Different blow-up sets

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Abstract

We consider the heat equation in the half-line, with Dirichlet boundary data which blow up in finite time. Though the blow-up set may be any interval $[0, a]$, $a \in [0, \infty]$ depending on the Dirichlet data, we prove that the *effective blow-up set*, that is, the set of points $x \geq 0$ where the solution behaves like $u(0, t)$, consists always only of the origin.

As an application of our results we consider a system of two heat equations with a nontrivial nonlinear flux coupling at the boundary. We show that by prescribing the nonlinearities the two components may have different blow-up sets. However, the effective blow-up sets do not depend on the coupling and coincide with the origin for both components.

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1 Introduction

For many parabolic equations or systems it is well known that solutions may develop singularities in finite time. A special case of this is when the L^∞-norm of the solution goes to infinity in finite time. This phenomenon is called *blow-up* and has been object of active research in recent years, see the survey [3], the book [7] and references therein.

An important aspect of blow-up problems is the spatial structure of the set where the solution becomes unbounded, that is, the *blow-up set*. More precisely, the blow-up set of a solution u that blows up at time T is defined as

$$B(u) = \{x;\ \text{there exist } x_n \to x,\ t_n \nearrow T,\ \text{with } u(x_n, t_n) \to \infty\}.$$

A problem which has attracted a lot of attention in the literature is the identification of possible blow-up sets. There are several examples where the blow-up set is a single point (single point blow-up), a proper subset of the spatial domain (regional blow-up) or the whole space (global blow-up), see [7].

Let us consider the problem

$$\begin{cases} u_t = u_{xx}, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ u(0,t) = \psi(t), & t \in (0,T), \\ u(x,0) = u_0(x) & x \in \mathbb{R}_+, \end{cases} \tag{1.1}$$

where blow-up is driven by a blowing-up boundary data ψ, $\psi(t) \to +\infty$ as $t \nearrow T$. It is well known that prescribing ψ one can fix the blow-up set to be any interval $[0,a]$, [7]. However, blow-up may take place at very different speeds at different points, see [7] for an example. This poses a severe difficulty when trying to perform numerical computations, since it is quite complicated to detect the singularity for points at which blow-up takes place more slowly than at the blowing up boundary. This leads us to introduce the *effective blow-up set* as the set of points x where u behaves like its boundary value ψ, that is,

$$B_e(u) = \overline{\{x \in \mathbb{R}_+;\ u(x,t) \sim \psi(t)\}},$$

where the notation $u \sim v$ means that there exist positive constants C_1, C_2 such that $C_1 v \le u \le C_2 v$. This is the set that one may expect to observe when performing numerical approximations, since $\tilde{u}(x,t) = u(x,t)/\psi(t)$ goes to zero in the open set $(B_e(u))^c$ along some sequences $\{t_n\}$, $t_n \nearrow T$, see [1], [5]. Let us remark that when $\psi(t)$ is increasing then the effective blow-up set consists of the set of points where the solution behaves like its maximum, $u(0,t)$. Though the blow-up set may be any interval, the effective blow-up set always consists of a single point. This is the content of our main result.

Theorem 1.1 *Given any positive blowing up ψ, $\lim_{t\nearrow T} \psi(t) = +\infty$, the effective blow-up set is a single point, $B_e(u) = \{0\}$.*

Let us remark that for other problems the effective blow-up set can be larger than the origin, for example when we are dealing with nonlinear diffusion like in the porous medium equation $u_t = (u^m)_{xx}$, $m > 1$, [4], [6]. This means that there is a remarkable difference between nonlinear diffusion and the usual Laplace operator. Though in the linear case the blow-up at the boundary may propagate to interior points, in doing so it loses some strength, and the blow-up rates are different from the blowing up boundary data. On the other hand, for the porous medium the diffusivity depends on u and becomes enormous for u large. This allows a propagation of blow-up to interior points without losing rate. Thus, the effective blow-up set coincides with the blow-up set and can be larger than a single point, see [4], [6].

For systems a new important phenomenon may occur: the blow-up sets for the components can be different. This was first observed by the authors in [6], where they study a system of two porous medium equations, $u_t = (u^m)_{xx}$, $v_t = (v^n)_{xx}$, with $m, n > 1$, in the half line, $\mathbb{R}_+ = [0, \infty)$, with nonlinear fluxes coupled at the boundary, $-(u^m)_x(0,t) = v^p(0,t)$, $-(v^n)_x(0,t) = u^q(0,t)$.

One natural question arises: Is this difference in the blow-up sets provoked by the nonlinear (and different) diffusion of the equations?

Here we show that this is not the case. In fact, we find that the same phenomenon appears in a system composed by two *linear* heat equations with nontrivially coupled fluxes at the boundary. Concretely, we consider the spatial structure of the set in which blow-up occurs for the reaction-diffusion system

$$\begin{cases} u_t = u_{xx}, \\ v_t = v_{xx}, \end{cases} \qquad (x,t) \in \mathbb{R}_+ \times (0,T), \tag{1.2}$$

with nonlinear boundary conditions given by

$$\begin{cases} -u_x(0,t) = f(v(0,t)), \\ -v_x(0,t) = g(u(0,t)), \end{cases} \qquad t \in (0,T), \tag{1.3}$$

and positive initial data $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, $x \in \mathbb{R}_+$. We find that this system has the possibility of different blow-up sets. More precisely one can adjust the coupling through the boundary to prescribe any pair of intervals, $[0,a]$, $[0,b]$, as blow-up sets for the components. Also one can prescribe any positive blow-up time, T. However, the effective blow-up sets for (1.2)–(1.3) are the same for all possible coupling functions f and g and consist of a single point, $x = 0$. This has to be contrasted with the case of nonlinear diffusion where one can prescribe any effective blow-up sets for the components of the system, see [6].

Theorem 1.2 *Given $0 < T < \infty$, $a, b \in [0, \infty]$ (a and b can be $+\infty$) and u_0, v_0 in L^∞, there exist f, g such that the solution of* (1.2)–(1.3) *blows up at time T with blow-up sets given by $B(u) = [0,a]$ and $B(v) = [0,b]$. However, for any blowing up solution of* (1.2)–(1.3) *the effective blow-up sets coincide and consist of a single point $B_e(u) = B_e(v) = \{0\}$.*

Remark. For power-like couplings, $f(v) = v^p$, $g(u) = u^q$, the blow-up set is known to be a single point for both components, $B(u) = B(v) = \{0\}$, see [2].

2 Proofs

Proof of Theorem 1.1. To clarify our arguments and for completeness, let us give first a proof of the fact that we can obtain any blow-up set, that is, given a we want to find $\psi(t)$ such that $B(u) = [0,a]$. To this end, we observe that, since the equation is linear, solutions of (1.1) can be decomposed as $u = u_1 + u_2$, where u_1 is the solution of (1.1) with initial datum $u_1(x,0) \equiv 0$ and u_2 is the solution of the heat equation with boundary condition $u_2(0,t) = 0$ and initial datum $u_2(x,0) = u_0(x)$. As u_2 is uniformly bounded, we have to focus our attention on u_1. Using the representation formula for solutions of the heat equation and taking into account that $u_1(x,0) = 0$ we obtain that

$$u_1(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t e^{-x^2/4(t-s)} (t-s)^{-3/2} \psi(s)\, ds. \tag{2.1}$$

Let us consider any function $\psi = \psi(t)$ with $\psi(t) \sim e^{\alpha(T-t)^{-q}}$, as $t \nearrow T$. A careful but easy analysis of (2.1), shows that

$$B(u) = B(u_1) = \begin{cases} \{0\} & \text{if } 0 < q < 1, \\ [0, 2\sqrt{\alpha}\,] & \text{if } q = 1, \\ \mathbb{R}_+ & \text{if } q > 1. \end{cases}$$

The desired result follows taking $2\sqrt{\alpha} = a$ when $0 < a < \infty$.

Now let ψ be any positive function with blow-up at T. We want to prove that $B_e(u) = \{0\}$. Hence we take a point $x_0 \neq 0$ and we prove that $x_0 \notin B_e(u)$. Let us observe that when $B(u) = \{0\}$ then $B_e(u) = \{0\}$, hence we can assume that $B(u)$ is larger than the origin. In this case, we claim that $\int^T \psi(s)\, ds = +\infty$. To see this fact just use the representation formula (2.1) to conclude that if $\int^T \psi(s)\, ds < +\infty$ then $B(u) = \{0\}$. Let us consider $z(x,t) = \int_0^t u_1(x,s)\, ds$. This function z is a solution of the heat equation with $z(0,t) = \int_0^t u_1(0,s)\, ds = \varphi(t)$. Note that $\varphi(t) \to +\infty$ as $t \nearrow T$. If $z(x_0,t) \le K$ for $t \in [0,T)$, there exists a sequence $t_n \to T$ with $u(x_0,t_n)/\psi(t_n) \to 0$ and hence $x_0 \notin B_e(u)$. Therefore we can assume that $z(x_0,t)$ is unbounded. Let us integrate in time again. Let $w(x,t) = \int_0^t z(x,s)\, ds$. This w is a solution of the heat equation with $w(0,t) = \theta(t)$. If $\theta(T) < +\infty$ then $B(z) = \{0\}$ but this is impossible since we are assuming that $z(x_0,t)$ is unbounded. Using (2.1) we obtain

$$\begin{aligned} w(x,t) &= \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{t}}^{+\infty} e^{-r^2} \theta(t - x^2/4r^2)\, dr \\ &= \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{t}}^{M(t)} e^{-r^2} \theta(t - x^2/4r^2)\, dr + \frac{2}{\sqrt{\pi}} \int_{M(t)}^{+\infty} e^{-r^2} \theta(t - x^2/4r^2)\, dr \\ &\le C\theta(t - x^2/4M(t)^2) + C\theta(t) \int_{M(t)}^{+\infty} e^{-r^2}\, dr. \end{aligned}$$

Using that φ is increasing, $\varphi'(t) = \psi(t) \ge 0$, if we choose $M(t) = \sqrt{x^2/8(T-t)}$, we get

$$\lim_{t \nearrow T} \frac{\theta(t - x^2/4M(t)^2)}{\theta(t)} = \lim_{t \nearrow T} \frac{\varphi(t - x^2/M(t)^2)}{\varphi(t)} \left(1 - \frac{x^2 M'(t)}{4M(t)^3}\right) = 0.$$

On the other hand, since $\lim_{t \nearrow T} M(t) = +\infty$, we have

$$\lim_{t \nearrow T} \int_{M(t)}^{+\infty} e^{-s^2}\, ds = 0.$$

This shows that

$$\lim_{t \nearrow T} \frac{w(x,t)}{\theta(t)} = 0$$

for all $x \neq 0$. Hence there exists $t_n \nearrow T$ such that $z(x_0, t_n)/\varphi(t_n) \to 0$, but then there exists another sequence $t_k \nearrow T$ such that $u(x_0, t_k)/\psi(t_k) \to 0$. This proves that $B_e(u) = \{0\}$. $\square$

The idea of the proof of Theorem 1.2 is to find the coupling functions f and g from the blow-up rates for u and v.

Proof of Theorem 1.2. We choose $\psi(t)$, $\varphi(t)$ two increasing functions such that the solutions u and v of the problem (1.1) with $\psi(t)$ and $\varphi(t)$ as boundary data respectively blow up at time T with blow-up sets $B(u) = [0, a]$, $B(v) = [0, b]$. Let us call P and Q the fluxes at the boundary, i.e., $P(t) = -u_x(0, t)$, $Q(t) = -v_x(0, t)$. Next let us define f and g by the formulas $f(\tau) = P(\varphi^{-1}(\tau))$, $g(\tau) = Q(\psi^{-1}(\tau))$. With this choice of f and g it is immediate that u, v verify (1.3). Hence (u, v) is a solution of (1.2)–(1.3) with $B(u) = [0, a]$, $B(v) = [0, b]$. To end the proof, we only have to observe that Theorem 1.1 shows that any blowing up solution of (1.1) has $B_e(u) = B_e(v) = \{0\}$. $\square$

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