THE BLOW-UP RATE FOR A SYSTEM OF HEAT EQUATIONS WITH NONTRIVIAL COUPLING AT THE BOUNDARY

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We study the blow-up rate of positive radial solutions of a system of two heat equations, $(u_1)_t = \Delta u_1$ $(u_2)_t = \Delta u_2$, in the ball $B(0,1)$, with boundary conditions

$$\frac{\partial u_1}{\partial r} = (u_1)^{p_{11}}(u_2)^{p_{12}} \qquad \frac{\partial u_2}{\partial r} = (u_1)^{p_{21}}(u_2)^{p_{22}}$$

Under some natural hypothesis on the matrix $P = (p_{ij})$ that guarrantee the blow-up of the solution at time T, and some assumptions on the initial data u_{0i}, we find that if $\|x_0\| = 1$ then $u_i(x_0,t)$ goes to infinity like $(T-t)^{\alpha_i/2}$ where the $\alpha_i < 0$ are the solutions of $(P - \mathrm{Id})(\alpha_1,\alpha_2)^t = (-1,-1)^t$.

As a corollary of the blow-up rate we obtain the localization of the blow-up set at the boundary of the domain.

I. Introduction.

In this paper we consider positive solutions of the following problem :

$$(1.1) \qquad \begin{cases} (u_1)_t = \Delta u_1 & \text{in } B(0,1)\times(0,T) \\ (u_2)_t = \Delta u_2 \end{cases}$$

$$(1.2) \qquad \begin{cases} \frac{\partial u_1}{\partial r} = (u_1)^{p_{11}}(u_2)^{p_{12}} & \text{on } \partial B(0,1)\times(0,T) \\ \frac{\partial u_2}{\partial r} = (u_1)^{p_{21}}(u_2)^{p_{22}} \end{cases}$$

$$(1.3) \qquad \begin{cases} u_1(x,0) = u_{1,0}(x) & \text{in } B(0,1) \\ u_2(x,0) = u_{2,0}(x) \end{cases}$$

Where the matrix $P = p_{ij}$ satisfies

P1) $P = (p_{ij})$ is a matrix with nonnegative entries such that $p_{11} < 1$, $p_{22} < 1$ and $\det(P - \mathrm{Id}) < 0$.

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Under this hypothesis there exists a unique vector (α_1, α_2) with $\alpha_i < 0$ such that

$$(P - Id)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{pmatrix}-1\\ -1\end{pmatrix}$$

We also suppose that the initial data, u_{i0}, are $C^3(\overline{B})$, strictly positives, satisfy the boundary condition and are radially symmetric (i.e. $u_{i0}(x) = v_{i0}(\|x\|)$). Moreover, we shall suppose that the first three derivatives of $v_{i0}(r)$ are nonnegative.

Under this hypothesis the solution u_i is also radially symmetric, $u_i(x,t) = v_i(\|x\|, t)$ where the functions $v_i : [0,1] \times [0,T) \mapsto$ are a solution of the problem

$$(1.4) \qquad \begin{cases} (v_i)_t = (v_i)_{rr} + \frac{N-1}{r}(v_i)_r \\ (v_i)_r(0,t) = 0 \qquad (v_i)_r(1,t) = (v_1)^{p_{i1}}(v_2)^{p_{i2}}(1,t) \\ (v_i)(r,0) = v_{i0}(r) \end{cases}$$

$$i = 1, 2$$

Also the solution v_i satisfies that its first three derivatives with respect to r are nonnegative.

Under the hypotesis **P1)** it is proved in [13] that the solution u_i blows-up in finite time T.

At that time T we have, by the result of [2] (see also [13])

$$\limsup_{t \nearrow T} \|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t)\|_{L^\infty(\Omega)} = +\infty$$

Here we can deduce that both functions, u_1 and u_2, go to infinity at time T. This is a consequence of the fact that p_{11} and p_{22} are less than one. In fact, suppose that u_1 remains bounded in $\overline{\Omega} \times [0,T)$ then u_2 is a solution of

$$\begin{cases} (u_2)_t = \Delta(u_2) \\ \frac{\partial u_2}{\partial \eta} \leq K(u_2)^{p_{22}} \end{cases}$$

where K is a bound for $(u_1)^{p_{21}}$ and it is well known that u_2 remains bounded up to time T (see [10], [11] or [15]) and hence T is not the blow-up time.

Blow-up for nonlinear equations has deserved a great deal of interest. In particular for this kind of problems the blow-up rate and the localization of blow-up points are not well known even in the case of a single heat equation with a nonlinear absorption through the boundary.

In [11] we prove a blow-up result for f such that $\int^{+\infty} \frac{1}{f} < +\infty$ (see also [9], [10] and [15] for blow-up results) and localize the blow-up set on the boundary of a general Ω with the condition $\Delta u_0 > 0$. If $f(u) = u^p$ in [7] the blow-up set is localized on the boundary and the asymptotic behavior near a blow-up point is obtained, but with restrictions on p. In [6] the blow-up rate is obtained for every

$p > 1$ but the restrictions are on the domain (a ball) and the initial data (the hypothesis are similar to ours).

We state here some of these results in order to make the paper self contained since we will be refering to them below (see Section II, Lemmas 1 and 2).

A work closely related to ours is [3] where the author studies blow-up phenomena for a system like (1.1)-(1.3) but with $p_{11} = p_{22} = 0$. He obtaines that $p_{21}p_{12} > 1$ is a necessary and sufficient condition in order to have a blow-up and also he localizes the blow-up set in the boundary of the domain if the solutions are radial and $p_{21} > 1$ and $p_{12} > 1$.

In [14] and [1] the blow-up problem is considered for more general nonlinearities, in the equation and in the boundary conditions, in a general smooth domain Ω.

A related paper that deal with the case $p_{11} = p_{22} = 0$ if the domain is a half space is [5].

In [12] the author localizes, in colaboration with Diego F. Rial, the blow-up set in the boundary of a smooth domain for a system of heat equations with more general nonlinear boundary conditions.

For the blow-up rate of (1.1)-(1.3) in the case of $p_{11} = p_{22} = 0$ we refer to [4].

In this paper we characterize the blow-up rate in the following sence, we prove :

Theorem 1. *If the hypothesis upon the matrix P holds (**P1**)) and the solution (v_1, v_2) of the problem (1.4) is such that v_r, v_t, v_{tr} are nonnegative then there exists positive constants C, c such that*

$$c \le v_1(1,t)(T-t)^{-\alpha_1/2} \le C \qquad (t \nearrow T)$$

$$c \le v_2(1,t)(T-t)^{-\alpha_2/2} \le C \qquad (t \nearrow T)$$

*where α_1, α_2 are given by **P1**).*

This is equivalent to

$$c \le \max u_1(t)(T-t)^{-\alpha_1/2} \le C \qquad (t \nearrow T)$$

$$c \le \max u_2(t)(T-t)^{-\alpha_2/2} \le C \qquad (t \nearrow T)$$

As a Corollary we obtain the localization of the blow-up set in the boundary of B.

Corollary 1. *Let u_i be a solution of the problem (1.1)-(1.3) such that*

$$\max u_1(t) \le C(T-t)^{\alpha_1/2} \qquad (t \nearrow T)$$

$$\max u_2(t) \le C(T-t)^{\alpha_2/2} \qquad (t \nearrow T)$$

(every solution that verifies Theorem 1 verifies this condition).

If $r < 1$ there exists a constant $C = C(r)$ such that

$$u_1(r,t) < C \qquad (t \in [0,T))$$

$$u_2(r,t) < C \qquad (t \in [0,T))$$

(i.e. the blow-up set is localized in $\partial\Omega$).

To prove Theorem 1 we need a result that gives us the asymptotic behavior of the solutions of

$$\text{(1.5)} \qquad \begin{cases} u_t = \Delta u & B(0,1) \times [0,T) \\ \frac{\partial u}{\partial \eta} = \frac{u^r}{(T-t)^s} & \partial B(0,1) \times [0,T) \\ u(x,0) = u_0(x) > 0 \end{cases}$$

Where ($s > 1/2,\ r < 1$). We state this result as Proposition 1.

Proposition 1. *Every positive solution of (1.5) blows-up at time T and*

$$c \le u(1,t)(T-t)^{\frac{s-1/2}{1-r}} \le C \qquad (t \nearrow T)$$

We believe that this result has independent interest.

The paper is organized as follows, in Section II we prove Proposition 1 and in Section III we prove Theorem 1 and Corollary 1.

II. The blow-up rate for a single equation.

Proposition 1 gives us the asymptotic behavior of the positive solutions of (1.5). For the proof we need the result of [6], one of [7] and use a comparison argument.

We begin this section with a result of [6] adapted to our interests.

Lemma 1. *Let v a positive solution of*

$$\begin{cases} v_t = v_{rr} + \frac{N-1}{r} v_r \\ v_r(0,t) = 0 \qquad v_r(1,t) \ge (v(1,t))^p \\ v(x,0) = v_0(x) \end{cases}$$

if v has v_r, v_t, v_{tr} nonnegative then there exists $c > 0$ such that

$$v(1,t)(T-t)^{\frac{1}{2(p-1)}} \le c$$

__Proof__. The proof is the same of [6] Lemma 2.1 with the first equality replaced by an inequality.

Next we state a result due to [7] to obtain the reverse inequality. We remark that this result holds for every solution (they do not use the symmetry of the domain).

Lemma 2. *Let u be a solution of*

$$\begin{cases} u_t = \Delta u \\ \frac{\partial u}{\partial \eta} \le u^p \\ u(x,0) = u_0(x) \end{cases}$$

such that u blows-up at time T, then there exists a constant $c > 0$ such that

$$\max_{x \in \overline{\Omega}} u(x,t) \ge \frac{c}{(T-t)^{\frac{1}{2(p-1)}}}$$

Proof. The proof follows by replacing the equality by an inequality in Theorem 3.2 of [7].

Remark 1. Given $\kappa > 1$ and $T > 0$ we claim that there exists z_0 radial with three positive derivatives such that the solution of

$$\begin{cases} z_t = \Delta z \\ \frac{\partial z}{\partial \eta} = z^{\kappa} \\ z(x,0) = z_0(x) > w_0(x) \end{cases} \tag{2.1}$$

has blow-up at time T. Also we may choose z_0 satisfying a compatibility condition of order one. Therefore the solution z to (2.1) will have three spatial derivatives continuous up to $t = 0$ and thus its first three spatial derivatives will be positive (see [8] Chap. IV, Theorem 5.3).

In fact, take $\overline{z}_0$ compatible up to order one, radial with three positive spatial derivatives. Also take $\overline{z}_0$ small enough so that $T_{\overline{z}_0}$ (its blow-up time) is greater than T. Let $\overline{z}$ be the solution to (2.1) with initial datum $\overline{z}_0$. Now $z_0(x) = \overline{z}(x, T_{\overline{z}_0} - T)$ will satisfy all our conditions.

With the previous Lemmas we can prove Proposition 1.

Proof of Proposition 1. First we consider solutions of

$$\begin{cases} w_t = \Delta w \\ \frac{\partial w}{\partial \eta} = c\frac{w^r}{(T-t)^s} \\ w(x,0) = w_0(x) \end{cases} \tag{2.2}$$

We claim that if c and w_0 are small enough, then the solution w blows-up at time T and also

$$\max_{\overline{B}} w(t) \le \frac{L_2}{(T-t)^{\frac{s-1/2}{1-r}}}$$

We choose z a solution of (2.1) with $\kappa = \frac{2s-r}{2s-1} > 1$ and z_0 radial with three derivatives positive and such that the solution z has blow-up at time T. (see Remark 1).

From one of the previous Lemmas (Lemma 1) we know that

$$z(x,t) \leq \frac{C}{(T-t)^{\frac{1}{2(\kappa-1)}}}$$

Next we observe that, if w_0 and c are small, we can obtain $w < z$, because $(z-w)$ is positive at time $t=0$ and satisfies

$$\begin{cases} (z-w)_t = \Delta(z-w) \\ \frac{\partial(z-w)}{\partial\eta} = z^\kappa - c\frac{w^r}{(T-t)^s} \end{cases}$$

Suppose that there exists a first time t_0 and a point x_0 such that $z(x_0,t_0) = w(x_0,t_0)$, x_0 must belong to $\partial\Omega$ and at that point we have

$$\frac{\partial(z-w)}{\partial\eta} = z^r(z^{\kappa-r} - \frac{c}{(T-t)^s})$$

By Lemma 2 we know that $z^{\kappa-r}(1,t) \geq \frac{c_1}{(T-t)^{\frac{\kappa-r}{2(\kappa-1)}}} > \frac{c}{(T-t)^s}$ (because of our choice of κ the powers of $(T-t)$ are that same and it rests to choose the constant c small). This is a contradiction to the maximum principle.

We have proved

$$\max_{\overline{B}} w(t) \leq z(1,t) \leq \frac{C}{(T-t)^{\frac{1}{2(\kappa-1)}}}$$

But, because of the choice of κ

$$\frac{1}{2(\kappa-1)} = \frac{s-1/2}{1-r}$$

and we obtain that if c and w_0 are small then one of the inequalities claimed by Proposition 1 holds.

To avoid the restrictions over c and the initial data let $\overline{w}$ be a solution of (2.1) with constant c and initial data arbitrary.

We multiply $\overline{w}$ by k with k small in order to obtain that $k\overline{w}$ is under the previous case. Then we have

$$k\overline{w}(t) \leq \frac{C}{(T-t)^{\frac{s-1/2}{1-r}}}$$

and this finish the first part of the proof.

We remark that the previous conclusion remains valid if we take as boundary condition an inequality

$$\frac{\partial w}{\partial\eta} \leq c\frac{w^r}{(T-t)^s}$$

We can obtain the other inequality

$$\frac{L_1}{(T-t)^{\frac{s-1/2}{1-r}}} \leq \max_{\overline{B}} w(t)$$

just by using a similar argument (using a comparison with z from below) to obtain

$$\max_{\overline{B}} w(t) \geq \max_{\overline{B}} z(t) \geq \frac{L_1}{(T-t)^{\frac{s-1/2}{1-r}}}$$

And we can observe again that this conclusion remains true if we change the boundary condition by an inequality

$$\frac{\partial w}{\partial \eta} \geq c\frac{w^r}{(T-t)^s}$$

We have proved Proposition 1.

II. Blow-up rate for the system.

We begin this section with a comparison result between the pair of functions u_1 and $(u_2)^\gamma$. (u_1, u_2) is the solution of (1.1)-(1.3). This comparison result allows us to reduce the problem to a single equation and then apply Proposition 1.

We can suppose that the α_i are such that

$$\alpha_1 \leq \alpha_2 < 0$$

Then we can prove the following Lemma,

Lemma 3. *There exists a constant $C > 0$ such that*

$$Cu_1 \geq (u_2)^{\frac{\alpha_1}{\alpha_2}}$$

Where (u_1, u_2) is a solution of (1.1)-(1.3).

Proof. We begin by choosing C big enough in order to verify

$$Cu_1(x,0) > (u_2)^{\frac{\alpha_1}{\alpha_2}}(x,0)$$

Next we observe that Cu_1 is a solution of the heat equation and, by the fact that $\frac{\alpha_1}{\alpha_2} \geq 1$, $(u_2)^{\frac{\alpha_1}{\alpha_2}}$ is a subsolution.

Suppose that there exists a first time t_0 with a point $x_0 \in \overline{\Omega}$ such that both functions are equal at (x_0, t_0). x_0 must belong to $\partial\Omega$ and at that point it holds

$$\frac{\partial(Cu_1 - (u_2)^{\frac{\alpha_1}{\alpha_2}})}{\partial \eta} = C^{1-p_{11}}(Cu_1)^{p_{11}}((u_2)^{\frac{\alpha_1}{\alpha_2}})^{p_{12}\frac{\alpha_2}{\alpha_1}} -$$

$$-\frac{\alpha_1}{C^{p_{21}}\alpha_2}(Cu_1)^{p_{21}}((u_2)^{\frac{\alpha_1}{\alpha_2}})^{\frac{\alpha_2}{\alpha_1}(p_{22}-1)+1} =$$

$$= C^{1-p_{11}}(Cu_1)^{p_{11}+p_{12}\frac{\alpha_2}{\alpha_1}} - \frac{\alpha_1}{C^{p_{21}}\alpha_2}(Cu_1)^{p_{21}+\frac{\alpha_2}{\alpha_1}(p_{22}-1)+1}$$

We can verify that the powers involved are equal because, by **P1)** it holds,

$$p_{11} + p_{12}\frac{\alpha_2}{\alpha_1} = p_{21} + \frac{\alpha_2}{\alpha_1}(p_{22}-1) + 1$$

Also we can choose C such that,

$$C^{1-p_{11}} - \frac{\alpha_1}{C^{p_{21}}\alpha_2} > 0$$

We have obtained that at (x_0, t_0)

$$\frac{\partial(Cu_1 - (u_2)^{\frac{\alpha_1}{\alpha_2}})}{\partial\eta} > 0$$

which is a contradiction.

We deduce that the time t_0 cannot exist and hence

$$Cu_1 \geq (u_2)^{\frac{\alpha_1}{\alpha_2}}$$

which was our claim.

Now we can prove Theorem 1.

Proof of Theorem 1. We begin by u_2. By Lemma 3 we can observe that u_2 satisfies

$$(u_2)_t = \Delta(u_2)$$

$$\frac{\partial(u_2)}{\partial\eta} = (u_1)^{p_{21}}(u_2)^{p_{22}} \geq C(u_2)^{\gamma}$$

where

$$\gamma = \frac{\alpha_1}{\alpha_2}p_{21} + p_{22} =$$

$$= 1 - \frac{1}{\alpha_2} > 1$$

Now we observe that we are under the hypothesis of Lemma 1 because the function u_2 is radial and by hypothesis it has three positive derivatives. Therefore we can conclude that there exists a constant c such that

$$u_2 \leq \frac{c}{(T-t)^{\frac{1}{2(\gamma-1)}}}$$

But

$$\frac{1}{2(\gamma-1)} = -\frac{\alpha_2}{2}$$

and then

$$u_2 \leq \frac{c}{(T-t)^{-\frac{\alpha_2}{2}}}$$

Using this bound for u_2 and Proposition 1 we can obtain the bound for u_1. u_1 satisfies

$$(u_1)_t = \Delta(u_1)$$

$$\frac{\partial(u_1)}{\partial\eta} = (u_1)^{p_{11}}(u_2)^{p_{12}} \leq$$

$$\leq C\frac{(u_1)^r}{(T-t)^s}$$

In this case

$$r = p_{11} \qquad \text{and} \qquad s = -\frac{\alpha_2 p_{12}}{2}$$

By hypothesis **P1)**, $s > 1/2$ and $r < 1$ and then we fall into the hypothesis of Proposition 1 that states that,

$$u_1 \leq \frac{c}{(T-t)^{\frac{s-1/2}{1-r}}}$$

We remark that

$$\frac{s-1/2}{1-r} = \frac{-\alpha_1}{2}$$

and we have obtained the upper bound for u_1

$$u_1 \leq \frac{c}{(T-t)^{\frac{-\alpha_1}{2}}}$$

Let us prove the reverse inequalities in order to finish the proof of Theorem 1.

Now we begin by u_1. Using Lemma 3, u_1 satisfies

$$(u_1)_t = \Delta(u_1)$$

$$\frac{\partial(u_1)}{\partial\eta} = (u_1)^{p_{11}}(u_2)^{p_{12}} \leq C(u_1)^\gamma$$

where

$$\gamma = \frac{1}{\alpha_1}\{\alpha_1 p_{11} + \alpha_2 p_{12}\} =$$

$$= \frac{-1}{\alpha_1} + 1 > 1$$

then Lemma 2 tells us that, as the maximum of $u_1(x,t) = v_1(\|x\|, t)$ it is attained at $\|x\| = 1$,

$$v_1(1,t) \geq \frac{c}{(T-t)^{\frac{1}{2(\gamma-1)}}}$$

that is

$$\max_{\overline{\Omega}} u_1(t) \geq \frac{c}{(T-t)^{\frac{1}{2(\gamma-1)}}}$$

but

$$\frac{1}{2(\gamma-1)} = \frac{-\alpha_1}{2}$$

and then

$$\max_{\overline{\Omega}} u_1(t) \geq \frac{c}{(T-t)^{-\frac{\alpha_1}{2}}}$$

Now for u_2. By the previous bound u_2 satisfies,

$$(u_2)_t = \Delta(u_2)$$

$$\frac{\partial(u_2)}{\partial\eta} = (u_1)^{p_{21}}(u_2)^{p_{22}} \geq C\frac{(u_2)^r}{(T-t)^{\frac{-\alpha_1 p_{21}}{2}}} =$$

$$= C\frac{(u_2)^r}{(T-t)^s}$$

in this case

$$r = p_{22} \qquad \text{and} \qquad s = \frac{-\alpha_1 p_{21}}{2}$$

and, using again **P1)** we have that $s > 1/2$ and $r < 1$ and then, using Proposition 1 we obtain

$$(v_2)(1,t) \geq \frac{c}{(T-t)^{\frac{s-1/2}{1-r}}}$$

We observe that

$$\frac{s-1/2}{1-r} = \frac{-\alpha_2}{2}$$

and then u_2 satisfies

$$\max_{\overline{\Omega}} u_2(t) \geq \frac{c}{(T-t)^{\frac{-\alpha_2}{2}}}$$

We have proved that

$$c \leq v_i(1,t)(T-t)^{-\alpha_i/2} \leq C \qquad (t \nearrow T)$$

that is equivalent to

$$c \leq \max_{\overline{\Omega}} u_i(t)(T-t)^{-\alpha_i/2} = u_i(x,t)\mid_{\|x\|=1}(T-t)^{-\alpha_i/2} \leq C \qquad (t \nearrow T)$$

and we have finished the proof of Theorem 1.

To finish the paper we observe that with this blow-up rate we can localize the blow-up set at the boundary of the domain.

Proof of Corollary 1. We just observe that we fall into the hypothesis of Theorem 4.1 of [7].

References

[1] G. Acosta and J. D. Rossi, *Blow-up vs. global existence for quasilinear parabolic systems with a nonlinear boundary condition*, Preprint (1995).

[2] H. Amann, *Dynamic Theory of Quasilinear Parabolic Systems III . Global Existence*, Math. Z **202(2)** (1989), 219–254.

[3] K. Deng, *Global Existence and Blow-up for a System of Heat Equations with a Nonlinear Boundary Condition*, Math. Meth. Appl. Sci **18** (1995), 307–315.

[4] K. Deng, *Blow-up rates for parabolic systems*, Z. Angew Math. Phys **46** (1995), 110–118.

[5] K. Deng, M. Fila and H. A. Levine, *On critical exponents for a system of heat equations coupled in the boundary conditions*, Acta Math. Univ Comenianae **LXIII** (1994), 169–192.

[6] M. Fila and P. Quitter, *The Blow-up Rate for the Heat Equation with a Nonlinear Boundary Condition*, Math. Meth. Appl. Sci **14** (1991), 197–205.

[7] Bei Hu and Hong-Ming Yin, *The Profile Near Blow-up Time for the Solution of the Heat Equation with a Nonlinear Boundary Condition*, Trans. Amer. Math. Soc **346 (1)** (1995), 117–135.

[8] O. Ladyzenskaja, V. Solonnikov y N. Uralceva, *Linear and quasilinear equations of parabolic type*, Trans. Math. Monographs **23** (1968).

[9] H. A. Levine and L. E. Payne, *Nonexistence Theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time*, Jour. Diff. Eq. **16** (1974), 319–334.

[10] J. Lopez Gomez, V. Marquez and N. Wolanski, *Blow-up Results and Localization of Blow-up Points for the Heat Equation with a Nonlinear Boundary Condition*, Jour. Diff. Eq. **92(2)** (1991), 384–401.

[11] D. F. Rial and J. D. Rossi, *Blow-up Results and Localization of Blow-up Points in an N-Dimensional Smooth Domain*, Preprint (1995).

[12] D. F. Rial and J. D. Rossi, *Localization of blow-up points for a parabolic system with nonlinear boundary condition*, In preparation.

[13] J. D. Rossi, *On Existence and Nonexistence in the Large for an N-Dimensional System of Heat Equations with Nontrivial Coupling at the Boundary*, Preprint (1995).

[14] J. D. Rossi and N. Wolanski, *Global Existence and Nonexistence for a Parabolic System with Nonlinear Boundary Conditions*, to appear in Diff. Int. Eq.

[15] W. Walter, *On Existence and Nonexistence in the Large of Solutions of Parabolic Differential Equations with a Nonlinear Boundary Condition*, SIAM J. Math. Anal. **6(1)** (1975), 85–90.