_1(x)$ with domain B_1 such that $\phi_1 > 0$ in B_1^0

$$\phi_1\Big|_{\partial B_1} \equiv 0$$
$$\Delta\phi_1(x) \ge \alpha > 0 \quad \text{if} \quad \phi_1(x) \le c_1$$

Next we dilate it by taking

$$\phi_k(x) = \phi_1\left(\frac{x}{k}\right) \qquad (k \in \mathbf{R}_{>0})$$

Now $\phi_k(x)$ satisfies

$$\begin{aligned}
&\phi_k(x) : B_k \to \mathbf{R}_+ \qquad \phi_k > 0 \text{ in } B_k^0 \\
&\phi_k\Big|_{\partial B_k} \equiv 0 \\
&\Delta\phi_k(x) \geq \frac{\alpha}{k^2} > 0 \qquad \text{if } \phi_k(x) \leq c_1 \\
&\Delta\phi_k(x) \geq -\frac{L}{k^2} \qquad (-L = \inf_{x\in B_1} \Delta\phi_1(x))
\end{aligned}$$

We propose

$$(3.3) \qquad \begin{cases} \underline{u}(x,t) = a(t)\phi_k(x) \\ \underline{v}(x,t) = b(t)\phi_k(x) \end{cases}$$

as a subsolution of (1.1)-(1.3) in B_k. We want to choose a, b increasing positive functions that tend to infinity as t increases with $a(0)$, $b(0)$ small.

We have to verify

$$(3.4) \qquad \begin{cases} \underline{u}_t(x,t) \leq \Delta\underline{u} + \underline{v}^p \cdot e^{\underline{u}} \\ \underline{v}_t(x,t) \leq \Delta\underline{v} + \underline{u}^q \cdot e^{\underline{v}} \end{cases}$$

this is,

$$(3.5) \qquad \begin{cases} a'(t)\phi_k(x) \leq a(t)\Delta\phi_k(x) + b(t)^p\phi_k(x)^p e^{a(t)\phi_k(x)} \\ b'(t)\phi_k(x) \leq b(t)\Delta\phi_k(x) + a(t)^q\phi_k(x)^q e^{b(t)\phi_k(x)} \end{cases}$$

If $x \in \{\phi_k(x) \leq c_1\}$ we impose,

$$(3.6) \qquad \begin{cases} a'(t) \cdot c_1 \leq \dfrac{\alpha}{k^2} \cdot a(0) \\ b'(t) \cdot c_1 \leq \dfrac{\alpha}{k^2} \cdot b(0) \end{cases}$$

And if x is such that $\phi_k(x) \geq c_1$ then $(c_2 = \sup \phi_k(x))$

$$(3.7) \qquad \begin{cases} a'(t) \cdot c_2 \leq -a(t)\dfrac{L}{k^2} + b^p(t) \cdot c_1^p \cdot e^{a(t)\cdot c_1} \\ b'(t) \cdot c_2 \leq -b(t)\dfrac{L}{k^2} + a^q(t) \cdot c_1^q \cdot e^{b(t)\cdot c_1} \end{cases}$$

Since $p \cdot q \leq 1$ either p or q should be smaller than 1. Let us assume that $p \leq 1$. Now if $b(t)^p = a(t)$ and $b(t) = \varepsilon t + b(0)$ using the fact that $p \cdot q \leq 1$, $(a(t), b(t))$ satisfy (3.6)-(3.7), provided that k is large enough independently of how small $a(0)$ and $b(0)$ should be.

In order to prove (3.6) it is enough to verify,

$$\begin{cases} \varepsilon \cdot p \cdot b(0)^{p-1} \cdot c_1 \leq \dfrac{\alpha}{k^2} \cdot b(0)^p \\ \varepsilon \cdot c_1 \leq \dfrac{\alpha}{k^2} \cdot b(0) \end{cases}$$

The constant k will be fixed later on. Once k is fixed it is clear that (3.6) will be satisfied if ε is small enough depending on $b(0)$.

Now let us consider (3.7). Let us take k large enough so that $\frac{L}{k^2} \leq \frac{c_1^p}{2}$. Then the first equation in (3.7) is satisfied if,

$$\varepsilon \leq \frac{b^{1-2p}(0) \cdot c_1^p}{2pc_2}$$

With this choice of a and b the second equation in (3.7) reads,

$$\varepsilon \cdot c_2 \leq -b(t) \cdot \frac{L}{k^2} + b^{pq} \cdot c_1^q \cdot e^{b(t) \cdot c_1}$$

Let $\kappa > 0$ be such that,

$$\frac{e^{c_1 \cdot x}}{x^{1-pq}} \geq \kappa \qquad \text{for } x > 0$$

Then it is enough to have,

$$\varepsilon \cdot c_2 \leq b(0) \cdot (\kappa \cdot c_1^q - \frac{L}{k^2})$$

so our last condition on k will be that $\frac{L}{k^2} \leq \kappa \frac{c_1^q}{2}$.

So that it is clear that once we have chosen k small enough there exist ε small (depending of this choice of k, $a(0)$ and $b(0)$) such that $b(t) = \varepsilon t + b(0)$ and $a(t) = b^p(t)$ verify (3.6)-(3.7).

With these subsolutions we can prove:

THEOREM 3.2. *If $p \cdot q \leq 1$ there exist thick domains Ω such that every solution of (1.1)-(1.3) blows-up in finite time.*

Proof. Suppose that Ω is a domain that contains a ball B_R with R large enough. We can construct a pair of subsolutions in B_R as before, such that

$$\left.\begin{array}{l} \underline{u}(x,0) < u(x,0) = u_0(x) \\ \underline{v}(x,0) < v(x,0) = v_0(x) \end{array}\right\} \quad \text{in } B_R$$

(This can be done because $a(0), b(0)$ are arbitrary and $u_0 > 0$, $v_0 > 0$ in Ω.)

We claim that

$$\text{(3.8)} \qquad \left\{\begin{array}{ll} u(x,t) > \underline{u}(x,t) & \\ v(x,t) > \underline{v}(x,t) & x \text{ in } B_R. \end{array}\right.$$

In order to prove it suppose that it is not true and let

$$\begin{aligned} t_0 = \sup \{t/\ & u(x,\tau) > \underline{u}(x,\tau) && \text{and} \\ & v(x,\tau) > \underline{v}(x,\tau) && \text{in } B_R, \text{ for } 0 < \tau < t\} \end{aligned}$$

If $t_0 < +\infty$ and $t_0 < T(u,v)$, either $u(x_0,t_0) = \underline{u}(x_0,t_0)$ or $v(x_0,t_0) = \underline{v}(x_0,t_0)$ for some point $x_0 \in \overline{B}_R$.

If $u(x_0,t_0) = \underline{u}(x_0,t_0)$ then by the positivity of u, x_0 must belong to B_R^0. But $(u - \underline{u})$ is a supersolution of the heat equation $B_R \times (0,t_0)$ and is positive (> 0) in the parabolic boundary, a contradiction.

Hence, as $\underline{u}, \underline{v}$ tends to infinity and (3.8) holds by the Remark after Theorem 3.1 we see that the solution $(u(x,t), v(x,t))$ blows-up in finite time. $\square$

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References

[1] M. Escobedo, M.A. Herrero "Boundedness and Blow-up for a Semilinear Reaction-Diffusion System". *Jour. Diff. Eq.* **89**, 176-202 (1991).

[2] M. Escobedo, M.A. Herrero "A Semilinear Parabolic System in a Bounded Domain". *Annali di Matematica pura ed applicata* (IV), Vol. CLXV, 315-336 (1993).

[3] M. Escobedo, H.A. Levine "Critical blow-up and global existence numbers for a weakly coupled system of reaction-diffusion equations", to appear in Arch. Rat. Mech. Analysis.

[4] A. Friedman, *"Partial Differential Equations of Parabolic Type"* Krieger Malabar, FL, 1983.

[5] D. Henry, *"Geometric theory of Semilinear Parabolic Equations"*,Lecture Notes in Mathematics Vol. 840, Springer (1981).