Uniqueness and non-uniqueness for a system of heat equations with non-linear coupling at the boundary

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Abstract

We study the uniqueness problem for non negative solutions of a system of two heat equations, $u_t = \Delta u$, $v_t = \Delta v$, in a bounded smooth domain Ω, with nonlinear boundary conditions, $\frac{\partial u}{\partial \eta} = v^p$, $\frac{\partial v}{\partial \eta} = u^q$.

We prove that for identically zero initial data, $(u(x,0), v(x,0)) \equiv (0,0)$, the zero solution is unique if and only if $pq \geq 1$. Moreover, in the case of non-negative non-trivial initial data the solution is always unique.

1 Introduction.

Let Ω be a bounded domain in R^N with smooth boundary and let p and q be two positive real numbers. In this article we are concerned with the uniqueness problem for non-negative solutions of the following system

$$\begin{array}{llll} u_t = \Delta u, & & v_t = \Delta v & \text{in } \Omega \times (0,T) \\ \frac{\partial u}{\partial \eta} = v^p, & & \frac{\partial v}{\partial \eta} = u^q & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0 & \text{and} & v(x,0) = v_0 & \text{in } \Omega. \end{array} \tag{1.1}$$

with smooth initial data, $u_0 \geq 0$ and $v_0 \geq 0$.

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Parabolic reaction-diffusion systems like (1.1) or of a more general form, allowing coupling through source terms or with different boundary conditions, appear in several branches of applied mathematics. They have been used to model, for example, chemical reactions, heat transfer or population dynamics and have been studied by several authors. See [P] and the references therein.

Local existence in time, regularity and comparison results for solutions of several types of systems, including (1.1), have been obtained among others in [A], [EH1] and [RW]. See also the corresponding references.

Positive solutions of (1.1) blow up in finite time if $pq > 1$, see [RW], [D]. See also [HY] and the references therein for the scalar equation. Throughout this note, since we are concerned with uniqueness, we can always assume that the solution is defined and bounded in a small interval of time $[0, T]$.

It has been proved in [A], that under the assumption that the initial data is strictly positive, the solution of (1.1) is unique. The main result of this note deals with the uniqueness problem for (1.1) in the case of identically null initial data. More precisely we prove

Theorem 1.1 *Assume* $(u_0, v_0) \equiv (0, 0)$*. Then*

a) If $pq \geq 1$*, then the unique solution of problem (1.1) is* $(u, v) \equiv (0, 0)$*.*

b) If $pq < 1$*, then there exists exactly one solution,* $(\widetilde{u}, \widetilde{v})$ *of problem (1.1), such that both* $\widetilde{u}$ *and* $\widetilde{v}$ *are strictly positive for every positive time. Moreover, any other nonnegative nontrivial solution of (1.1) is of the form*

$$(\widetilde{u}((t-\tau)_+), \widetilde{v}((t-\tau)_+))$$

for some fixed $\tau > 0$*. Here* $r_+ = \max(r, 0)$*.*

Part *b)* of Theorem 1.1 is a generalization to several space dimensions of a result of [DFL], where non-uniqueness is proved in the case of $\Omega = (0, +\infty)$. It is also conjectured in that paper that the solution is unique if $pq > 1$.

In the case of non-trivial initial data we prove the following

Theorem 1.2 *If* $(u_0, v_0) \not\equiv (0, 0)$*, then the solution of (1.1) is unique.*

A related result can be found in [EH1]. In that paper the authors deal with non-negative solutions of the Cauchy problem, in all of R^N, for the system

$$\begin{array}{lcl} u_t = \Delta u + v^p, & & v_t = \Delta v + u^q, \\ u(x,0) = u_0 & \text{and} & v(x,0) = v_0 \end{array}$$

with $u_0 \geq 0$ and $v_0 \geq 0$. They prove that if $(u_0, v_0) \not\equiv (0, 0)$, then the solution is unique and, that for $(u_0, v_0) \equiv (0, 0)$ the solution is unique if and only if

$pq \geq 1$. Moreover, if $pq < 1$ then any non-trivial solution is of the form $(f((t-\tau)_+), g((t-\tau))_+)$, where $(f(t), g(t))$ is the space variable independent solution that can be calculated explicitly. See also [EH2] for the corresponding result for the Dirichlet problem on bounded domains. In the case of the scalar equation with a source term, this type of results were already obtained in [FW].

The rest of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. The method of the proof of part a), which is standard, relies in a fixed point argument and uses the representation of the solution through a variation of constants formula and, some estimates for the Green's function that can be found in [F]. The proof of part b) is based upon a classical monotonicity argument. The key step in this proof is the construction of a non trivial sub-solution of (1.1) in the case that $pq < 1$ and $(u_0, v_0) \equiv (0, 0)$. This construction uses ideas already used in [CDE].

In the last section we prove Theorem 1.2 using ideas from [EH2] and part b) of Theorem 1.1.

2 Proof of Theorem 1.1.

Proof of part a).

Fix $T > 0$. For any $r > 0$ define the operator

$$N_r : C(\partial\Omega \times [0, T]) \to C(\partial\Omega \times [0, T])$$

by

$$N_r(h) = w_r \mid_{\partial\Omega \times [0,T]}$$

where w_r is the solution of

$$\begin{aligned} w_t &= \Delta w \\ \frac{\partial w}{\partial \eta} &= (h)^r \\ w(x, 0) &= 0. \end{aligned}$$

Observe that if (u, v) is a solution of (1.1), then $h = v \mid_{\partial\Omega \times [0,T]}$ is a fixed point of the operator $N_q \circ N_p$. Also, by continuity, $v(x, t) \leq 1$ if t is small. Therefore we will prove the uniqueness of the zero solution as soon as we prove that there exists $T > 0$ and $0 < k < 1$ such that

$$||N_q \circ N_p(h)||_\infty \leq k||h||_\infty \tag{2.1}$$

for any h with $||h||_\infty \leq 1$.

Let Γ be the fundamental solution of the heat equation, that is

$$\Gamma(x, t) = \frac{1}{(4\pi t)^{n/2}} \exp[\frac{-||x||^2}{4t}].$$

Then, see Theorem 2 chap. 5 of [F], $w_r(x,t)$ can be written in the form

$$w_r(x,t) = \int_0^t \int_{\partial\Omega} (h(y,\tau))^r\, \Gamma(x-y, t-\tau)\ dS_y\ d\tau$$

$$-\int_0^t \int_{\partial\Omega} w_r(y,\tau) \frac{\partial\Gamma}{\partial\eta}(x-y,t-\tau)\ dS_y\ d\tau.$$

Letting $x \to \partial\Omega$, using Theorem 1 chap. 5 of [F], we obtain

$$\frac{1}{2} w_r(x,t) = \int_0^t \int_{\partial\Omega} (h(y,\tau))^r\, \Gamma(x-y, t-\tau)\ dS_y\ d\tau$$
$$-\int_0^t \int_{\partial\Omega} w_r(y,\tau) \frac{\partial\Gamma}{\partial\eta}(x-y,t-\tau)\ dS_y\ d\tau. \tag{2.2}$$

Pick $0 < \alpha < 1$ and fix μ such that $1 - \frac{\alpha}{2} < \mu < 1$. Using inequalities (1.4) and (2.12) of chapter 5 in [F], one obtains for any $x, y \in \partial\Omega$

$$\mid \Gamma(x-y, t-\tau) \mid \leq \frac{C}{(t-\tau)^{\frac{3}{4}} \mid x-y \mid^{n-\frac{3}{2}}} \tag{2.3}$$

and

$$\mid \frac{\partial\Gamma}{\partial\eta}(x-y,t-\tau) \mid \leq \frac{C}{(t-\tau)^{\mu} \mid x-y \mid^{n+1-2\mu-\alpha}}. \tag{2.4}$$

Let $M_r = \max_{\partial\Omega\times[0,T]} w_r(x,t)$. Let $(x_0,t_0) \in \partial\Omega \times [0,T]$ be a point such that $M_r = w_r(x_0,t_0)$. Then from (2.2)

$$\frac{1}{2} M_r \leq \|h\|_{L^\infty}^r \int_0^{t_0} \int_{\partial\Omega} \Gamma(x_0-y, t_0-\tau)\ dS_y\ d\tau$$

$$+\int_0^{t_0} \int_{\partial\Omega} w_r(y,\tau) \mid \frac{\partial\Gamma}{\partial\eta}(x_0-y,t_0-\tau) \mid\ dS_y\ d\tau.$$

Or, using (2.3) and (2.4),

$$\frac{1}{2} M_r \leq C T^{1/4} \|h\|_{L^\infty}^r + C T^{1-\mu} M_r$$

where C depends on n, μ and Ω.

So, if $CT^{1-\mu} < \frac{1}{4}$, we have

$$\|N_r(h)\|_{L^\infty} \leq C T^{1/4} \|h\|_{L^\infty}^r.$$

Consequently

$$\|N_q(N_p(h))\|_{L^\infty} \le CT^{1/4}\|N_p(h)\|_{L^\infty}^q \le CT^{(q+1)/4}\|h\|_{L^\infty}^{pq} \le CT^{(q+1)/4}\|h\|_{L^\infty}$$

as $pq \ge 1$ and $\|h\|_{L^\infty} \le 1$.

Finally, if T is such that $CT^{(q+1)/4} < k$, then (2.1) follows. This ends the proof of part a) of Theorem 1.1.

Proof of part b).

We will construct, in the case $pq < 1$, a non trivial non negative sub-solution of (1.1). This will imply via a standard monotonicity argument that the zero solution is not unique. For this argument we will need the following comparison lemma that can be found in [A] and [RW].

Lemma 2.1 *If $(\underline{u}, \underline{v})$ is a sub-solution and $(\overline{u}, \overline{v})$ a super-solution of (1.1) such that $\underline{u}(x,0) < \overline{u}(x,0)$ and $\underline{v}(x,0) < \overline{v}(x,0)$ for all $x \in \overline{\Omega}$, then $\underline{u} < \overline{u}$ and $\underline{v} < \overline{v}$ in $\overline{\Omega} \times [0,T)$ provided both exist up to time T.*

Before we can define the sub-solution we need some previous results.

Lemma 2.2 *Given $\gamma > 0$ and $0 < \xi_1 < 1$, then there exists $\xi_0(\gamma) < \xi_1$ and a positive solution, z_γ, of*

$$\gamma z(\xi) - \frac{1}{2} z'(\xi)\xi = z''(\xi) \text{ on } [0, \xi_0(\gamma)) \tag{2.5}$$

with $z_\gamma(\xi_0(\gamma)) = 0$, $z_\gamma(0) = 1$ and such that $z'_\gamma < 0$ and $z''_\gamma > 0$ on $[0, \xi_0(\gamma)]$.

Proof:
Set

$$\begin{aligned} x(\xi) &= z(\xi) \\ y(\xi) &= z'(\xi). \end{aligned}$$

Now (2.5) reads

$$\begin{aligned} x'(\xi) &= y(\xi) \\ y'(\xi) &= \gamma x(\xi) - \frac{1}{2} y(\xi)\xi \end{aligned}$$

with initial conditions

$$x(0) = 1 \text{ and } y(0) = y_0$$

where y_0 is to be determined later.

The lemma will be proved as soon as we find $y_0 < 0$ such that the first zero, $\xi_0 = \xi_0(\gamma)$, of x is such that $\xi_0 < \xi_1$ and, $x(\xi) > 0$, $y(\xi) < 0$ whenever $0 < \xi < \xi_0$.

To do this we first observe that, since $y(\xi)$ is negative, $x(\xi) \leq 1$ and hence

$$y'(\xi) \leq \gamma - \frac{1}{2} y(\xi)\xi.$$

By integration we obtain that if $0 < \xi < \xi_1 < 1$, then

$$y(\xi) \leq \gamma \int_0^1 e^{s^2/4} ds + y_0 < 0$$

and, therefore

$$x'(\xi) \leq \gamma \int_0^1 e^{s^2/4} ds + y_0.$$

Integrating the last inequality we get

$$x(\xi_1) \leq 1 + \xi_1 (\gamma \int_0^1 e^{s^2/4} ds + y_0).$$

Hence $\xi_0 < \xi_1$ provided that $|y_0|$ is large enough. This proves the lemma. $\square$

Let (α, β) be the solution of

$$\begin{pmatrix} -1 & p \\ q & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

We observe that since $pq < 1$ then α and β are both positive.

Lemma 2.3 *There exist positive constants k_1 and k_2 such that the functions*

$$a(s,t) = k_1 t^{\alpha} z_{\alpha}\left(\frac{s}{\sqrt{t}}\right)$$

and

$$b(s,t) = k_2 t^{\beta} z_{\beta}\left(\frac{s}{\sqrt{t}}\right)$$

satisfy

$$\begin{array}{ll} a_t = a_{ss} \;\; on \;\; 0 \leq s \leq \xi_0(\alpha)\sqrt{t} & \qquad b_t = b_{ss} \;\; on \;\; 0 \leq s \leq \xi_0(\beta)\sqrt{t} \\ -a_s(0,t) = b^p(0,t) & \qquad -b_s(0,t) = a^q(0,t) \\ a(\xi_0(\alpha)\sqrt{t}, t) = 0 & \qquad a(\xi_0(\beta)\sqrt{t}, t) = 0. \end{array}$$

Proof:

The lemma is an immediate consequence of Lemma 2.2. $\square$

We can define now the sub-solution. In order to do it consider the following change of variables in a neighborhood of $\partial\Omega$. Let $\bar{x}$ a point in $\partial\Omega$. We denote by $\hat{n}(\bar{x})$ the inner unit normal to $\partial\Omega$ at the point $\bar{x}$. Since the boundary is smooth it is well known that there exists $\delta > 0$ such that the mapping $\varphi : \partial\Omega \times [0,\delta) \rightarrow R^N$ given by $\varphi(\bar{x}, s) = \bar{x} + s\hat{n}(\bar{x})$ defines new coordinates $(\bar{x}, s)$ in a neighborhood U of $\partial\Omega$ in $\overline{\Omega}$.

Let (μ, ν) be the solution of

$$\begin{pmatrix} -1 & p \\ q & -1 \end{pmatrix}\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Since $pq < 1$, then μ and ν are both positive.

Fix δ such that $0 < \delta < 1$. Define

$$\underline{u}_\varepsilon(\bar{x}, s, t) = \begin{cases} \varepsilon^\mu a(\frac{s}{\varepsilon}, (1-\varepsilon^\delta)\frac{t}{\varepsilon^2}) & \text{if} \quad s \leq \xi_0(\alpha)\sqrt{(1-\varepsilon^\delta)t} \\ 0 & \text{if} \quad \xi_0(\alpha)\sqrt{(1-\varepsilon^\delta)t} \leq s \end{cases}$$

$$\underline{v}_\varepsilon(\bar{x}, s, t) = \begin{cases} \varepsilon^\nu b(\frac{s}{\varepsilon}, (1-\varepsilon^\delta)\frac{t}{\varepsilon^2}) & \text{if} \quad s \leq \xi_0(\beta)\sqrt{(1-\varepsilon^\delta)t} \\ 0 & \text{if} \quad \xi_0(\beta)\sqrt{(1-\varepsilon^\delta)t} \leq s \end{cases}$$

in U, and extend $\underline{u}_\varepsilon$, $\underline{v}_\varepsilon$ as zero to the whole of Ω.

We can state now

Proposition 2.1 *There exists ϵ_0 and $T_0 > 0$ such that the pair of functions $(\underline{u}_{\varepsilon_0}(\bar{x}, s, t), \underline{v}_{\varepsilon_0}(\bar{x}, s, t))$ is a sub-solution of (1.1) in $\Omega \times [0, T_0]$.*

Proof:

We claim that the boundary conditions are satisfied. We have

$$-\frac{\partial \underline{u}_\varepsilon}{\partial s}(\bar{x}, 0, t) = -\varepsilon^{\mu-1}\frac{\partial a}{\partial s}(0, (1-\varepsilon^\delta)\frac{t}{\varepsilon^2}) =$$

$$\varepsilon^{\mu-1}(b(0, (1-\varepsilon^\delta)\frac{t}{\varepsilon^2}))^p = \varepsilon^{\mu-1}\varepsilon^{-\nu p}(\varepsilon^\nu b(0, (1-\varepsilon^\delta)\frac{t}{\varepsilon^2}))^p =$$

$$= \varepsilon^{\mu-1-\nu p}(\underline{v}_\varepsilon(\bar{x}, 0, t))^p = (\underline{v}_\varepsilon(\bar{x}, 0, t))^p$$

because of the choice of μ and ν.

Analogously we obtain

$$-\frac{\partial \underline{v}_\varepsilon}{\partial s}(\bar{x}, 0, t) = (\underline{u}_\varepsilon(\bar{x}, 0, t))^q$$

and the claim is proved.

We will see now how to compute the Laplacian in coordinates $(\bar{x}, s)$. Fix $\bar{x} \in \partial\Omega$. After a rotation and a translation we can assume that $\bar{x}$ is the origin and that the tangent space to $\partial\Omega$ at $\bar{x}$ is given by the equation $x_N = 0$. Moreover we can assume that the coordinates $(x_1, ..., x_{N-1})$ in $x_N = 0$ point in the principal directions at $\bar{x}$. We can also assume that, in a neighborhood of $\bar{x}$, $\partial\Omega$ is given by the equation $x_N = h(x_1, ..., x_{N-1})$ where $h \in C^4$. For $j = 1, ..., N-1$ we set $p_j = \frac{\partial h}{\partial x_j}$ and we denote by $H_j(\bar{x})$ the principal curvatures of $\partial\Omega$ at $\bar{x}$.

A straightforward computation shows that Δ applied to a function f evaluated at the point $(\bar{x}, s)$ is given by

$$\Delta(f) = L_s(f) + L_{\bar{x}}(f) \tag{2.6}$$

where

$$L_s f(\bar{x}, s) = \frac{\partial^2 f}{\partial s^2}(\bar{x}, s) - \sum_{j=1}^{N-1} \frac{H_j(\bar{x})}{(1 - H_j(\bar{x})s)} \frac{\partial f}{\partial s}(\bar{x}, s) \tag{2.7}$$

and

$$L_{\bar{x}} f(\bar{x}, s) = \sum_{j=1}^{N-1} \frac{1}{(1 - sH_j(\bar{x}))^2} \frac{\partial^2 f}{\partial x_j^2}(\bar{x}, s) +$$
$$s \sum_{i,j=1}^{N-1} \frac{1}{(1 - sH_j(\bar{x}))(1 - sH_i(\bar{x}))^2} \frac{\partial^2 p_j}{\partial x_i^2} \frac{\partial f}{\partial x_j}(\bar{x}, s).$$

If (s, t) is such that $0 < s \leq \xi_0(\alpha)\sqrt{(1 - \varepsilon^\delta)t}$ we have

$$(\underline{u}_\varepsilon)_t(s, t) - \Delta \underline{u}_\varepsilon(s, t) =$$
$$-\epsilon^{\mu+\delta-2} a_t(\frac{s}{\varepsilon}, (1 - \varepsilon^\delta)\frac{t}{\varepsilon^2})) + \epsilon^{\mu-1} \sum_{j=1}^{N-1} \frac{H_j(\bar{x})}{(1 - H_j(\bar{x})s)} a_s(\frac{s}{\varepsilon}, (1 - \varepsilon^\delta)\frac{t}{\varepsilon^2})) =$$
$$-\epsilon^{\mu+\delta-2} a_{ss}(\frac{s}{\varepsilon}, (1 - \varepsilon^\delta)\frac{t}{\varepsilon^2})) + \epsilon^{\mu-1} \sum_{j=1}^{N-1} \frac{H_j(\bar{x})}{(1 - H_j(\bar{x})s)} a_s(\frac{s}{\varepsilon}, (1 - \varepsilon^\delta)\frac{t}{\varepsilon^2})).$$

Or, using the definition of a,

$$(\underline{u}_\varepsilon)_t(s, t) - \Delta \underline{u}_\varepsilon(s, t) =$$

$$-\epsilon^{\mu+\delta-2}((1 - \varepsilon^\delta)\frac{t}{\varepsilon^2})^{\alpha-1} k_1 z''_\alpha(s/\sqrt{(1 - \varepsilon^\delta)t})$$

$$\times \left[1 - \epsilon^{1-\delta}((1 - \varepsilon^\delta)\frac{t}{\varepsilon^2})^{\frac{1}{2}} \sum_{j=1}^{N-1} \frac{H_j(\bar{x})}{(1 - \epsilon H_j(\bar{x})s)} \frac{z'_\alpha(s/\sqrt{(1 - \varepsilon^\delta)t})}{z''_\alpha(s/\sqrt{(1 - \varepsilon^\delta)t})} \right].$$

Since the function $\xi \to z'_\alpha(\xi)/z''_\alpha(\xi)$ is bounded and $z''_\alpha(\xi) > 0$ for $\xi \in [0, \xi_0(\alpha)]$, we obtain that if ϵ is small enough then

$$(\underline{u}_\varepsilon)_t(s, t) - \Delta \underline{u}_\varepsilon(s, t) \leq 0 \tag{2.8}$$

if $0 < s \leq \xi_0(\alpha)\sqrt{(1-\varepsilon^\delta)t}$.

In a similar fashion one proves that for small enough ε

$$(\underline{v}_\varepsilon)_t(s,t) - \Delta \underline{v}_\varepsilon(s,t) \leq 0 \tag{2.9}$$

if $0 < s \leq \xi_0(\beta)\sqrt{(1-\varepsilon^\delta)t}$.

After choosing $\varepsilon = \varepsilon_0$ small enough such that (2.8) and (2.9) hold, pick s_0 such that $\{(\bar{x}, s)/0 < s < s_0\}$ is contained in U. Choose T_0 such that for all $0 < t < T_0$ one has $\xi_0(\alpha)\sqrt{(1-\varepsilon^\delta)t} \leq s_0$ and $\xi_0(\beta)\sqrt{(1-\varepsilon^\delta)t} \leq s_0$.

Since $(0,0)$ is a solution of (1.1), the proposition follows now by the standard argument that proves that the maximum among sub-solutions is a sub-solution. $\square$

We can end now the proof of part b) of the theorem.

The argument that follows is classical. Let (u_n, v_n) be the solution of (1.1) in $\overline{\Omega} \times [0, T_0)$ with initial data $u_0 = v_0 = 1/n$. By lemma 2.1, the sequence (u_n, v_n) is decreasing.

We define

$$\widetilde{u} = \lim_{n \nearrow \infty} u_n$$

and

$$\widetilde{v} = \lim_{n \nearrow \infty} v_n.$$

Passing to the limit in the variation of constant formula we obtain that $(\widetilde{u}, \widetilde{v})$ is a solution of (1.1).

Again by comparison, $u_n > \underline{u}_{\varepsilon_0}$ and $v_n > \underline{v}_{\varepsilon_0}$ for all n, so the same holds for $\widetilde{u}$ and $\widetilde{v}$. Hence we have constructed a non trivial solution of (1.1) with $\widetilde{u}(x,0) \equiv \widetilde{v}(x,0) \equiv 0$.

It remains to prove the statement about uniqueness. Suppose, for a contradiction, that there exists another solution (w,z) of (1.1) such that is strictly positive for every positive time. By the comparison principle $\widetilde{u}(t) \leq w(t+\tau)$ and $\widetilde{v}(t) \leq z(t+\tau)$ for every $\tau > 0$. Letting $\tau \to 0$ we obtain $\widetilde{u}(t) \leq w(t)$ and $\widetilde{v}(t) \leq z(t)$. Analogously $\widetilde{u} \geq w$ and $\widetilde{v} \geq z$ and part b) of Theorem 1.1 is proved.

3 Proof of Theorem 1.2.

We note that if the initial condition is non-negative and non-trivial, then a non-negative solution of (1.1) is such that $u > 0$ and $v > 0$ for all positive times as long as the solution exists. This can be seen by comparison with the solution of the corresponding problem with homogeneous Neumann boundary data.

We use a variation of the ideas in [EH2]. Observe that there exists a maximal solution (u_M, v_M) of (1.1) in the sense that if (u,v) is any solution of (1.1), then $u \leq u_M$ and $v \leq v_M$. This maximal solution is constructed, in a standard fashion, as the limit as $n \to \infty$ of the unique solutions, (u_n, v_n), of (1.1) with initial condition $(u_0 + \frac{1}{n}, v_0 + \frac{1}{n})$.

In order to prove uniqueness it suffices to show that if (u, v) is any solution of (1.1), then

$$u_M \leq u \text{ and } v_M \leq v. \tag{3.1}$$

We distinguish three cases.

Case 1. If $p \geq 1$ and $q \geq 1$.

In this case uniqueness follows from the results in [A], since the non linearities involved are locally Lipschitz continuous.

Case 2. If $p < 1$ and $q \leq 1$.

Let $w = u_M - u$ and $z = v_M - v$. Since $p < 1$ and $q \leq 1$ we have that (w, z) satisfies

$$\begin{array}{llll} w_t = \Delta w, & & z_t = \Delta z & \text{in } \Omega \times (0,T) \\ \frac{\partial w}{\partial \eta} \leq z^p, & & \frac{\partial z}{\partial \eta} \leq w^q & \text{on } \partial\Omega \times (0,T) \\ w(x,0) \equiv 0 & \text{and} & z(x,0) \equiv 0 & \text{in } \Omega. \end{array}$$

As $pq < 1$, by Theorem 1.1 there exists a unique solution (a, b) of

$$\begin{array}{llll} a_t = \Delta a, & & b_t = \Delta b & \text{in } \Omega \times (0,T) \\ \frac{\partial a}{\partial \eta} = b^p, & & \frac{\partial b}{\partial \eta} = a^q & \text{on } \partial\Omega \times (0,T) \\ a(x,0) \equiv 0 & \text{and} & b(x,0) \equiv 0 & \text{in } \Omega \end{array}$$

such that $a > 0$ and $b > 0$ for all $t > 0$.

We note that $a \geq w$ and $b \geq z$. Now set $f = a - w$ and $g = b - z$. Then

$$\begin{array}{llll} f_t = \Delta f, & & g_t = \Delta g & \text{in } \Omega \times (0,T) \\ \frac{\partial f}{\partial \eta} \geq g^p, & & \frac{\partial g}{\partial \eta} \geq f^q & \text{on } \partial\Omega \times (0,T) \\ f(x,0) \equiv 0 & \text{and} & g(x,0) \equiv 0 & \text{in } \Omega. \end{array}$$

Here we have used the fact that if $q \leq 1$ and a, u_M and u are non negative numbers such that $a + u \geq u_M$, $a \leq u_M$ and $u \leq u_M$, then

$$a^q - u_M^q + u^q \geq (a - u_M + u)^q.$$

Similarly for $p \leq 1$ and b, v_M and v. See [EH2] or [AE].

We claim that $f > 0$ and $g > 0$ for all $t > 0$. Indeed, if not, by the observation at the beginning of this section, there exists t_0 such that $f(x,t) \equiv 0$ and $g(x,t) \equiv 0$ for all $0 \leq t \leq t_0$. But in this case we have for $0 < t < t_0$

$$(b + v)^p = v_M^p = \frac{\partial u_M}{\partial \eta} = \frac{\partial a}{\partial \eta} + \frac{\partial u}{\partial \eta} = b^p + v^p.$$

This is a contradiction since $b > 0$, $v > 0$ and $p < 1$. The claim is proved.

By comparison it follows that $f \geq a$ and $g \geq b$. This implies (3.1) and the theorem holds under the assumption $p < 1$ and $q \leq 1$.

Case 3. If $0 < p < 1 < q$.

As in [EH2] we reduce this case to Case 2. Set

$$w = \alpha(u_M - u) \text{ and } z = \beta(v_M - v)$$

with α and β to be determined later.

Since $p < 1$,

$$\frac{\partial w}{\partial \eta} = \alpha(v_M^p - v^p) \leq \alpha(v_M - v)^p = \alpha\beta^{-p} z^p.$$

Also, since $q > 1$,

$$\frac{\partial z}{\partial \eta} = \beta(u_M^q - u^q) = \alpha^{-1}\beta \frac{u_M^q - u^q}{u_M - u} w \leq \alpha^{-1}\beta K w$$

for some constant $K > 0$ that depends on the bounds for u_M on $\Omega \times [0, T]$. It is clear that α and β can be chosen to be positive and such that (w, z) satisfies

$$\begin{array}{lll} w_t = \Delta w, & z_t = \Delta z & \text{in } \Omega \times (0,T) \\ \frac{\partial w}{\partial \eta} \leq z^p, & \frac{\partial z}{\partial \eta} \leq w & \text{on } \partial\Omega \times (0,T) \\ w(x,0) \equiv 0 \quad \text{and} & z(x,0) \equiv 0 & \text{in } \Omega. \end{array}$$

Now the arguments of Case 2 give (3.1) and the theorem is proved.

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