Decay estimates for a nonlocal $p-$Laplacian evolution problem with mixed boundary conditions

Raúl Ferreira and Julio D. Rossi

Abstract. In this paper we prove decay estimates for solutions to a nonlocal $p-$Laplacian evolution problem with mixed boundary conditions, that is,

$$u_t(x,t) = \int_{\Omega\cup\Omega_0} J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy$$

for $(x,t)\in\Omega\times\mathbb{R}^+$ and $u(x,t)=0$ in $\Omega_0\times\mathbb{R}^+$. The proof of these estimates is based on bounds for the associated first eigenvalue.

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1. Introduction

In this paper we consider the following evolution problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\Omega\cup\Omega_0} J(x-y)\Psi_p(u(y,t)-u(x,t))\,dy & \Omega\times\mathbb{R}^+, \\ u(x,t) = 0 & \Omega_0\times\mathbb{R}^+, \\ u(x,0) = u_0(x) & \Omega, \end{cases} \tag{1.1}$$

where $\Psi_p : \mathbb{R}\to\mathbb{R}$ is given by $\Psi_p(s) = |s|^{p-2}s$, $p>1$, Ω is a connected and bounded domain, the kernel J is a nonnegative smooth symmetric radial function supported in the unit ball, $\overline{B(0,1)}$, which is strictly positive in $B(0,1)$ (therefore, nonlocal problems governed with the fractional Laplacian are not included in this article) and Ω_0 is a measurable set (with positive measure) included in $\{x\in\mathbb{R}^N\setminus\Omega : d(x,\partial\Omega)<1\}$.

Nonlocal evolution problems like (1.1) have been recently used to model diffusion processes in applied mathematics (for example, in population dynamics, phase transitions, elasticity models, etc.), see the survey [11] and the references [4], [5], [9], [10], [13], [16], [17] and [18].

This problem can be viewed as a nonlocal analogous to the well known $p-$Laplacian diffusion problem $v_t = \mathrm{div}(|\nabla v|^{p-2}\nabla v)$, we refer to [3] for a proof of the fact that the local $p-$Laplacian can be approximated by these kind of problems. Concerning the boundary condition, if $\Omega_0 = \emptyset$, problem (1.1) is written as

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)\Psi_p(u(y,t)-u(x,t))\,dy & \Omega\times\mathbb{R}^+, \\ u(x,0) = u_0(x) & \Omega. \end{cases}$$

Observe that the diffusion takes place only in Ω. Moreover, using the symmetry of J we can integrate the equation to obtain that the total mass of the solution is constant in time. So, in this case, we say that we get a homogeneous Neumann boundary condition. This problem was studied in [2] (see also [1], [6], [7] and [8] for the linear case $p = 2$).

On the other hand, to get homogeneous Dirichlet boundary condition we take $\Omega_0 = \mathbb{R}^N \setminus \Omega$. In this case problem (1.1) becomes

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)\Psi_p(u(y,t)-u(x,t))\,dy & \Omega\times\mathbb{R}^+, \\ u(x,t) = 0 & \mathbb{R}^N\setminus\Omega\times\mathbb{R}^+, \\ u(x,0) = u_0(x) & \Omega. \end{cases}$$

Observe that in this case the diffusion also takes place in Ω_0 where u is zero. Since J is supported in the unit ball, we also obtain homogeneous Dirichlet boundary condition taking $\Omega_0 = \{x \in \mathbb{R}^N \setminus \Omega \,:\, d(x,\partial\Omega) < 1\}$. See [3] and [6] for a study of this problem.

In our case, $\emptyset \neq \Omega_0 \subset \{x \in \mathbb{R}^N \setminus \Omega \,:\, d(x,\partial\Omega) < 1\}$, and hence we face Dirichlet boundary condition in Ω_0 and Neumann boundary conditions in $(\mathbb{R}^N \setminus \Omega) \setminus \Omega_0$. Then, (1.1) is analogous to a problem with mixed boundary conditions.

Global well-posedness of this problem for $u_0 \in L^p(\Omega)$ as well as a contraction principle can be found in [3]. Moreover, it is proved there, see also the previously mentioned references, that for homogeneous Neumann boundary conditions the solution converge to the mean value of the initial condition while for homogeneous Dirichlet boundary conditions solutions converge to zero.

Our main aim here is to obtain upper bounds for the asymptotic decay of the solutions as t goes to infinity. Our main result is the following.

Theorem 1.1. *Assume that $u_0 \in L^\infty(\Omega)$ then*

1. *For $p > 2$, we have a polynomial decay, for every $0 \le r < \infty$ there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^{r+1}(\Omega)} \le \left(\|u_0\|^{2-p}_{L^{r+1}(\Omega)} + Ct\right)^{-\frac{1}{p-2}} .$$

2. *For* $1 < p \leq 2$, *we have an exponential decay, for every* $0 \leq r < \infty$ *there exists* $\gamma > 0$ *such that*

$$\|u(\cdot,t)\|_{L^{r+1}(\Omega)} \leq \|u_0\|_{L^{r+1}(\Omega)} e^{-\gamma t}\,.$$

For recent references dealing with decay rates for nonlocal evolution equations we refer to [6], [14] and the book [3].

The proof of our decay estimates is based on the positivity of the following infimum (that we will call form now on the first eigenvalue associated with our problem),

$$\lambda_{1,p}(\Omega_0) = \inf_{u\in V} \frac{\displaystyle\int_{\Omega\cup\Omega_0}\int_{\Omega\cup\Omega_0} J(x-y)|u(y)-u(x)|^p\,dy\,dx}{\displaystyle\int_\Omega |u|^p\,dx}\,.$$

Here $V = \{u : \Omega\cup\Omega_0 \to \mathbb{R} \,:\, u \in L^p(\Omega)\,,\ u\,|_{\Omega_0} = 0\}$. In fact, for the linear case the L^2-norm decay exponentially with a rate given by the first eigenvalue, that is, in (2) $\gamma = \lambda_{1,2}(\Omega_0)$, for $p = 2$ and $r = 1$.

This constant $\lambda_{1,p}(\Omega_0)$ is the nonlocal analogous to the first eigenvalue for the local $p-$La-placian operator, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, given by

$$\tilde{\lambda}_1 = \inf_{u\in W^{1,p}(\Omega)\,:\,u|_{\Gamma_0}=0} \frac{\displaystyle\int_\Omega |\nabla u(x)|^p\,dx}{\displaystyle\int_\Omega |u(x)|^p\,dx},$$

where Γ_0 is a nontrivial (in terms of $p-$capacity) subset of $\partial\Omega$. Remark that, in contrast to what happens with $\tilde{\lambda}_1$, it is not known, due to lack of compactness, if the infimum in $\lambda_{1,p}(\Omega_0)$ is attained or not. This fact, the possible nonexistence of eigenfunctions for $\lambda_{1,p}(\Omega_0)$, forces us to work directly with the infimum.

In the linear case with Dirichlet boundary conditions, that is, $p = 2$ and $\Omega_0 = \{x \in \mathbb{R}^N \setminus \Omega \,:\, d(x,\partial\Omega) < 1\}$, the infimum is achieved thanks to the results in [15]. In this linear case the first eigenvalue (and its asymptotic behaviour in expanding domains) was studied in [12].

For general $1 \leq p < \infty$, the positivity of $\lambda_{1,p}(\Omega_0)$ was proved in [3] for the Dirichlet case. This positivity result is analogous to the existence of a positive constant for the classical Poincare inequality $\int_\Omega |v|^p \leq C\int_\Omega |\nabla v|^p$ valid for functions $v \in W_0^{1,p}(\Omega)$. Here we show how to adapt the arguments for the Dirichlet case from [3] to obtain that $\lambda_{1,p}(\Omega_0)$ is strictly positive under mixed boundary conditions. Note that $\lambda_{1,p}(\Omega_0)$ is the best constant in a sort of nonlocal Poincare inequality.

In addition to positivity, we prove further properties of $\lambda_{1,p}(\Omega_0)$. In particular, we show the continuity with respect to Ω_0 and we find some bounds in terms of the measure of Ω_0 which implies that $\lim_{|\Omega_0|\to 0}\lambda_{1,p}(\Omega_0) = 0$. In fact, we have

Theorem 1.2. *If* $(\Omega_0)_n \to \Omega_0$ *in the sense that* $\|\chi_{(\Omega_0)_n} - \chi_{\Omega_0}\|_{L^1} \to 0$*, that is,*

$$|(\Omega_0)_n \Delta\, \Omega_0| := |((\Omega_0)_n \cup \Omega_0) \setminus ((\Omega_0)_n \cap \Omega_0)| \to 0$$

then

$$\lambda_{1,p}((\Omega_0)_n) \to \lambda_{1,p}(\Omega_0).$$

Moreover, there exist constants C_1*,* C_2 *depending on* J*,* Ω *and* p *but not on* Ω_0 *such that*

$$C_1[H(|\Omega_0|)]^2 \le \lambda_{1,p}(\Omega_0) \le \frac{1}{|\Omega|}\int_{\Omega_0}\int_{\Omega} J(x-y)\,dx\,dy \le C_2|\Omega_0|$$

where

$$H(|\Omega_0|) = \min_{\{x\in\Omega_0\,:\,d(x,\partial\Omega)<1-\delta\}} \int_{\{x\in\Omega\,:\,d(x,\partial\Omega)<\delta/2\}} J(x-y)\,dx\,,$$

with $\delta \in (0,1)$ *such that* $|\Omega_0 \cap \{x \in \mathbb{R}^N \setminus \Omega \,:\, d(x,\partial\Omega) < 1-\delta\}| = |\Omega_0|/2.$

The rest of the paper is organized as follows: In Section 2, we prove the properties of $\lambda_{1,p}(\Omega_0)$ stated in Theorem 1.2, in particular we prove a Poincare type inequality which implies the positivity of the first eigenvalue; and, in Section 3, we show the decay bound for the evolution problem.

2. The first eigenvalue

In this section we study some properties of $\lambda_{1,p}(\Omega_0)$ in terms of Ω_0. Since,

$$\begin{aligned}
&\int_{\Omega\cup\Omega_0}\int_{\Omega\cup\Omega_0} J(x-y)|u(y)-u(x)|^p\,dy\,dx \\
&= \int_{\Omega}\int_{\Omega} J(x-y)|u(y)-u(x)|^p\,dy\,dx + 2\int_{\Omega_0}\int_{\Omega} J(x-y)|u(y)|^p\,dy\,dx \\
&= \int_{\Omega}\int_{\Omega} J(x-y)|u(y)-u(x)|^p\,dy\,dx + 2\int_{\Omega}\left(\int_{\Omega_0} J(x-y)\,dx\right)|u(y)|^p\,dy,
\end{aligned}$$

we get that the dependence of $\lambda_{1,p}(\Omega_0)$ in Ω_0 comes from the weight

$$\Xi_{\Omega_0}(y) = \int_{\Omega_0} J(x-y)\,dx$$

defined for $y \in \Omega$. Notice that this weight is continuous with respect to Ω_0, thus we have the following continuity result for $\lambda_{1,p}(\Omega_0)$.

Lemma 2.1. *Assume that a sequence of sets* $(\Omega_0)_n$ *verifies* $(\Omega_0)_n \to \Omega_0$ *in the sense that* $|(\Omega_0)_n \Delta\, \Omega_0| := |((\Omega_0)_n \cup \Omega_0) \setminus ((\Omega_0)_n \cap \Omega_0)| \to 0$ *then*

$$\lambda_{1,p}((\Omega_0)_n) \to \lambda_{1,p}(\Omega_0).$$

Proof. We only have to remark that

$$\begin{aligned}
\Xi_{(\Omega_0)_n}(y) = &\int_{(\Omega_0)_n} J(x-y)\,dx = \int_{\mathbb{R}^N} J(x-y)\chi_{(\Omega_0)_n}(x)\,dx \\
= &\; J * \chi_{(\Omega_0)_n}(y) \to \Xi_{(\Omega_0)}(y)
\end{aligned}$$

uniformly for $y \in \overline{\Omega}$. Therefore, for any u with $\|u\|_{L^p(\Omega)} = 1$, we get

$$\left| \int_\Omega \left(\int_{(\Omega_0)_n} J(x-y)\,dx \right) |u(y)|^p\,dy - \int_\Omega \left(\int_{\Omega_0} J(x-y)\,dx \right) |u(y)|^p\,dy \right|$$

$$\leq \left\| \left(\int_{(\Omega_0)_n} J(x-y)\,dx \right) - \left(\int_{\Omega_0} J(x-y)\,dx \right) \right\|_{L^\infty(\Omega)} \to 0,$$

as $n \to \infty$. ☐

Now, we look for lower and upper estimates of $\lambda_{1,p}(\Omega_0)$. To do that, we need the following Poincare type inequality that shows that $\lambda_{1,p}(\Omega_0)$ is strictly positive.

Lemma 2.2. *There exists a positive constant* λ *such that*

$$\int_{\Omega\cup\Omega_0} \int_{\Omega\cup\Omega_0} J(x-y)|u(y)-u(x)|^p\,dy\,dx \geq \lambda \int_\Omega |u|^p\,dx,$$

for every $u \in V = \{u : \Omega\cup\Omega_0 \to \mathbb{R} \,:\, u \in L^p(\Omega)\,,\ u\,|_{\Omega_0} = 0\}$.

Proof. Following [3] we cover the domain $\Omega\cup\Omega_0$ with a finite family of disjoint sets, B_j $j = 0, 1, \cdots, L$ and define

$$\alpha_j = \frac{1}{2^p} \min_{x\in B_j} \int_{B_{j-1}} J(x-y)\,dy\,, \qquad \beta = \int_{\mathbb{R}^N} J(s)\,ds\,. \tag{2.1}$$

Now,

$$\begin{aligned} \int_{\Omega\cup\Omega_0} \int_{\Omega\cup\Omega_0} & J(x-y)|u(y)-u(x)|^p\,dy\,dx \\ \geq & \int_{B_j} \int_{B_{j-1}} J(x-y)|u(y)-u(x)|^p\,dy\,dx \end{aligned}$$

for $j = 1, \cdots, L$, and

$$\begin{aligned} \int_{B_j} \int_{B_{j-1}} & J(x-y)|u(y)-u(x)|^p\,dy\,dx \\ \geq\ & \frac{1}{2^p} \int_{B_j} \left(\int_{B_{j-1}} J(x-y)\,dy \right) |u(x)|^p\,dx \\ & - \int_{B_{j-1}} \left(\int_{B_j} J(x-y)\,dx \right) |u(y)|^p\,dy \\ \geq\ & \alpha_j \int_{B_j} |u(x)|^p\,dx - \beta \int_{B_{j-1}} |u(y)|^p\,dy\,. \end{aligned}$$

Then, iterating this inequality and using that $u = 0$ in B_0 we get that

$$\int_{B_j} |u(x)|^p\,dx \leq C_j \int_{\Omega\cup\Omega_0} \int_{\Omega\cup\Omega_0} J(x-y)|u(y)-u(x)|^p\,dy\,dx$$

where,

$$C_1 = \frac{1}{\alpha_1}\,, \qquad C_j = \frac{1}{\alpha_j}(1 + \beta C_{j-1}) \quad j = 2, \cdots, L\,.$$

Therefore, adding in j, we have the Poincare type inequality

$$\int_{\Omega\cup\Omega_0}\int_{\Omega\cup\Omega_0} J(x-y)|u(y)-u(x)|^p\,dy\,dx \geq \lambda \int_{\Omega} |u(x)|^p\,dx, \tag{2.2}$$

with

$$\lambda = \left(\sum_{j=0}^{L} C_j\right)^{-1} \sim \prod_{j=1}^{L} \alpha_j\,.$$

Now, we construct the family B_j. To do that, we define the sets

$$\Gamma_\delta = \{x\in\mathbb{R}^N\setminus\Omega\,:\,d(x,\partial\Omega)<1-\delta\},\qquad \Theta_\delta = \{x\in\Omega\,:\,d(x,\partial\Omega)<\delta\}\,.$$

Observe that the function $g(\delta)=|\Omega_0\cap\Gamma_\delta|$ is continuous and $g(0)=|\Omega_0|$ and $g(1)=0$, then we can fix a δ such that, $|\Omega_0\cap\Gamma_\delta|=|\Omega_0|/2$. Notice also that,

$$\frac{1}{2}|\Omega_0| = |\Omega_0\cap(\Gamma_1\setminus\Gamma_\delta)| \sim \delta\,.$$

Now, we define the two first sets of the family,

$$B_0=\Omega_0,\qquad B_1=\{x\in\Theta_{\delta/2}\,:\,\int_{B_0} J(x-y)\,dy > a\}$$

for some a given bellow. Observe that by definition we can take $\alpha_1=a$.

In order to see that B_1 has a positive measure, we note that

$$\begin{aligned}
\int_{\Theta_{\delta/2}}\int_{B_0} J(x-y)\,dy\,dx \geq\ & \int_{\Theta_{\delta/2}}\int_{B_0\cap\Gamma_\delta} J(x-y)\,dy\,dx\\
=\ & \int_{B_0\cap\Gamma_\delta}\int_{\Theta_{\delta/2}} J(x-y)\,dx\,dy\\
\geq\ & |B_0\cap\Gamma_\delta| \min_{x\in B_0\cap\Gamma_\delta}\int_{\Theta_{\delta/2}} J(x-y)\,dx\\
=\ & \tfrac{1}{2}|B_0|H(\delta)\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{\Theta_{\delta/2}}\int_{B_0} J(x-y)\,dy\,dx =\ & \int_{B_1}\int_{B_0} J(x-y)\,dy\,dx\\
& + \int_{\Theta_{\delta/2}\setminus B_1}\int_{B_0} J(x-y)\,dy\,dx\\
\leq\ & \|J\|_\infty|B_0||B_1| + a|\Theta_{\delta/2}\setminus B_1|\\
\leq\ & \|J\|_\infty|B_0||B_1| + a|\Theta_{\delta/2}|\\
\leq\ & \|J\|_\infty|B_0||B_1| + aC|\partial\Omega|\delta\\
\leq\ & \|J\|_\infty|B_0||B_1| + aC|B_0|\,.
\end{aligned}$$

Thus, taking $a=H(\delta)/4C$, we get

$$|B_1| \geq \frac{1}{4\|J\|_\infty}H(\delta)\,.$$

In order to localize a piece of B_1 with positive measure we take a finite family of balls of radius $\rho = 1/16$ which cover $\overline{\Theta}_{\delta/2}$. We denote this family as E_i for $i = 1, \cdots, l$. As $B_1 \subset \Theta_{\delta/2} \subset \cup_{i=1}^{l} E_i$ we have that

$$|B_1| \leq \sum_{i=1}^{l} |B_1 \cap E_i| \leq l \max_{i=1,\cdots,l} |B_1 \cap E_i| .$$

Then, there exists E_{i_0} such that

$$|B_1 \cap E_{i_0}| \geq CH(\delta) .$$

Now, we define

$$B_2 = \{x \in \Omega \setminus B_1 \,:\, B_1 \cap E_{i_0} \subset B\,(x, 1/4)\} .$$

Since the distance between $x \in B_2$ and $y \in E_{i_0}$ is small than $1/4$ we have that

$$\int_{B_1} J(x-y)\,dy \geq \int_{B_1 \cap E_{i_0}} J(x-y)\,dy \geq |B_1 \cap E_{i_0}| \min_{s \in B(0,\frac{1}{4})} J(s) \geq CH(\delta) .$$

Thus,

$$\alpha_2 = \min_{x \in B_2} \int_{B_1} J(x-y)\,dy \geq CH(\delta) .$$

Moreover, the measure of B_2 is bounded from bellow independent of δ.

The rest of the sets is defined as

$$B_j = \left\{ x \in \Omega \setminus \cup_{i=1}^{j-1} B_i \,:\, d(x, B_{j-1}) < \frac{1}{4} \right\} , \qquad j = 3, \cdots, L$$

and it is easy to see that there exists a positive constant $K = K(\Omega)$ such that,

$$\alpha_j = \min_{x \in B_j} \int_{B_{j-1}} J(x-y)\,dy \geq K .$$

Summing up, we get that there exists three constants such that

$$\alpha_1 \geq K_1 H(\delta)\,,\ \alpha_2 \geq K_2 H(\delta)\,,\ \alpha_j \geq K_3\,,\ j = 3, \cdots, L .$$

Moreover $\delta \sim |\Omega_0|$. Thus, it is easy to see that

$$\lambda \sim \prod_{j=1}^{L} \alpha_j \geq C(H(\Omega_0))^2 . \tag{2.3}$$

$\square$

Remark 2.3. In the particular case when J is the characteristic function, that is $J(s) = \chi_{B(0,1)}(s)$, we get that for $x \in \Omega_0 \cap \Gamma_\delta$

$$\int_{\Theta_{\delta/2}} J(x-y)\,dy \sim |\Lambda_\delta| ,$$

where Λ_δ is spherical cap of height δ. Then,

$$H(\delta) \sim \delta^{\frac{N+1}{2}} \sim |\Omega_0|^{\frac{N+1}{2}} .$$

Lemma 2.4. *The first eigenvalue satisfies*

$$C(H(|\Omega_0|))^2 \le \lambda_{1,p}(\Omega_0) \le \frac{1}{|\Omega|}\int_{\Omega_0}\int_{\Omega} J(x-y)\,dx\,dy.$$

Proof. The lower estimate is a consequence of the Poincare inequality (Lemma 2.2) and (2.3), while the upper estimate is obtained choosing as a test function

$$v(x) = \begin{cases} 1 & x \in \Omega, \\ 0 & x \in \Omega_0. \end{cases}$$

□

This result and Remark 2.3 give us the following corollary.

Corollary 2.5. *If* $J(s) = \chi_{B(0,1)}(s)$ *we have that*

1. *for a general* Ω_0 *we have that there exists two positive constants such that*
$$C_1|\Omega_0|^{N+1} \le \lambda_{1,p}(\Omega_0) \le C_2|\Omega_0|\,;$$
2. *in the particular case* $\Omega_0 = \{x \in \mathbb{R}^N \setminus \Omega \,:\, 1-\delta < d(x,\Omega) < \delta\}$ *for some small* $\delta > 0$*, we get*
$$C_1|\Omega_0|^{N+1} \le \lambda_{1,p}(\Omega_0) \le C_2|\Omega_0|^{\frac{N+3}{2}}\,.$$

Proof. In the general case, we only observe that J is integrable, then

$$\int_{\Omega_0}\int_{\Omega} J(x-y)\,dx\,dy \le C|\Omega_0|\,.$$

For the particular case, we get that

$$\int_{\Omega_0}\int_{\Omega} J(x-y)\,dx\,dy = \int_{\Omega_0}\int_{\Theta_\delta} J(x-y)\,dy\,dx$$

and we can estimate the interior integral by the measure of a spherical cap of height δ. Moreover $|\Omega_0| \sim |\partial\Omega|\delta$. Then,

$$\int_{\Omega_0}\int_{\Theta_\delta} J(x-y)\,dy\,dx \sim \int_{\Omega_0} \delta^{\frac{N+1}{2}}\,dx \sim |\partial\Omega|\,\delta^{1+\frac{N+1}{2}}\,,$$

as we wanted to prove. □

3. Estimates for the decay of the associated evolution problem

In this section we study the asymptotic behavior of the solution of the problem (1.1).

Proof of Theorem 1.1. We want to obtain the decay rate in $L^{r+1}(\Omega)$. To this end, we multiply the equation by u^r, with $0 < r < \infty$ (to obtain a decay in

$L^1(\Omega)$ we multiply by the sign of u, we leave the details to the reader), and integrate to obtain

$$\begin{aligned}
&\partial_t \int_\Omega \frac{u^{r+1}(x,t)}{r+1}\,dx \\
&= \int_{\Omega\cup\Omega_0}\int_{\Omega\cup\Omega_0} J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))u^r(x,t)\,dy\,dx \\
&= -\frac{1}{2}\int_{\Omega\cup\Omega_0}\int_{\Omega\cup\Omega_0} J(x-y)|u(y,t)-u(x,t)|^{p-1}|u^r(y,t)-u^r(x,t)|\,dy\,dx \\
&\le -C(p,r)\int_{\Omega\cup\Omega_0}\int_{\Omega\cup\Omega_0} J(x-y)|u^\alpha(y,t)-u^\alpha(x,t)|^p\,dy\,dx
\end{aligned}$$

with $\alpha = (p+r-1)/p$. The last inequality follows from the fact that for any $p>1$, $r>0$, there is a positive constant $C = C(p,r)$ such that

$$|x-1|^{p-1}||x|^{r-1}x-1| \ge C(p,r)||x|^\alpha - 1|^p, \qquad x\in\mathbb{R}.$$

Note that for $p=2$, $r=1$ we get $C(2,1)=1$. Unfortunately $C(p,r)\to 0$ as $r\to\infty$, then we can not obtain an estimate for the L^∞ norm.

On the other hand, using the Poincare inequality ($\lambda_{1,p}(\Omega_0)$ is best constant in such inequality) we get

$$\partial_t \int_\Omega \frac{u^{r+1}(x,t)}{r+1}\,dx \le -C(p,r)\lambda_{1,p}(\Omega_0)\int_\Omega |u(x,t)|^{\alpha p}\,dx\,.$$

Notice that for $p=2$, $\alpha p = r+1$, then we have exponential decay of the L^r norm,

$$\|u\|_{L^{r+1}(\Omega)} \le \|u_0\|_{L^{r+1}(\Omega)} e^{-C(p,r)\lambda_{1,p}(\Omega_0)t}\,.$$

For the case $p>2$ we can use Jensen inequality to obtain

$$\partial_t \int_\Omega \frac{u^{r+1}(x,t)}{r+1}\,dx \le -C\left(\int_\Omega |u(x,t)|^{r+1}\,dx\right)^{\frac{\alpha p}{r+1}}\,.$$

Thus, we get

$$\|u\|_{L^{r+1}(\Omega)} \le \left(\|u_0\|^{2-p}_{L^{r+1}(\Omega)} + C\frac{p-2}{r+1}t\right)^{-\frac{1}{p-2}}\,.$$

Finally, in the case $1<p<2$, since we assume that $u_0\in L^\infty(\Omega)$, by the maximum principle, we get that for every $t>0$, $|u(x,t)|\le \|u_0\|_{L^\infty(\Omega)}$ a.e. in Ω. Then, as $p<2$ have that

$$-|u|^{\alpha p} = -|u|^{r+1}|u|^{p-2} \le -C(\|u_0\|_{L^\infty(\Omega)})|u|^{r+1},$$

then

$$\begin{aligned}
\partial_t \int_\Omega \frac{u^{r+1}(x,t)}{r+1}\,dx &\le -C(p,r,\|u_0\|_{L^\infty(\Omega)})\lambda_{1,p}(\Omega_0)\int_\Omega |u(x,t)|^{\alpha p}\,dx \\
&\le -C(p,r,\|u_0\|_{L^\infty(\Omega)})\lambda_{1,p}(\Omega_0)\int_\Omega |u(x,t)|^{r+1}\,dx,
\end{aligned}$$

and we obtain an exponential decay. $\square$

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Raúl Ferreira
Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040, Madrid, SPAIN.
e-mail: `raul_ferreira@mat.ucm.es`

Julio D. Rossi
Departamento de Análisis Matemático, Universidad de Alicante
Ap. correos 99, 03080
Alicante, SPAIN.
On leave from Dpto. de Matemáticas, FCEyN, Universidad de Buenos Aires
1428, Buenos Aires, ARGENTINA.
e-mail: `jrossi@dm.uba.ar`
`Web page: http://mate.dm.uba.ar/~jrossi/`