REGULARITY RESULTS FOR THE BLOW-UP TIME AS A FUNCTION OF THE INITIAL DATA

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Abstract. We study the dependence of the finite blow-up time with respect to the initial data for solutions of the equation $u_t = \Delta u^m + u^p$. We obtain Lipschitz continuity for a certain class of initial data and Holder regularity for wider classes.

Introduction.

In this paper we study the dependence with respect to the initial data of the blow-up time for solutions of the following problem

$$\begin{cases} u_t = \Delta(u^m) + u^p, & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $m \geq 1$, $p > 1$ and u_0 is nonnegative and smooth in its positivity domain.

A remarkable fact is that the solution of parabolic problems may develop singularities in finite time, no matter how smooth the initial data are. It is well known that for many differential equations or systems the solutions can become unbounded in finite time (a phenomena that is known as blow-up). The study of blow-up solutions has attracted a considerable attention in recent years, see [7], [12], [14] and the references therein.

For our problem, if the solution is defined on a maximal time interval, $[0,T)$ with $T < +\infty$, then $\lim_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty} = +\infty$. We say that T is the blow-up time. The existence of blowing up solutions for (1.1) has been proved in [5], [14]. In [8] it is proved that when $1 < p \leq m + 2/N$ every nontrivial solution blows up in finite time, while if $p > m + 2/N$ there are blowing up solutions for initial data large enough and also global solutions for small initial data. The speed at which solutions blow up (blow-up rate) and the spatial structure of the set in which the solution becomes unbounded (blow-up sets) are well known for this

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problem. The blow-up rate is given by $\|u(\cdot,t)\|_{L^\infty} \sim (T-t)^{-1/(p-1)}$. Concerning the blow-up set, there are three different cases according to $p < m$, $p = m$ or $p > m$. In fact, the blow-up set is, typically, the whole $\mathbb{R}^N$ if $p < m$ (global blow-up), a bounded subset if $p = m$ (regional blow-up) and a single point if $p > m$ (single point blow-up). See [2], [3], [14] and references therein.

Our main interest here is to investigate the dependence of the blow-up time with respect to the initial data. We assume that we are dealing with an initial datum u_0 which produces a solution u that blows up at time $T = T(u_0)$ and an arbitrary small perturbation $h(x)$, such that the solution u_h with initial datum $u_{0,h} = u_0(x) + h(x)$ also blows up in finite time, that we call $T_h = T(u_{0,h})$. Our concern is to obtain bounds for $|T_h - T|$ in terms of h.

For the semilinear case, $m = 1$, it is known that the blow-up time is continuous with respect to the initial data in L^∞ when $1 < p < (N+2)/(N-2)$, see [1], [11], [13], if Ω is bounded (with Dirichlet boundary conditions) and [4] if $\Omega = \mathbb{R}^N$. That is, $T_h \to T$ as $\|h\|_{L^\infty} \to 0$. Remark that the restriction on p is not technical. Indeed, if it does not hold, then the blow-up time is not even continuous as a function of the initial data, see [6]. Moreover, in [9] it is proved that T is almost Lipschitz in the following sense, $|T_h - T| \le C\|h\|_{L^\infty} |\ln(\|h\|_{L^\infty})|^{\frac{N+2}{2}+\varepsilon}$. The one dimensional case was treated in [10] where it was shown that T is Lipschitz for some special initial data and some particular perturbations h.

Here we improve the above mentioned results in two ways. On the one hand, we show that, under certain conditions on u_0, the blow-up time T is Lipschitz without any restriction on the perturbations h. On the other hand, the main ideas of the previous works, [1], [9], [11], [13], heavily rely on the linearity of the Laplace operator, while our approach, allows to deal with nonlinear operators such as the porous medium equation ($m > 1$). For this equation, up to our knowledge, this is the first analysis of the dependence of the blow-up time with respect to the initial data.

One of the tools involved in the analysis presented relies on the natural scaling invariance of the problem. We mean the following, if $u(x,t)$ is a solution of (1.1) then $u_\lambda(x,t) = \lambda^{-\alpha} u(\lambda^{-\beta}x, \lambda^{-1}t)$ is also a solution if we choose $\alpha = 1/(p-1)$ and $\beta = (p-m)/2(p-1)$. This solution u_λ blows up at time $T_\lambda = \lambda T$ and has initial data $u_\lambda(x,0) = \lambda^{-\alpha}u_0(\lambda^{-\beta}x)$. From the explicit form of T_λ and $u_\lambda(x,0)$ it is not difficult to derive

Lispschitz continuity in this case. The result for this particular perturbation and standard comparison arguments allow us to handle a general perturbation h.

Theorem 1.1. *Assume that u_0 verifies*

$$(1.2)\quad -\frac{\partial(u_\lambda(x,0))}{\partial\lambda}\Big|_{\lambda=1} = \alpha u_0(x) + \beta x\nabla u_0(x) \geq c > 0,\ \ x \in supp(u_0).$$

Then, the blow-up time is Lispchitz with respect to the initial data in the following sense, there exist a positive constant C such that

$$|T - T_h| \leq C dist(u_0; u_{0,h}),$$

for every $u_{0,h}$ with $dist(u_0, u_{0,h}) = (\|u_0 - u_{0,h}\|_{L^\infty} + |a - a_h|)$ small. Here a and a_h denote the corresponding interfaces for u_0 and $u_{0,h}$ and $|a - a_h|$ stands for the usual distance between sets.

We note that the Lipschitz constant cannot be uniform. This follows just by looking at the ODE $u_t = u^p$. Now we want to comment on the hypotheses that appear in Theorem 1.1. The distance involves a term related to the interfaces, which is natural since solutions of the porous medium equation have finite speed of propagation of their supports. Obviously, if u_0 is positive, this term does not appear. Concerning our assumption (1.2), we first note that, since the problem is invariant under spatial translations (keeping the same blow-up time), one can check it on any translation of u_0. Using this fact, one can see that, after an appropriate spatial translation, (1.2) holds for a wide class of initial data, including positive or compactly supported bell shaped ones when $1 < p < m$. Moreover, in this range of exponents the blow-up behavior is described by means of a self-similar solution with a profile that verifies (1.2). Therefore, one can expect that solutions satisfy (1.2) for times close to their blow-up time. If $p \geq m$, since $\beta \geq 0$, (1.2) implies that u_0 must be positive. Also in this case, one can check the existence of initial data verifying (1.2) with a suitable behavior at ∞.

The ideas involved in the proof, also give Holder continuity results for wider classes of initial data, imposing higher order conditions on u_0.

Theorem 1.2. *Let u_0 be such that*

$$\frac{\partial^k(u_\lambda(x,0))}{\partial\lambda^k}\Big|_{\lambda=1} \neq 0,$$

when x verifies $\partial^j(u_\lambda(x,0))/\partial\lambda^j = 0$ for $1 \leq j \leq k-1$. Assume that T is continuous at u_0, then T is Holder continuous with exponent $1/k$, that is

$$|T - T_h| \leq C(dist(u_0; u_{0,h}))^{1/k}.$$

This result improves continuity to Holder continuity when (1.2) does not hold. Remark that for certain initial data even continuity may fail, [6]. As a particular application, we note that Theorem 1.2 provides with Holder 1/2 continuity, for instance, for concave compactly supported data.

One can consider other scalings that leave invariant the equation, for instance $u_\lambda(x,t) = u(x,t+\lambda)$. In this case, imposing $u_t > 0$ one gets again Lipschitz continuity. We will comment on these extensions after the proof of Theorem 1.2.

Finally, we remark that the same ideas can be used to deal with equations involving other operators and/or source terms like $u_t = \Delta u + e^u$, $u_t = div(|\nabla u|^{q-2}\nabla u) + u^p$, etc. We only need a scaling invariance law together with a comparison result.

Proof of the results

Proof of Theorem 1.1.

To begin our analysis let us prove the result for the one-parameter family of initial data obtained by using the scaling invariance of the equation. As we mentioned in the introduction, if $u(x,t)$ is a solution of $u_t = \Delta u^m + u^p$ a straightforward calculation shows that

$$u_\lambda(x,t) = \lambda^{-\alpha} u(\lambda^{-\beta} x, \lambda^{-1} t),$$

is also a solution when α and β are the self-similar exponents associated to the problem under consideration, namely,

$$\alpha = \frac{1}{p-1}, \qquad \beta = \frac{p-m}{2(p-1)}.$$

The solution u_λ has initial datum $u_\lambda(x,0) = \lambda^{-\alpha} u_0(\lambda^{-\beta} x)$, and blow-up time $T_\lambda = T\lambda$. Therefore, we get

$$|T - T_\lambda| = T|1-\lambda|$$

and

$$\begin{aligned} dist(u_0; u_{0,\lambda}) = \ & |1-\lambda| \left(\|\alpha u_0(x) + \beta x \nabla u_0(x)\|_{L^\infty} + |\beta||a_0| \right) \\ & + o(|1-\lambda|). \end{aligned}$$

Hence

$$|T - T_\lambda| \le C dist(u_0; u_{0,\lambda}),$$

where the constant C can be chosen as

$$C = \frac{T}{\left(\|\alpha u_0(x) + \beta x \nabla u_0(x)\|_{L^\infty} + |\beta||a_0| \right)} + o(1),$$

for $|1-\lambda|$ small. Therefore, Theorem 1.1 follows for the special family $u_{0,\lambda}$.

In order to deal with general perturbations, the main idea is to use comparison arguments between u_h (the solution with initial datum $u_{0,h}$) and a suitable u_λ.

Given $u_{0,h}$ with $dist(u_0, u_{0,h})$ small, let us define

$$\lambda^+ = \sup\{\lambda < 1;\ u_{0,\lambda}(x) \geq u_{0,h}(x)\},$$

and

$$\lambda_- = \inf\{\lambda > 1;\ u_{0,\lambda}(x) \leq u_{0,h}(x)\}.$$

Note that both λ^+ and λ_- are well defined by our hypothesis (1.2).

From the definition of λ^+ and λ_- it is clear that

$$u_{0,\lambda_-}(x) \leq u_{0,h}(x) \leq u_{0,\lambda^+}(x).$$

By using a well known comparison argument, we get

$$u_{\lambda_-}(x,t) \leq u_h(x,t) \leq u_{\lambda^+}(x,t).$$

Then, if we denote by T_{λ^+}, T_h and T_{λ_-} the blow-up times for u_{λ^+}, u_h and u_{λ_-} respectively, we obtain

$$T_{\lambda^+} \leq T_h \leq T_{\lambda_-}.$$

Therefore

$$|T - T_h| \leq \max\{T_{\lambda_-} - T\ ;\ T - T_{\lambda^+}\}.$$

Hence we want to obtain bounds on $T_{\lambda_-} - T$ and $T - T_{\lambda^+}$ in terms of $dist(u_0, u_{0,h})$. We deal with $T_{\lambda_-} - T$ in detail. The bound for $T - T_{\lambda^+}$ can be handled in a similar way, but some differences appear. We perform the details when appropriate.

Using the previous result for the family u_λ, we get

$$T_{\lambda_-} - T \leq C dist(u_0; u_{0,\lambda_-}).$$

If u_0 is positive, in order to estimate the distance $dist(u_0; u_{0,\lambda_-})$ in terms of $dist(u_0; u_{0,h})$ we remark that $u_{0,h}$ and u_{0,λ_-} must have at least a contact point, x_{λ_-}, that may be ∞. Then

$$\begin{aligned} dist(u_0, u_{0,h}) &\geq \|u_0 - u_{0,h}\|_{L^\infty} \geq |u_0(x_{\lambda_-}) - u_{0,h}(x_{\lambda_-})| \\ &= |u_0(x_{\lambda_-}) - u_{0,\lambda_-}(x_{\lambda_-})| \geq C|1 - \lambda_-|, \end{aligned}$$

where we can take

$$C = \inf |\alpha u_0 + \beta x \nabla u_0| + o(1).$$

Remark that $C > 0$ by our hypothesis on the initial data u_0, (1.2).

On the other hand,

$$|1-\lambda_-| \geq C dist(u_0, u_{0,\lambda_-}).$$

Collecting all these bounds we get

$$T_{\lambda_-} - T \leq C dist(u_0, u_{0,h}).$$

If u_0 is compactly supported, to bound $T_{\lambda_-} - T$ the previous calculations remain valid since again in this case the contact point verify $x_{\lambda_-} \in supp(u_0)$.

When considering $T - T_{\lambda^+}$, the only difference in the arguments appears if u_0 is compactly supported since for $u_0 > 0$ the contact point always belong to its support. Also in this case $u_{0,h}$ and u_{0,λ^+} must have a contact point, x_{λ^+}. However x_{λ^+} must not necessarily belong to the support of u_0. First, if $x_{\lambda^+} \in supp(u_0)$, the case is analogous to the previous one and can be handled in a similar fashion. It remains to deal with $x_{\lambda^+} \notin supp(u_0)$. If $u_{0,h}(x_{\lambda^+}) \geq |1-\lambda^+|$ we get

$$\begin{aligned} dist(u_0, u_{0,h}) &\geq \|u_0 - u_{0,h}\|_{L^\infty} \geq |u_0(x_{\lambda^+}) - u_{0,h}(x_{\lambda^+})| \\ &= |u_{0,\lambda^+}(x_{\lambda^+})| \geq |1-\lambda^+|, \end{aligned}$$

and we finish the argument as above. Finally, if $u_{0,h}(x_{\lambda^+}) < |1-\lambda^+|$, the condition involving the interfaces comes into account and we have, using our hypothesis on u_0, (1.2),

$$dist(u_0, u_{0,h}) \geq |a - a_h| \geq C|1-\lambda^+|.$$

This ends the proof of Theorem 1.1. □

Proof of Theorem 1.2.

Now we want to show how to obtain Holder regularity by imposing higher order conditions. Let us analyze in detail the case $k = 2$. Assume that u_0 verifies $\partial^2(u_\lambda(x,0))/\partial\lambda^2 \neq 0$ for x where (1.2) fails and that T is continuous at u_0. Under these assumptions, by using appropriate modifications of the ideas above, we get that T is Holder continuous with exponent 1/2. It will be convenient to deal first with regular perturbations of the form $u_{0,\varepsilon}^+ = u_0 + \varepsilon$ if $x \in supp(u_0)$ compactly supported and such that $dist(u_0, u_{0,\varepsilon}^+) \leq 2\varepsilon$ and $u_{0,\varepsilon}^- = (u_0 - \varepsilon)_+$. This special family of initial data will play the same role as the family u_λ in the previous case when dealing with general perturbations. Once we have proved Holder continuity for the family $u_{0,\varepsilon}$, from $u_{0,\varepsilon}^+ \geq u_{0,h} \geq u_{0,\varepsilon}^-$, we obtain the general result.

In order to get the Holder 1/2 regularity for this special perturbations, we use again the family u_λ obtained by scaling. For a given ε, let us consider u_ε^+ the solution with initial data $u_{0,\varepsilon}^+$ and T_ε^+ its blow-up time. We select the only value λ^* such that $T_{\lambda^*} = \lambda^* T = T_\varepsilon^+$. From this choice, it is clear that both data $u_{0,\varepsilon}^+$ and u_λ^* must have an intersection point x^* (otherwise they will be ordered and their blow-up times will be different). If $x^* \in supp(u_0)$, at this point we have, for ε small (since the continuity of the blow-up time implies $|1-\lambda^*|$ small),

$$\varepsilon = u_{\lambda^*}(x^*,0) - u_0(x^*) = \frac{\partial u_\lambda(x^*,0)}{\partial \lambda}\Big|_{\lambda=1}(\lambda^*-1)$$
$$+\frac{\partial^2 u_\lambda(x^*,0)}{\partial \lambda^2}\Big|_{\lambda=1}(\lambda^*-1)^2 + o((\lambda^*-1)^2).$$

Therefore

$$(dist(u_0, u_{0,\varepsilon}^+))^{1/2} = (\varepsilon)^{1/2} \geq C|1-\lambda^*|,$$

and hence

$$|T_\varepsilon^+ - T| = |T_{\lambda^*} - T| = T|1-\lambda^*| \leq C(dist(u_0, u_{0,\varepsilon}^+))^{1/2}.$$

The same procedure gives an analogous estimate when considering $u_{0,\varepsilon}^-$. On the other hand, if x does not belong to the support of u_0, the condition on the interface appears by using similar arguments to those in Theorem 1.1.

For a general $u_{0,h}$ we take a value of ε of order $(dist(u_0, u_{0,h}))$ such that $u_{0,\varepsilon}^- \leq u_{0,h} \leq u_{0,\varepsilon}^+$ and proceed as before. Indeed, from $T_\varepsilon^+ \leq T_h \leq T_\varepsilon^-$ we obtain

$$|T_h - T| \leq \max\{T - T_\varepsilon^+,\ T_\varepsilon^- - T\} \leq C(\varepsilon)^{1/2} \leq C(dist(u_0, u_{0,h}))^{1/2}.$$

In a similar way we deal with higher order conditions,

$$\frac{\partial^k (u_\lambda(x,0))}{\partial \lambda^k} \neq 0$$

at points where the first $k-1$ derivatives vanish, obtaining Holder continuity with exponent $1/k$. $\Box$

Further results.

Finally, we consider the family u_λ given by time translations,

$$u_\lambda(x,t) = u(x, t+\lambda).$$

Assume that u_0 is an initial data such that the solution $u(x,t)$ is defined for $t \in (-a,a)$ and that $u_t(x,0) \geq c > 0$. Let $u_{0,h}$ be a perturbation of u_0 and assume that $T_h \leq T$. Define

$$\tau^+ = \inf\{\tau > 0;\ u_{0,h}(x) \leq u(x,\tau)\}.$$

This τ^+ is well defined due to our assumption $u_t > 0$. We have

$$T - T_h \leq T - (T - \tau^+) = \tau^+.$$

Moreover, as $u(x,\tau^+)$ and $u_{0,h}$ must have a contact point, we get, as before
$$dist(u_0,u_{0,h}) \geq C\tau^+.$$
Therefore we obtain a Lipschitz estimate.

If $T_h > T$, we use
$$\tau_- = \sup\{\tau < 0;\ u_{0,h}(x) \geq u(x,\tau)\}.$$
As above, τ_- is well defined due to our assumption $u_t > 0$ and the fact that the initial datum u_0 corresponds to a solution defined for negative small times. We have
$$T_h - T \leq (T-\tau_-) - T = -\tau_-.$$
Moreover, as $u(x,\tau_-)$ and $u_{0,h}$ must have a contact point, we get, as before
$$dist(u_0,u_{0,h}) \geq C(-\tau_-).$$
Therefore we get the result.

Also for this family we may impose higher order conditions obtaining Holder regularity results.

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