UNIFORM STABILITY OF THE BALL WITH RESPECT TO THE FIRST DIRICHLET AND NEUMANN $\infty-$EIGENVALUES

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ABSTRACT. In this note we analyze how perturbations of a ball $\mathfrak{B}_r \subset \mathbb{R}^n$ behaves in terms of their first (non-trivial) Neumann and Dirichlet $\infty-$eigenvalues when a volume constraint $\mathcal{L}^n(\Omega) = \mathcal{L}^n(\mathfrak{B}_r)$ is imposed. Our main result states that Ω is uniformly close to a ball when it has first Neumann and Dirichlet eigenvalues close to the ones for the ball of the same volume $\mathfrak{B}_r$. In fact, we show that, if

$$|\lambda^D_{1,\infty}(\Omega) - \lambda^D_{1,\infty}(\mathfrak{B}_r)| = \delta_1 \quad \text{and} \quad |\lambda^N_{1,\infty}(\Omega) - \lambda^N_{1,\infty}(\mathfrak{B}_r)| = \delta_2,$$

then there are two balls such that

$$\mathfrak{B}_{\frac{r}{\delta_1 r+1}} \subset \Omega \subset \mathfrak{B}_{\frac{r+\delta_2 r}{1-\delta_2 r}}.$$

In addition, we also obtain a result concerning stability of the Dirichlet $\infty-$eigenfunctions.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (connected open subset) with smooth boundary, $1 < p < \infty$ and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ (the standard p-Laplacian operator). Historically (cf. [13]), it well-known that the first eigenvalue (referred as *the principal frequency* in physical models) of the $p-$Laplacian Dirichlet eigenvalue problem

$$\text{(1.1)} \qquad \left\{ \begin{array}{rcll} -\Delta_p u & = & \lambda^D_{1,p}(\Omega)|u|^{p-2}u & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \right.$$

can be characterized variationally as the minimizer of the following (normalized) problem:

$$(\textbf{p-Dirichlet}) \qquad \lambda^D_{1,p}(\Omega) := \inf_{u \in W^{1,p}_0(\Omega)\setminus\{0\}} \left\{ \int_\Omega |\nabla u|^p dx : \int_\Omega |u|^p dx = 1 \right\}.$$

In the theory of shape optimization and nonlinear eigenvalue problems obtaining (sharp) estimates for the eigenvalues in terms of geometric quantities of the domain (e.g. measure, perimeter, diameter, among others) plays a fundamental role due to several applications of these problems in pure and applied sciences. We recall that the explicit value to (**p-Dirichlet**) is known only for some specific values of p or for very particular domains Ω. Notice that upper bounds for $\lambda^D_{1,p}(\Omega)$ are usually obtained by selecting particular test functions in (**p-Dirichlet**). Nevertheless, lower bounds are a more challenging task. In this direction we have the remarkable

2010 *Mathematics Subject Classification.* 35B27, 35J60, 35J70.

Key words and phrases. $\infty-$eigenvalues estimates, $\infty-$eigenvalue problem, approximation of domains.

Faber-Krahn inequality: *Among all domains of prescribed volume the ball minimizes* (**p-Dirichlet**). More precisely,

$$\lambda^D_{1,p}(\Omega) \geq \lambda^D_{1,p}(\mathfrak{B}), \tag{1.2}$$

where $\mathfrak{B}$ is the n-dimensional ball such that $\mathcal{L}^n(\Omega) = \mathcal{L}^n(\mathfrak{B})$ (along this paper $\mathcal{L}^n(\Omega)$ will denote the Lebesgue measure of Ω that is assumed to be fixed). Using isoperimetric or isodiametric inequality similar lower bounds for (**p-Dirichlet**) in terms of the perimeter (resp. diameter) of Ω are also available (cf. [1] and [14, page 224], and the references therein). Recently, stability estimates for certain geometric inequalities were established in [10], thereby providing an improved version of (1.2) by adding a suitable remainder term, i.e.,

$$\lambda^D_{1,p}(\Omega) \geq \lambda^D_{1,p}(\mathfrak{B})\left(1 + \gamma_{p,n}(\mathcal{S}(\Omega))^{2+p}\right),$$

where $\mathcal{S}(\Omega)$ is the so-called *Fraenkel asymmetry* of Ω, which is precisely defined as

$$\mathcal{S}(\Omega) := \inf_{x_0\in\mathbb{R}^n}\left\{\frac{\mathcal{L}^n(\Omega \triangle \mathfrak{B}_r(x_0))}{\mathcal{L}^n(\Omega)} : \mathcal{L}^n(\mathfrak{B}_r(x_0)) = \mathcal{L}^n(\Omega)\right\},$$

and $\gamma_{p,n}$ is a constant. Observe that $\mathcal{S}$ measures the distance of a set Ω from being a ball. For such quantitative estimates and further related topics we quote [2], [4], [9] and references therein.

Our main goal here is to find stability results for the limit case $p = \infty$.

First, we introduce what is known for the limit as $p \to \infty$ in the eigenvalue problem for the $p-$Laplacian. When one takes the limit as $p \to \infty$ in the minimization problem (**p-Dirichlet**), one obtains

$$(\infty\textbf{-Dirichlet}) \qquad \lambda^D_{1,\infty}(\Omega) := \lim_{p\to\infty} \sqrt[p]{\lambda^D_{1,p}(\Omega)} = \inf_{u\in W^{1,\infty}_0(\Omega)\setminus\{0\}} \|\nabla u\|_{L^\infty(\Omega)} > 0,$$

see [11]. Concerning the limit equation, also in [11] it is proved that any family of normalized eigenfunctions $\{u_p\}_{p>1}$ to (**p-Dirichlet**) converges (up to a subsequence) locally uniformly to $u_\infty \in W^{1,\infty}_0(\Omega)$, a minimizer for ∞**-Dirichlet** with $\|u_\infty\|_{L^\infty(\Omega)} = 1$. Moreover, the pair $(u_\infty, \lambda^D_{1,\infty}(\Omega))$ is a nontrivial solution to

$$\left\{\begin{array}{rcll} \min\left\{-\Delta_\infty v_\infty, |\nabla v_\infty| - \lambda^D_{1,\infty}(\Omega)v_\infty\right\} & = & 0 & \text{in} \quad \Omega \\ v_\infty & = & 0 & \text{on} \quad \partial\Omega. \end{array}\right. \tag{1.3}$$

Solutions to (1.3) must be understood in the viscosity sense (cf. [6] for a survey) and $\Delta_\infty u(x) := \nabla u(x)^T D^2u(x)\cdot\nabla u(x)$ is the well-known $\infty-$*Laplace operator*. In addition, also in [11], it is given an interesting and useful geometrical characterization for (∞**-Dirichlet**):

$$\lambda^D_{1,\infty}(\Omega) = \left(\max_{x\in\Omega}\operatorname{dist}(x,\partial\Omega)\right)^{-1}. \tag{1.4}$$

Such an information means that the "principal frequency" for the ∞-eigenvalue problem can be detected from the geometry of the domain: it is precisely the reciprocal of radius $\mathfrak{r}_\Omega > 0$ of the largest ball inscribed in Ω. For more references concerning the first eigenvalue (1.3) we refer to [12], [15] and [18].

Now, let us turn our attention to Neumann boundary conditions and consider the following eigenvalue problem:

$$\begin{cases} -\Delta_p u &= \lambda_{1,p}^N(\Omega)|u|^{p-2}u \quad \text{in} \quad \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= 0 \qquad\qquad\qquad \text{on} \quad \partial\Omega. \end{cases} \tag{1.5}$$

As before, we stress that the first non-zero eigenvalue of (1.5) can also be characterized variationally as the minimizer of the following normalized problem:
(**p-Neumann**)

$$\lambda_{1,p}^N(\Omega) := \inf_{u\in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p dx : \int_\Omega |u|^p dx = 1 \text{ and } \int_\Omega |u|^{p-2}u dx = 0 \right\}.$$

The celebrated *Payne-Weinberger inequality* provides a lower bound (on any convex domain $\Omega \subset \mathbb{R}^n$) for the first (non-trivial) Neumann $p-$eigenvalue (cf. [8] and [17])

$$\lambda_{1,p}^N(\Omega) \geq (p-1)\left(\frac{2\pi}{p\,\mathrm{diam}(\Omega)\,\sin(\frac{\pi}{p})}\right)^p. \tag{1.6}$$

For a stability estimate for this problem with $p=2$ we refer to [2].

When $p\to\infty$, the minimization problem (**p-Neumann**) becomes

$$(\infty\textbf{-Neumann}) \qquad \lambda_{1,\infty}^N(\Omega) := \lim_{p\to\infty} \sqrt[p]{\lambda_{1,p}^N(\Omega)} = \inf_{\substack{u\in W^{1,\infty}(\Omega) \\ \max_\Omega u = -\min_\Omega u = 1}} \|\nabla u\|_{L^\infty(\Omega)},$$

see [7] and [16]. Concerning the limit equation, also in [7] and [16], it is proved that any family of normalized eigenfunctions $\{u_p\}_{p>1}$ to (**p-Neumann**) converges (up to subsequence) locally uniformly to $u_\infty \in W_0^{1,\infty}(\Omega)$ with $\|u_\infty\|_{L^\infty(\Omega)} = 1$. Moreover, the pair $(u_\infty, \lambda_{1,\infty}^N(\Omega))$ is a nontrivial solution to

$$\begin{cases} \min\left\{ -\Delta_\infty v_\infty, |\nabla v_\infty| - \lambda_{1,\infty}^N(\Omega)v_\infty \right\} = 0 & \text{in} \quad \Omega\cap\{v>0\} \\ \max\left\{ -\Delta_\infty v_\infty, -|\nabla v_\infty| - \lambda_{1,\infty}^N(\Omega)v_\infty \right\} = 0 & \text{in} \quad \Omega\cap\{v<0\} \\ -\Delta_\infty v_\infty = 0 & \text{in} \quad \Omega\cap\{v=0\} \\ \dfrac{\partial v_\infty}{\partial\nu} = 0 & \text{in} \quad \partial\Omega. \end{cases} \tag{1.7}$$

In addiction, we have the following geometrical characterization for $\lambda_{1,\infty}^N(\Omega)$:

$$\lambda_{1,\infty}^N(\Omega) = \frac{2}{\mathrm{diam}(\Omega)}, \tag{1.8}$$

where the intrinsic diameter of Ω is defined as

$$\mathrm{diam}(\Omega) := \max_{\bar\Omega\times\bar\Omega} d_\Omega(x,y) = \max_{\partial\Omega\times\partial\Omega} d_\Omega(x,y),$$

being $d_\Omega(x,y)$ the geodesic distance given by $d_\Omega(x,y) = \inf_\gamma \mathrm{Long}(\gamma)$, where the infimum is taken over all possible Lipschitz curves in $\bar\Omega$ connecting x and y.

We remark that in the limit case $p=\infty$, the geometrical characterization (1.8) of (∞**-Neumann**) yields several interesting consequences:

✓ If $\mathcal{L}^n(\Omega) = \mathcal{L}^n(\mathfrak{B})$, $\mathfrak{B}$ being a ball, then $\lambda_{1,\infty}^N(\Omega) \leq \lambda_{1,\infty}^N(\mathfrak{B})$, which establishes a *Szegö-Weinberger type inequality*: among all domains of prescribed volume the ball maximizes (∞**-Neumann**).

✓ $\lambda^N_{1,\infty}(\Omega) \le \lambda^D_{1,\infty}(\Omega)$ for any convex Ω with equality if and only if Ω is a ball.

✓ The Payne-Weinberger inequality, (1.6), becomes an equality when $p = \infty$.

Taking account the previous historic overview, we arrive to our main result, which establishes the stability of the ball with respect to small perturbations of their first Dirichlet and Neumann $\infty-$eigenvalues. More precisely, if a domain $\Omega \subset \mathbb{R}^n$ has Dirichlet and Neumann $\infty-$eigenvalues close enough to those of the ball $\mathfrak{B}_r$ of the same Lebesgue measure, then Ω is uniformly "almost" ball-shaped.

Theorem 1.1. *Let Ω be an open domain satisfying $\mathcal{L}^n(\Omega) = \mathcal{L}^n(\mathfrak{B}_r)$. If for some $\delta_i > 0$ $(i = 1, 2)$ small enough it holds that*

$$|\lambda^D_{1,\infty}(\Omega) - \lambda^D_{1,\infty}(\mathfrak{B}_r)| = \delta_1 \quad \textit{and} \quad |\lambda^N_{1,\infty}(\Omega) - \lambda^N_{1,\infty}(\mathfrak{B}_r)| = \delta_2,$$

then there are two balls such that

$$\mathfrak{B}_{\frac{r}{\delta_1 r+1}} \subset \Omega \subset \mathfrak{B}_{\frac{r+\delta_2 r}{1-\delta_2 r}}.$$

The previous theorem implies the following convergence result.

Theorem 1.2. *Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a family of uniformly bounded domains satisfying $\mathcal{L}^n(\Omega_k) = \mathcal{L}^n(\mathfrak{B}_r)$. If*

$$|\lambda^D_{1,\infty}(\Omega_k) - \lambda^D_{1,\infty}(\mathfrak{B}_r)| = o(1) \quad \textit{and} \quad |\lambda^N_{1,\infty}(\Omega) - \lambda^N_{1,\infty}(\mathfrak{B}_r)| = o(1) \quad \textit{as } k \to \infty,$$

then

$$\Omega_k \to \mathfrak{B}_r$$

in the sense that the Hausdorff distance between Ω and a ball $\mathfrak{B}_r$ goes to zero, i.e.,

$$d_{\mathcal{H}}(\Omega_k, \mathfrak{B}_r) := \max\left\{ \sup_{x\in\Omega_k} \inf_{y\in\mathfrak{B}_r} d(x,y),\ \sup_{y\in\mathfrak{B}_r} \inf_{x\in\Omega_k} d(x,y) \right\} \to 0.$$

Note that our results imply that

$$\max\left\{\mathcal{L}^n\left(\Omega \triangle \mathfrak{B}_{\frac{r}{\delta_1 r+1}}\right), \mathcal{L}^n\left(\Omega \triangle \mathfrak{B}_{\frac{r+\delta_2 r}{1-\delta_2 r}}\right)\right\} \le \mathfrak{C}(n, \delta_i, r) r^n. \tag{1.9}$$

where $\mathfrak{C}(n, \delta_i, r) = \omega_n \max\{(\delta_1 r + 1)^n - 1, (n-1)\delta_2\} \to 0$ as $\delta_i \to 0$. Hence, we can control the Fraenkel asymmetry of the set, $S(\Omega)$. But our results give much more since we have a sort of uniform control on how far the set is from being a ball (for instance, we have convergence in Hausdorff distance in Theorem 1.2).

Another important question in this theory consists on how the corresponding $\infty-$ground states (solutions to (1.3)) behave in relation to perturbations of the $\infty-$eigenvalues of the ball. The next result provides an answer for this issue, showing that Dirichlet $\infty-$eigenfunctions are uniformly close to a cone when the first Dirichlet and Neumann $\infty-$eigenvalues are close to those for the ball. Note that, in general, the $\infty-$eigenvalue problem (1.3) may have multiple solutions (the first eigenvalue may not be simple), see [5] and [18].

Theorem 1.3. *Let Ω be an open domain satisfying $\mathcal{L}^n(\Omega) = \mathcal{L}^n(\mathfrak{B}_r)$. Given $\varepsilon > 0$ there are $\delta_i(\varepsilon) > 0$ $(i = 1, 2)$ small enough such that: if*

$$|\lambda^D_{1,\infty}(\Omega) - \lambda^D_{1,\infty}(\mathfrak{B}_r)| < \delta_1 \quad \textit{and} \quad |\lambda^N_{1,\infty}(\Omega) - \lambda^N_{1,\infty}(\mathfrak{B}_r)| < \delta_2,$$

then

$$|u(x) - v_\infty(x)| < \varepsilon \ \ in \ \ \Omega \cap \mathfrak{B}_r,$$

where

$$v_\infty(x) = 1 - \frac{|x|}{r}$$

is the normalized $\infty-$ground state to (1.3) *in* $\mathfrak{B}_r$.

Theorem 1.3 can be rewritten as follows:

Corollary 1.4. *Let $\{u_k\}_{k\in\mathbb{N}}$ be a family of normalized solutions to* (1.3) *in* Ω_k *such that*

$$|\lambda^D_{1,\infty}(\Omega_k) - \lambda^D_{1,\infty}(\mathfrak{B}_r)| = o(1) \quad and \quad |\lambda^N_{1,\infty}(\Omega_k) - \lambda^N_{1,\infty}(\mathfrak{B}_r)| = o(1) \quad as\ k \to \infty.$$

Then,

$$u_k \to v_\infty \quad locally\ uniformly \quad in \quad \mathfrak{B}_r,$$

where

$$v_\infty(x) = 1 - \frac{|x|}{r}$$

is the normalized $\infty-$ground state to (1.3) *in* $\mathfrak{B}_r$.

Our approach can be applied for other classes of operators with $p-$Laplacian type structure. We can deal with $p-$Laplacian type problems involving an *anisotropic p-Laplacian operator*

$$-\mathcal{Q}_p u := -\mathrm{div}(\mathbb{F}^{p-1}(\nabla u)\mathbb{F}_\xi(\nabla u)),$$

where $\mathbb{F}$ is an appropriate (smooth) norm of $\mathbb{R}^n$ and $1 < p < \infty$. The necessary tools for studying the anisotropic Dirichlet eigenvalue problem, as well as its limit as $p \to \infty$ can be found in [3]. Here, to obtain results similar to ours, one has to replace Euclidean balls with balls in the norm $\mathbb{F}$.

The paper is organized as follows: in Section 2 we prove our main stability results including the behavior of the corresponding $\infty-$eigenfunctions and in Section 3 we collect several examples that illustrate our results.

2. Proof of the Main Theorems

Before proving our main result we introduce some notations which will be used throughout this section. Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a ball $\mathfrak{B}_r \subset \mathbb{R}^n$ of radius $r > 0$ we denote $\lambda^D_{1,\infty}(\Omega)$ and $\lambda^D_{1,\infty}(\mathfrak{B}_r)$ the first Dirichlet eigenvalues (1.4) in Ω and in $\mathfrak{B}_r$, respectively; analogously, $\lambda^N_{1,\infty}(\Omega)$ and $\lambda^N_{1,\infty}(\mathfrak{B}_r)$ stand for the first nontrivial Neumann eigenvalues (1.8) in Ω and in $\mathfrak{B}_r$.

We introduce the following class of sets which will play an important role in our approach. For non-negative constants δ_1 and δ_2 we define the class:

$$\Xi_{\delta_1,\delta_2}(\mathfrak{B}_r) := \left\{ \begin{array}{c} \Omega \subset \mathbb{R}^n \\ \text{bounded domain with} \\ \mathcal{L}^n(\Omega) = \mathcal{L}^n(\mathfrak{B}_r) \end{array} : \begin{array}{rcl} |\lambda^D_{1,\infty}(\Omega) - \lambda^D_{1,\infty}(\mathfrak{B}_r)| & = & \delta_1 \\ & & \\ |\lambda^N_{1,\infty}(\Omega) - \lambda^N_{1,\infty}(\mathfrak{B}_r)| & = & \delta_2 \end{array} \right\}.$$

Notice that, $\Xi_{0,0}(\mathfrak{B}_r)$ consists of the family of all balls with radius $r > 0$. Another important remark is that the elements of $\Xi_{\delta_1,\delta_2}(\mathfrak{B}_r)$ are invariant by rigid movements (rotations, translations, etc).

Similarly, we can define the class $\Xi^D_{\delta_1}(\mathfrak{B}_r)$ (resp. $\Xi^N_{\delta_2}(\mathfrak{B}_r)$) as being $\Xi_{\delta_1,\delta_2}(\mathfrak{B}_r)$ with the restriction on the Dirichlet (resp. Neumann) eigenvalues only.

In the next lemma we show that a control on the difference of the first Dirichlet eigenvalue implies that Ω contains a large ball.

Lemma 2.1. *If $\Omega \in \Xi^D_{\delta_1}(\mathfrak{B}_r)$ then there exists a ball such that*

$$\mathfrak{B}_{\frac{r}{\delta_1 r+1}} \subset \Omega.$$

Moreover,

$$\mathcal{L}^n\left(\Omega \triangle \mathfrak{B}_{\frac{r}{\delta_1 r+1}}\right) \le \mathfrak{c}(n,\delta_1,r)r^n.$$

where $\mathfrak{c} = o(1)$ as $\delta_1 \to 0$.

Proof. According to (1.4) we have that

$$\delta_1 = |\lambda^D_{1,\infty}(\Omega) - \lambda^D_{1,\infty}(\mathfrak{B}_r)| = \left|\frac{1}{r_\Omega} - \frac{1}{r}\right|.$$

It follows that

$$r_\Omega \ge \frac{r}{\delta_1 r+1}.$$

and then there is ball such that

$$\mathfrak{B}_{\frac{r}{\delta r+1}} \subset \Omega.$$

Finally,

$$\begin{aligned}\mathcal{L}^n(\Omega\triangle\mathfrak{B}_{\frac{r}{\delta r+1}}) &= \mathcal{L}^n(\Omega) - \mathcal{L}^n(\mathfrak{B}_{\frac{r}{\delta r+1}})\\ &= \omega_n r^n\left(1 - \frac{1}{(\delta r+1)^n}\right)\\ &\le \omega_n r^n\left((\delta r+1)^n - 1\right)\\ &= \mathfrak{c}(n,\delta,r)r^n\end{aligned}$$

and the lemma follows. $\square$

Now, we show that a control on the difference of the first Neumann eigenvalue implies that Ω is contained in a small ball.

Lemma 2.2. *If $\Omega \in \Xi^N_{\delta_2}(\mathfrak{B}_r)$ then there is a ball such that*

$$\Omega \subset \mathfrak{B}_{\frac{r}{1-\delta_2 r}}.$$

Moreover,

$$\mathcal{L}^n\left(\Omega \triangle B_{\frac{r}{1-\delta_2 r}}\right) \le (n-1)\omega_n r^n \delta_2.$$

Proof. Using (1.8) we have that

$$\delta_2 = |\lambda^N_{1,\infty}(\Omega) - \lambda^N_{1,\infty}(\mathfrak{B}_r)| = \left|\frac{2}{\mathrm{diam}(\Omega)} - \frac{1}{r}\right|.$$

It follows that

$$\mathrm{diam}(\Omega) \le \frac{2r}{1-\delta_2 r} = r + \frac{r(1+\delta r)}{1-\delta_2 r}$$

and then there exists a ball such that

$$\Omega \subset \mathfrak{B}_{\frac{\mathrm{diam}(\Omega)}{2}} = \mathfrak{B}_{\frac{r}{1-\delta_2 r}}.$$

Moreover,

$$\begin{aligned}\mathcal{L}^n\left(\Omega \triangle \mathfrak{B}_{\frac{\mathrm{diam}(\Omega)}{2}}\right) &= \mathcal{L}^n\left(\mathfrak{B}_{\frac{\mathrm{diam}(\Omega)}{2}}\right) - \mathcal{L}^n(\Omega)\\ &= \omega_n r^n\left(\left(1+\frac{\delta_2}{1-\delta_2 r}\right)^n - 1\right)\\ &= \omega_n r^n \delta_2 \sum_{k=2}^{n}\left(\frac{\delta_2}{1-\delta_2 r}\right)^k\\ &\leq (n-1)\omega_n \delta_2 r^n\end{aligned}$$

and the lemma follows. $\square$

Proof of Theorem 1.1. The proof of Theorem 1.1 follows as an immediate consequence of Lemmas 2.1 and 2.2. $\square$

Next, we will prove Theorem 1.2.

Proof of Theorem 1.2. The hypothesis implies that $\Omega_k \in \Xi_{\delta_k,\varepsilon_k}(\mathfrak{B}_r)$ for $\delta_k, \varepsilon_k = \mathrm{o}(1)$ as $k \to \infty$. For this reason, by Theorem 1.1 there are two balls such that

$$\mathfrak{B}_{\frac{r}{\delta_k r+1}} \subset \Omega_k \subset \mathfrak{B}_{\frac{r+\varepsilon_k r}{1-\varepsilon_k r}}.$$

Now, using that all these balls are centered at points that are bounded (since we assumed that the family Ω_k is uniformly bounded), we can extract a subsequence such that the centers converge and therefore we conclude that there is a ball $\mathfrak{B}_r$ such that $\Omega_k \to \mathfrak{B}_r$ as $k \to \infty$. $\square$

Proof of Theorem 1.3. The proof follows by contradiction. Let us suppose that there exists an $\varepsilon_0 > 0$ such that the thesis of Theorem fails to hold. This means that for each $k \in \mathbb{N}$ we might find a domain Ω_k and u_k, a normalized $\infty-$ground state to (1.3) in Ω_k, such that $\Omega_k \in \Xi_{\gamma_k,\zeta_k}(\mathfrak{B}_r)$ with $\gamma_k, \zeta_k = \mathrm{o}(1)$ as $k \to \infty$, that is,

$$|\lambda^D_{1,\infty}(\Omega_k) - \lambda^D_{1,\infty}(\mathfrak{B}_r)| < \gamma_k \quad \text{and} \quad |\lambda^N_{1,\infty}(\Omega_k) - \lambda^N_{1,\infty}(\mathfrak{B}_r)| < \zeta_k,$$

with $\gamma_k, \zeta_k = \mathrm{o}(1)$ as $k \to \infty$, together with

$$|u_k(x) - v_\infty(x)| > \varepsilon_0 \quad \text{in } \Omega_k \cap \mathfrak{B}_r, \tag{2.1}$$

for every $k \in \mathbb{N}$.

Using our previous results, we can suppose that every $\Omega_k \subset \mathfrak{B}_{2r}$. Then, by extending u_k to zero outside of Ω_k, we may assume that $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}_0(\mathfrak{B}_{2r})$. In this context, standard arguments using viscosity theory show that, up to a subsequence, $u_k \to u_\infty$ uniformly in $\overline{\mathfrak{B}_{2r}}$, being the limit u_∞ a normalized eigenfunction for some domain $\hat{\Omega}$ with $\hat{\Omega} \Subset \mathfrak{B}_{2r}$. Moreover, we have that $\lambda^D_{1,\infty}(\Omega_k) \to \lambda^D_{1,\infty}(\hat{\Omega})$.

According to Theorem 1.2, $\Omega_k \to \mathfrak{B}_r$ as $k \to \infty$. By the previous sentences we conclude that $\hat{\Omega} = \mathfrak{B}_r$. Now, by uniqueness of solutions to (1.3) in $\mathfrak{B}_r$ we conclude that $u_\infty = v_\infty$. However, this contradicts (2.1) for $k \gg 1$ (large enough). Such a contradiction proves the theorem. $\square$

3. Examples

Given a fixed ball $\mathfrak{B}$ and a domain Ω having both of them the same volume, Theorem 1.1 says that if the $\infty-$eigenvalues are close each other then Ω is almost ball-shaped uniformly. The following examples illustrate Theorem 1.1 and 1.2.

Example 3.1. The reciprocal in Theorem 1.1 (and Theorem 1.2) is not true: given a fixed ball $\mathfrak{B}$, clearly, there are domains Ω fulfilling (1.9) such that the difference between the Neumann (and Dirichlet) eigenvalues in Ω and in $\mathfrak{B}$ is not small. Let us present some illustrative examples.

(1) A stadium. Let $\mathfrak{B}$ be the unit ball in $\mathbb{R}^2$ and Ω the stadium domain given in Figure 3 (a) with $\ell = \frac{\pi(1-\varepsilon^2)}{2\varepsilon}$. In this case $\mathcal{L}^n(\mathfrak{B}) = \mathcal{L}^n(\Omega) = \pi$ for any $0 < \varepsilon < 1$. However,
$$\lambda^N_{1,\infty}(\mathfrak{B}) = 1, \qquad \lambda^N_{1,\infty}(\Omega) = \frac{2}{\operatorname{diam}(\Omega)} = \frac{4\varepsilon}{\pi + \varepsilon^2(4-\pi)} < \frac{1}{3} \quad \text{if } \varepsilon < \frac{1}{4}.$$

(2) A ball with holes. If $\Omega = B(0, \sqrt{1+\varepsilon^2}) \setminus B(0, \varepsilon)$ is the domain given in Figure 3 (b), then $\mathcal{L}^n(\mathfrak{B}) = \mathcal{L}^n(\Omega) = \pi$, however
$$\lambda^D_{1,\infty}(\mathfrak{B}) = 1, \qquad \lambda^D_{1,\infty}(\Omega) = \frac{1}{\sqrt{1+\varepsilon^2}} > \frac{3}{2} \quad \text{if } \frac{3}{4} < \varepsilon < 1.$$

(3) A ball with thin tubular branches. If Ω is the domain given in Figure 3 (c), the condition $\mathcal{L}^n(\mathfrak{B}) = \mathcal{L}^n(\Omega)$ gives the relation
$$r(r+\varepsilon) + \varepsilon(\tfrac{1}{\pi} + \tfrac{\varepsilon}{2}) = 1, \qquad \operatorname{diam}(\Omega) = 1 + r + \pi(1+r).$$
For instance, if we take $\varepsilon = 10^{-3}$ it follows that $r \sim 0.999465$ and then
$$\lambda^N_{1,\infty}(\mathfrak{B}) = \frac{2}{\operatorname{diam}(\mathfrak{B})} = 1, \qquad \lambda^N_{1,\infty}(\Omega) = \frac{2}{\operatorname{diam}(\Omega)} \sim 0.2415.$$

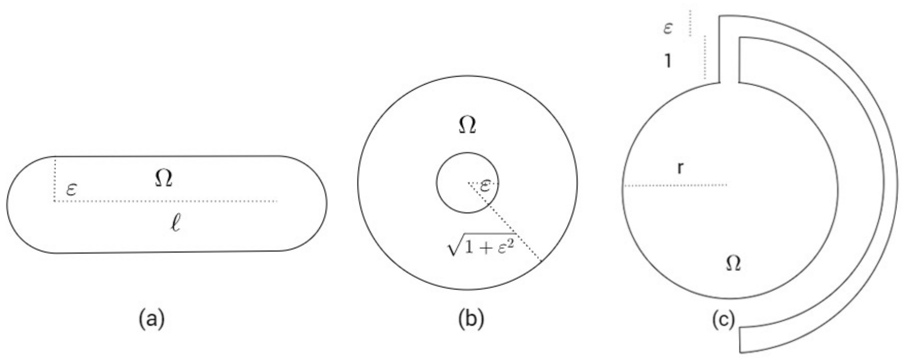

Figure 1. Three examples of domains

Hence, in view of these examples we conclude that a domain that has Dirichlet and Neumann $\infty-$eigenvalues close to the ones for the ball is close to a ball not only in the sense that $\mathcal{L}^n(\Omega \triangle \mathfrak{B}_r)$ is small but it can not contain holes deep inside (small holes near the boundary are allowed) and can not have thin tubular branches.

Example 3.2. The regular polygon $\mathbb{P}_k$ of $k-$sides ($k \geq 3$) centered at the origin such that $\mathcal{L}^n(\mathbb{P}_k) = \mathcal{L}^n(\mathfrak{B}_r)$ satisfies

$$|\lambda^D_{1,\infty}(\mathbb{P}_k) - \lambda^D_{1,\infty}(B_r)| = \delta_1 \quad \text{and} \quad |\lambda^N_{1,\infty}(B_r) - \lambda^N_{1,\infty}(\mathbb{P}_k)| = \delta_2,$$

where

$$\delta_1 = \frac{1}{r\sqrt{\frac{\pi}{k\tan(\frac{\pi}{k})}}} - \frac{1}{r} \quad \text{and} \quad \delta_2 = \frac{1}{r} - \frac{1}{r\sqrt{\frac{2\pi}{k\sin(\frac{2\pi}{k})}}}.$$

Therefore, we can recover the well known convergence $\mathbb{P}_k \to \mathfrak{B}_r$ as $k \to \infty$.

Example 3.3. Given $k \in \mathbb{N}$ and positive constants $\mathfrak{a}^k_1, \cdots, \mathfrak{a}^k_n$, the $n-$dimensional ellipsoid given by

$$\mathcal{E}_k := \left\{ (x_1, \cdots, x_n) \mid \sum_{i=1}^n \left(\frac{x_i}{\mathfrak{a}^k_i}\right)^2 < 1 \right\}$$

such that $\mathcal{L}^n(\mathcal{E}_k) = \mathcal{L}^n(\mathfrak{B}_r)$ satisfies

$$|\lambda^D_{1,\infty}(\mathcal{E}_k) - \lambda^D_{1,\infty}(B_r)| = \delta_1 \quad \text{and} \quad |\lambda^N_{1,\infty}(B_r) - \lambda^N_{1,\infty}(\mathcal{E}_k)| = \delta_2,$$

where

$$\delta_1 = \frac{1}{\min_i\{\mathfrak{a}^k_i\}} - \frac{1}{r}, \quad \text{and} \quad \delta_2 = \frac{1}{r} - \frac{1}{\max_i\{\mathfrak{a}^k_i\}}.$$

Therefore, we recover the fact that if $\min_i \mathfrak{a}^k_i \to r$ and $\max_i \mathfrak{a}^k_i \to r$ as $k \to \infty$, then $\mathcal{E}_k \to \mathfrak{B}_r$.

Example 3.4. Given $r > 0$ let $k_0 \in \mathbb{N}$ such that $\frac{1}{2\pi}\sqrt{\frac{4}{k^2} + 4\pi^2 r^2} > \frac{1}{k\pi}$ for all $k \geq k_0$. For each $k \in \mathbb{N}$ let Ω_k be the planar stadium domain from Figure 1 (a) with $l_k = \frac{1}{k}$ and $\varepsilon_k = \frac{1}{2\pi}\sqrt{\frac{4}{k^2} + 4\pi^2 r^2} - \frac{1}{k\pi}$. It is easy to check that $\Omega_k \in \Xi_{\frac{1}{\varepsilon_k} - \frac{1}{r}, \frac{2}{2\varepsilon_k + \frac{1}{k}} - \frac{1}{r}}(\mathfrak{B}_r)$. Furthermore, in this case we have that the eigenfunctions are explicit and given by

$$u_k(x) = \frac{1}{\varepsilon_k}\mathrm{dist}(x, \partial\Omega_k).$$

Finally, form Corollary 1.4

$$u_k(x) \to v_\infty(x) = \frac{1}{r}\mathrm{dist}(x, \partial\mathfrak{B}_r) \quad \text{locally uniformly} \quad \text{in} \quad \mathfrak{B}_r \quad \text{as} \quad k \to \infty.$$

Acknowledgments. This work was supported by Consejo Nacional de Investigaciones Cien-tíficas y Técnicas (CONICET-Argentina). JVS would like to thank the Dept. of Math. and FCEyN Universidad de Buenos Aires for providing an excellent working environment and scientific atmosphere during his Postdoctoral program.

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