A complete classification of simultaneous blow-up rates

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Abstract

We study the simultaneous blow-up rates of a system of two heat equations coupled through the boundary in a nonlinear way. We complete the previous known results by covering the whole range of possible parameters.

1 Introduction

We devote our attention to the parabolic system

$$u_t = u_{xx}, \quad v_t = v_{xx}, \qquad (x,t) \in (0,L) \times (0,T),$$

with a nonlinear coupling at one of the ends of the interval

$$-u_x(0,t) = u^{p_{11}}(0,t)v^{p_{12}}(0,t), \quad -v_x(0,t) = u^{p_{21}}(0,t)v^{p_{22}}(0,t), \quad t \in (0,T),$$

zero flux at the other end, $u_x(L,t) = 0$, $v_x(L,t) = 0$, $t \in (0,T)$ and initial data $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, $x \in (0,L)$, which are smooth and compatible with the boundary conditions. We consider all possible parameters satisfying $p_{ij} \geq 0$. Moreover, we will restrict to decreasing in space and increasing in time solutions.

The time T denotes the maximal existence time for the solution (u,v). If it is infinite we say that the solution is *global*. If it is finite we say that the

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solution *blows up*. Nontrivial solutions of our problem blow up if and only if the exponents p_{ij} verify any of the following conditions, $p_{11} > 1$, $p_{22} > 1$ or $p_{12}p_{21} > (1 - p_{11})(1 - p_{22})$, [10] (see also [11], [12]). In this case we have

$$\limsup_{t \nearrow T} \{\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty\} = \infty.$$

However, a priori there is no reason why both components, u and v, should go to infinity simultaneously at time T. Indeed, if $p_{11} > p_{21} + 1$ there are solutions for which u blows up while v remains bounded. Analogously, if $p_{22} > p_{12} + 1$ there are solutions for which v blows up while u remains bounded, [6]. If $p_{11} > p_{21} + 1$ and $p_{22} \le p_{12} + 1$, or $p_{22} > p_{12} + 1$ and $p_{11} \le p_{21} + 1$, then blow-up is always non-simultaneous, while if $p_{11} \le p_{21} + 1$ and $p_{22} \le p_{12} + 1$, blow-up is always simultaneous. It is also possible that simultaneous and non-simultaneous blow-up coexist. This happens if $p_{11} > p_{21} + 1$ and $p_{22} > p_{12} + 1$. See [1].

When blow-up is non-simultaneous, the blow-up rate for the blow-up component coincides with the rate for the scalar problem in which the bounded component is substituted by a constant. For instance, if u blows up while v remains bounded then $u(0, t) \sim (T - t)^{-1/2(p_{11}-1)}$, [1]. By $f \sim g$ we mean that there exist constants $c, C > 0$ such that $cf \le g \le Cf$.

What is the blow-up rate when blow-up is simultaneous? There are some partial results. Let

$$\alpha_1 = \frac{1 + p_{12} - p_{22}}{2(p_{12}p_{21} - (1 - p_{11})(1 - p_{22}))}, \quad \alpha_2 = \frac{1 + p_{21} - p_{11}}{2(p_{12}p_{21} - (1 - p_{11})(1 - p_{22}))}.$$

The case $p_{11} < 1 + p_{21}$, $p_{22} < 1 + p_{12}$, $p_{12}p_{21} > (1 - p_{11})(1 - p_{22})$ has been studied in [5], where the authors show that

$$u(0, t) \sim (T - t)^{-\alpha_1}, \quad v(0, t) \sim (T - t)^{-\alpha_2}, \tag{1.1}$$

provided $p_{11} < 1$ when $p_{11} \le p_{22} + p_{21} - p_{12}$ or $p_{22} < 1$ when $p_{22} \le p_{11} + p_{12} - p_{21}$. This includes the particular case $p_{11} < 1$, $p_{22} < 1$, $p_{12}p_{21} > (1 - p_{11})(1 - p_{22})$, previously studied in [9] under additional assumptions on the initial data. Very recently [13] have proved, adapting the scaling method from [4] to systems, see also [2], [8], [14], that the simultaneous blow-up rate is also given by (1.1) when $p_{11} \ge 1$ and $p_{22} \ge 1$ with $\alpha_1, \alpha_2 > 0$.

The above results do not cover the whole range of parameters for which simultaneous blow-up is possible. Our aim is to fill in all the gaps, namely

(i.a) $p_{11} < 1$ and $1 \le p_{22} < p_{11} + p_{12} - p_{21}$ if $p_{12} > p_{21}$ or

(i.b) $p_{22} < 1$, $1 \le p_{11} < p_{22} + p_{21} - p_{12}$ if $p_{21} > p_{12}$;

(ii) $p_{11} = p_{21} + 1$ and $p_{22} \le p_{12} + 1$;

(iii) $p_{22} = p_{12} + 1$ and $p_{11} \le p_{21} + 1$.

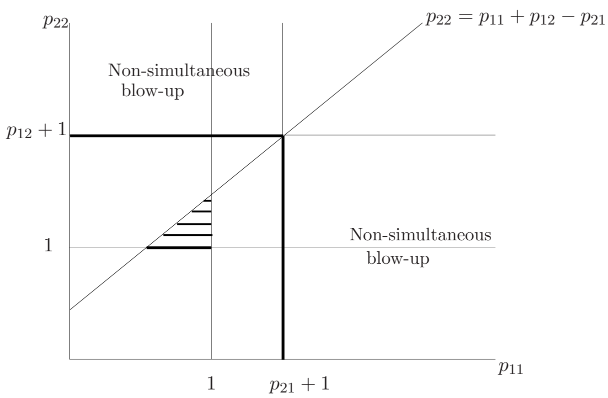

Figure 1:— Gaps for $p_{12} > p_{21}$

We prove the following theorem, covering the whole range of parameters.

Theorem 1.1 *When blow-up is simultaneous, $u(0,t) \sim x(t)$, $v(0,t) \sim y(t)$, where x and y solve*

$$x' = x^{2p_{11}-1}y^{2p_{12}}, \qquad y' = x^{2p_{21}}y^{2p_{22}-1}. \tag{1.2}$$

Thus, a straightforward integration shows that the blow-up rate is given by (1.1) if $\alpha_1, \alpha_2 > 0$, whenever blow-up is simultaneous. However, when one of the α_i vanishes a logarithmic blow-up rate appears. This happens precisely in the borderline cases between simultaneous and non-simultaneous blow-up. For instance, when the parameters go through the critical line $p_{11} = p_{21}+1$ (with $p_{22} < 1+p_{12}$), v passes from a pure power blow-up rate to being bounded; in between, α_2 becomes zero and we have a weaker form of blow-up given by

$$v(0,t) \sim (-\ln(T-t))^{1/(2(p_{12}+1-p_{22}))}. \tag{1.3}$$

The u component also has a logarithmic correction on that line,

$$u(0,t) \sim (T-t)^{-1/(2(p_{11}-1))}(-\ln(T-t))^{p_{12}/(2(p_{12}+1-p_{22})(p_{11}-1))}. \tag{1.4}$$

Notice that the pure power component of the blow-up rate of u on the critical line coincides with the one for non-simultaneous blow-up. Moreover, $\alpha_1 \to 1/(2(p_{11}-1))$ as $p_{11} \nearrow p_{21}+1$. At the point where both critical lines meet, we recover a pure power behaviour

$$u(0,t) \sim (T-t)^{-1/(2(p_{11}-1+p_{12}))}, \quad v(0,t) \sim (T-t)^{-1/(2(p_{22}-1+p_{21}))}. \tag{1.5}$$

2 Proof of Theorem 1.1

We first fill in the gap (i.a). The case (i.b) is similar.

Lemma 2.1 *If* $p_{11} < 1$, $1 \leq p_{22} < p_{11} + p_{12} - p_{21}$, *then* (1.1) *holds if* $p_{12} > p_{21}$.

Proof. If $p_{22} \leq p_{11} + p_{12} - p_{21}$, we have the one-sided blow-rates

$$u(0,t) \geq C(T-t)^{-\alpha_1}, \qquad v(0,t) \leq C(T-t)^{-\alpha_2}, \tag{2.6}$$

see [5]. Then, $u_t = u_{xx}$ with $-u_x(0,t) \leq Cu^{p_{11}}(0,t)(T-t)^{-\alpha_2 p_{12}}$ and $u_x(L,t) = 0$. Using Proposition 1 in [9] we get

$$u(0,t) \leq C(T-t)^{-\alpha_1}.$$

To obtain the rate from below for v, instead of using its equation we use again the equation satisfied by u. Using the well-known representation formula and the jump relation, [3], we have

$$u(0,t) \sim \int_0^t u^{p_{11}}(0,s) \frac{v^{p_{12}}(0,s)}{(t-s)^{1/2}}\, ds.$$

Since $u(0,t) \sim (T-t)^{-\alpha_1}$,

$$(T-t)^{-\alpha_1} \sim \int_0^t (T-s)^{-\alpha_1 p_{11}} \frac{v^{p_{12}}(0,s)}{(t-s)^{1/2}}\, ds.$$

Integrating by parts, since v is increasing,

$$\begin{aligned}
(T-t)^{-\alpha_1} \quad & \leq Cv^{p_{12}}(0,t) \int_0^t \frac{(T-s)^{-\alpha_1 p_{11}}}{(t-s)^{1/2}}\, ds \\
& \leq Cv^{p_{12}}(0,t) \int_0^t (T-s)^{-\alpha_1 p_{11} - 1/2}\, ds \\
& \leq Cv^{p_{12}}(0,t)(T-t)^{-\alpha_1 p_{11} + 1/2}.
\end{aligned}$$

Hence $v(0,t) \geq C(T-t)^{-\alpha_2}$. The obtained blow-up rates coincide with the behaviour of the solutions of (1.2). □

Next, we fill in the gap (ii). Gap (iii) can be handled in a similar way.

Lemma 2.2
(a) *Let* $p_{11} = p_{21} + 1$ *and* $p_{22} < p_{12} + 1$, *then* (1.3) *and* (1.4) *hold.*
(b) *Let* $p_{11} = p_{21} + 1$ *and* $p_{22} = p_{12} + 1$, *then* (1.5) *holds.*

Proof. (a) Following [7], define $M(t) = u(0,t)$ and $N(t) = v(0,t)$ and set, for $t < T$ and $y > 0$, $-t < bs, ds < 0$

$$\varphi_M(y,s) = \frac{u(ay, bs+t)}{M(t)}, \quad \psi_N(y,s) = \frac{v(cy, ds+t)}{N(t)},$$

with $a = M^{1-p_{11}}N^{-p_{12}}, b = a^2, c = N^{1-p_{22}}M^{-p_{21}}, d = c^2$. Since $p_{11} > 1$, a and b go to zero as $t \nearrow T$. We want that c and d also go to zero. This is true, if $p_{22} \geq 1$. Hence, let us assume that $p_{22} < 1$.

We claim that for $\gamma < \min\{1, p_{21}/(1-p_{22})\}$, there exists a constant K large enough such that $Ku^\gamma > v$. Indeed, let $w = Ku^\gamma$. Since $\gamma < 1$, $w_t - w_{xx}$ is a supersolution of the heat equation. As K is large we have $w(x, t_0) > v(x, t_0)$, for a fixed t_0 close to T. Now, we argue by contradiction. Let t_1 be the first time, such that there exists $x_1 \in [0, L]$ with $w(x_1, t_1) = v(x_1, t_1)$. From the maximum principle it follows that $x_1 = 0$. At this point the flux boundary conditions satisfied by w and v lead to a contradiction. Therefore, $w = Ku^\gamma > v$, for t close to T. The claim implies that $d^{1/2} = c \leq CM^{\gamma(1-p_{22})-p_{21}} \to 0$.

Using the technique described in [4] (see also [7]), which is based in the use of well-known Schauder estimates to pass to the limit as $t \nearrow T$, it is easy to show that

$$c \leq (\varphi_M)_s(0,0) \leq C, \qquad c \leq (\psi_N)_s(0,0) \leq C. \tag{2.7}$$

Writing (2.7) in terms of M and N, we get that solutions behave as those of (1.2).

(b) The proof of this case is similar to the previous one. The same calculations used to prove the claim taking $\gamma = 1$ show that $u \sim v$. The use of the ideas of [4] is even easier, since $p_{11}, p_{22} > 1$ imply that $a, b, c, d \to 0$. The relation between u and v together with (2.7) provides us with the desired rates. □

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