AN ADAPTIVE NUMERICAL SCHEME FOR A PARABOLIC PROBLEM WITH BLOW-UP

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Abstract. In this paper we present an adaptive method for the numerical study of positive solutions of the following problem,

$$\begin{cases} u_t = u_{xx} & (x,t) \in (0,1) \times [0,T), \\ u_x(0,t) = 0 & t \in [0,T), \\ u_x(1,t) = u^p(1,t) & t \in [0,T), \\ u(x,0) = u_0(x) & x \in (0,1), \end{cases}$$

with $p > 1$. We describe an adaptive numerical method that adds mesh points as the solution becomes bigger in a precise way that allows us to recover the blow-up rate and the blow-up set of the continuous problem.

1. Introduction.

In this paper we deal with a numerical approximation for the following problem,

$$\begin{cases} u_t = u_{xx} & (x,t) \in (0,1) \times [0,T), \\ u_x(0,t) = 0 & t \in [0,T), \\ u_x(1,t) = u^p(1,t) & t \in [0,T), \\ u(x,0) = u_0(x) & x \in (0,1). \end{cases} \tag{1.1}$$

We assume that u_0 is positive and compatible in order to have a regular solution. If $p > 1$ it is well known that every positive solution becomes unbounded in finite time, a phenomena that is known as blow-up. If T is the maximum time of existence of the solution u then

$$\lim_{t \nearrow T} \|u(\cdot,t)\|_\infty = +\infty,$$

1991 *Mathematics Subject Classification.* 35B40, 35K20, 65M50, 65N50.

Key words and phrases. numerical blow-up, heat equation, nonlinear boundary conditions, adaptive mesh.

RF supported by DGES Project PB94-0153, EU Programme TMR FMRX-CT98-0201 and AECI. PG and JDR supported by ANPCyT PICT No. 03-05009, CONICET (Argentina) and AECI.

see [20], [24], [27] and also [13], [19], [23], [26] for general references on blow-up problems. For (1.1) it is proved in [11], [14] that the blow-up rate is given by,

- $\|u(\cdot,t)\|_{L^\infty} \sim (T-t)^{-\frac{1}{2(p-1)}}$,

The blow-up set, $B(u)$, i.e., the set of points $x \in [0,1]$ where $u(x,t)$ becomes unbounded, is given by,

- $B(u) = \{1\}$, single point blow-up, see [17].

Here we are interested in numerical approximations of (1.1). Since the solution u develops a singularity in finite time, it is an interesting question what can be said about numerical approximations of this kind of problems. For previous work on numerical approximations of blowing up solutions we refer to [1], [2], [6], [7], [8], [9], [15], [18], [21], [22] the survey [5] and references therein. For approximations of (1.1) we refer to [3], [4] and [12]. Those papers deal with the behaviour of a semidiscrete numerical approximation in a fixed mesh of size h. It is observed there that significant differences appear between the continuous and the discrete problems. First, it is proved that positive solutions of the numerical problem blow up if and only if $p > 1$, see [12], this is the same blow-up condition that holds for the continuous problem. Next, it is shown in [3] and [4] that the blow-up rate for the numerical approximation, U, is given by

- $\|U(t)\|_\infty \sim (T_h - t)^{-\frac{1}{p-1}}$.

Therefore the blow-up rate does not coincide with the expected for the continuous problem. Concerning the blow-up set it holds,

- $B(U) = [1-Kh, 1]$ if $p > 1$,

see [4], [16]. The constant K depends only on p, K is the only integer that verifies

$$\frac{K+1}{K} < p \le \frac{K}{K-1}.$$

Hence the numerical blow-up set can be larger than a single point, $x = 1$, but as K does not depend on h,

$$B(U) \to B(u), \qquad \text{as } h \to 0.$$

Collecting all these results, we observe that when computing numerical approximations of a blow-up problem with a fixed mesh significant differences appear concerning the behaviour of the numerical and the continuous solutions. For this problem the blow-up rate and the blow-up set can be different.

In this paper we introduce an adaptive method that gives the right blow-up rate and the exact blow-up set. This numerical method is based on a semidiscretization in space and adds points near the boundary, $x = 1$, when the numerical solution becomes large. This adaptive procedure leads to a non-uniform mesh that concentrates near the singularity and gives the precise blow-up rate and set. Some references that use adaptive numerical methods are [3], [7] and [8].

Our numerical scheme is based on the scale invariance of the heat equation in the half line with the nonlinear boundary condition placed a $x = 0$, $-u_x(0,t) = u^p(0,t)$. If $u(x,t)$ is a solution then $u_\lambda(x,t) = \lambda^{\frac{1}{2(p-1)}} u(\lambda^{\frac{1}{2}} x, \lambda t)$ also is a solution. In [14] it is proved that there exists a self-similar blowing up solution in the half-line of the self-similar form

$$u_S(x,t) = (T-t)^{-\frac{1}{2(p-1)}} \varphi(\xi), \qquad \xi = x(T-t)^{-\frac{1}{2}}.$$

For an explicit form of the profile φ, see [14]. This solution $u_S(x,t)$ gives the behaviour near the blow-up time T for solutions of (1.1) in the following sense

$$u(x,t) \sim (T-t)^{-\frac{1}{2(p-1)}} \varphi(\xi), \tag{1.2}$$

for $x = 1 - \xi(T-t)^{1/2}$, $|\xi| \le C$. So the behaviour near the blow-up point $(1,T)$ is given by the self-similar solution in the half-line. Our numerical scheme uses this fact to add points near $x = 1$ trying to reproduce the scaling invariance in the half-line. This produces a method that gives the correct blow-up rate and set. Our result reads as follows.

Theorem 1.1. *Let u be an increasing in x, smooth solution of* (1.1) *and u_h be the numerical solution obtained by the adaptive scheme described in section 2. Then, for every $\tau > 0$, the numerical solution u_h converges to the continuous one uniformly in $[0,1]\times[0,T-\tau]$, in fact there exists a constant $C = C(\tau)$ such that*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \le C h^{\frac{3}{2}}.$$

The numerical solution u_h blows up if and only if $p > 1$ in the sense that there exists a finite time T_h with

$$\lim_{t \nearrow T_h} u_h(1,t) = +\infty.$$

This numerical blow-up time converges to the continuous one when $h \to 0$, in fact there exist $\alpha > 0$ and $C > 0$ such that

$$|T_h - T| \le C h^\alpha.$$

Moreover, the numerical blow-up rate is given by

$$\lim_{t \nearrow T_h} (T_h - t)^{\frac{1}{2(p-1)}} \|u_h(\cdot,t)\|_\infty = \Gamma = \varphi(0)$$

and the numerical blow-up set is

$$B(u_h) = \{1\}.$$

Organization of the paper: In section 2 we describe our numerical adaptive procedure. In sections 3 to 6 we develop the proofs of our main result, Theorem 1.1. To begin our analysis we prove in section 3 that numerical approximations converge uniformly in sets of the form $[0,1]\times[0,T-\tau]$. In section 4 we prove that the scheme reproduces the blow-up rate. In section 5 we find the numerical blow-up set that coincides with the continuous one. In section 6 we prove that the numerical blow-up time converges to the continuous one. Finally, in section 7 we present some numerical experiments. We leave for the Appendix the proof of a technical result needed in section 6.

2. The adaptive numerical scheme

In this section we present the numerical method. We discretize using piecewise linear finite elements with mass lumping in a mesh for the space variable.

Consider a partition (that can be non-uniform), $x_1, ..., x_{N+1}$ of $(0,1)$ of size h ($h = \max(x_i - x_{i-1})$) and its associated standard piecewise linear finite element space V_h. Let $\{\varphi_j\}_{1\le j\le N}$ be the usual Lagrange basis of V_h. The semidiscrete approximation $u_h(t)\in V_h$ obtained by the finite element method with mass lumping is defined by:

$$\int_0^1 (u_h' v)^I dx + \int_0^1 (u_h)_x v_x dx = u^p(1,t)v(1,t), \qquad \forall v\in V_h, \forall t\in(0,T)$$

where the superindex I denotes the Lagrange interpolation. We denote with $U(t) = (u_1(t),\dots,u_{N+1}(t))$ the values of the numerical approximation at the nodes x_k, at time t. Writing,

$$u_h(t) = \sum_{j=1}^{N} u_j(t)\varphi_j,$$

a simple computation shows that $U(t)$ satisfies the system of ordinary differential equations (see [10]):

$$\begin{cases} MU'(t) = -AU(t) + BU^p(t), \\ U(0) = u_0^I, \end{cases}$$

where M is the mass matrix obtained with lumping, A is the stiffness matrix and u_0^I is the Lagrange interpolation of the initial data, u_0.

Let us pay special attention to the numerical solution at the point $x=1$ (at this point is where the continuous solution develops the singularity). Writing the equation satisfied by u_{N+1} explicitly we obtain the following ODE,

$$u'_{N+1}(t) = \frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t),$$

where $h_N = 1 - x_N$. Now, let us describe our adaptive procedure. As we want the numerical blow-up rate to be

$$u_h(1,t) \sim (T_h - t)^{-\frac{1}{2(p-1)}}$$

we impose that the last node $u_{N+1}(t) = u_h(1,t)$ satisfies

$$c_1 u_{N+1}^{2p-1}(t) \le u'_{N+1}(t) \le c_2 u_{N+1}^{2p-1}(t). \tag{2.1}$$

Imposing (2.1) is equivalent to

$$c_1 u_{N+1}^{2p-1}(t) \le \frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t) \le c_2 u_{N+1}^{2p-1}(t). \tag{2.2}$$

As it is proved in [12] that the scheme with a fixed mesh blows up at the last node, we have that

$$R(t;h_N) = \frac{\frac{2}{h_N^2}(u_N(t) - u_{N+1}(t)) + \frac{2}{h_N} u_{N+1}^p(t)}{u_{N+1}^{2p-1}(t)} \to 0 \qquad \text{as } t \text{ increases.}$$

Let t_1 be the first time such that $R(t;h_N) = c_1$, that is the first time where $R(t;h_N)$ meets the tolerance c_1. At that time t_1 we add a point z between x_N and $x_{N+1} = 1$ to the mesh and give the value of $u_h(z,t_1)$ such that the slope of the line between $(z, u_h(z,t_1))$ and $(1, u_h(1,t_1))$ is the same as the slope between $(x_N, u_h(x_N,t_1))$ and $(1, u_h(1,t_1))$, see Figure 1.

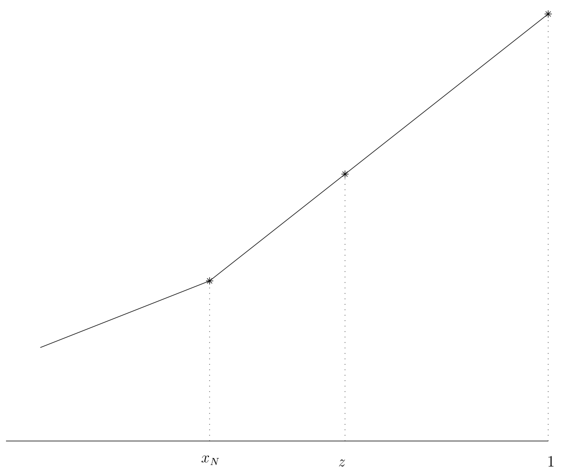

Figure 1.

Hence we have a new value for the length of the last interval, $[z,1]$,

$$h_{N,1} = 1 - z < h_N = 1 - x_N.$$

We remark that $u_h(z,t_1) = u_z(t_1)$ satisfies,

$$\frac{1}{h_N}(u_N(t_1) - u_{N+1}(t_1)) = \frac{1}{h_{N,1}}(u_z(t_1) - u_{N+1}(t_1)).$$

Let us see what happens with $R(t_1; h_{N,1})$.

$$\begin{aligned}
R(t_1; h_{N,1}) &= \frac{\frac{2}{h_{N,1}^2}(u_z(t_1) - u_{N+1}(t_1)) + \frac{2}{h_{N,1}} u_{N+1}^p(t_1)}{u_{N+1}^{2p-1}(t_1)} \\
&= \frac{\frac{2}{h_{N,1}}(u_z(t_1) - u_{N+1}(t_1)) + 2u_{N+1}^p(t_1)}{h_{N,1}\ u_{N+1}^{2p-1}(t_1)} \\
&= \frac{\frac{2}{h_N}(u_N(t_1) - u_{N+1}(t_1)) + 2u_{N+1}^p(t_1)}{h_{N,1}\ u_{N+1}^{2p-1}(t_1)} \\
&= \frac{h_N}{h_{N,1}} R(t_1; h_N) = \frac{h_N}{h_{N,1}}\ c_1 > c_1.
\end{aligned}$$

Therefore with this new mesh $x_1, ..., x_N, z, x_{N+1}$ we have that

$$R(t_1; h_{N,1}) > c_1,$$

and we can apply the method with initial data $u_h(x,t_1)$ in the new mesh to continue. This gives us a solution u_h that verifies (2.2) in a time interval $[t_1,t_2]$ where at time t_2, the function $R(t_2; h_{N,1})$ reaches the tolerance c_1. At that time t_2 we have to add another point in the last interval. As before this increases $R(t_2; h_{N,2})$ and we can continue with initial data $u_h(x,t_2)$ in a new mesh that is the old one plus the point that we have added near the boundary $x = 1$. This procedure generates an increasing sequence of times t_i and a decreasing sequence $h_i = h_{N,i}$, at which the tolerance $R(t_i; h_i) = c_1$ is reached, a sequence of added points accumulating at $x = 1$ and a numerical solution $u_h(x,t)$.

It remains to be more concrete on the election of the sequence h_i, c_1 and c_2. In fact if we take,

$$c_1 = \Gamma^{-2(p-1)}(2(p-1))^{-1},$$

and h_i and c_2 such that

$$\frac{h_i}{h_{i+1}} \to 1 \qquad \text{and} \qquad c_2 = c_2(t) \sim c_1, \tag{2.3}$$

integrating (2.1) we get that

$$\Gamma(T_h - t)^{-\frac{1}{2(p-1)}} \geq u_{N+1}(t) = \max u_h(\cdot, t)$$
$$\geq \left(2(p-1)\int_t^{T_h} c_2(s)\, ds\right)^{-\frac{1}{2(p-1)}},$$

Hence, as $t \nearrow T_h$,

$$u_{N+1}(t) = \max u_h(\cdot, t) \sim \Gamma(T_h - t)^{-\frac{1}{2(p-1)}},$$

and we have recovered the continuous asymptotic behaviour (1.2). So we want that h_i and c_2 verify (2.3). On the other hand, as we have mentioned in the introduction, the continuous problem verifies that

$$u(x,t) \sim (T-t)^{-\frac{1}{2(p-1)}}\varphi(\xi)$$

for $x = 1 - \xi(T-t)^{1/2} \sim C(u(1,t))^{-(p-1)}$. In order to recover this self-similar scaling we choose h_i and c_2 such that

$$h_{i+1}(u_{N+1}(t_i))^{p-1} = \frac{2}{c_1} - \frac{A}{u_{N+1}(t_i)} \quad \text{and} \quad c_2 = c_1\left(\frac{1}{1 - \frac{c_1 A}{2u_{N+1}(t_i)}}\right).$$

And hence,

$$1 - x_N = h_{i+1} \sim \frac{2}{c_1}(u_{N+1}(t_i))^{-(p-1)} \sim C(T_h - t)^{1/2}.$$

This procedure gives an approximation of the curve $x = 1 - \xi(T-t)^{1/2}$. Notice that if h_{i+1} and c_2 are given as above, relations (2.3) hold. So we recover the self-similar behavior (1.2).

3. Convergence of the numerical scheme

In this section we prove a uniform convergence result. We use ideas from [12] and we include the arguments here to make the paper self-contained. For any $\tau > 0$ we want that $u_h \to u$ (when $h \to 0$) uniformly in $[0,1] \times [0, T-\tau]$. This is a natural requirement since on such an interval the exact solution is regular.

In particular, uniform convergence can be obtained by using standard inverse inequalities. In the following Theorem we give a proof of the L^2 convergence for a problem like (1.1) considering mass lumping. As a corollary, we will obtain uniform convergence for problem (1.1).

Theorem 3.1. *Let u be the solution of* (1.1) *and let u_h its semidiscrete approximation obtained by the adaptive scheme. If $u \in C^{2,1}([0,1] \times [0, T-\tau])$ for some $\tau > 0$ then, there exists a constant C depending on τ such that:*

$$\|u - u_h\|_{L^\infty([0,T-\tau],L^2)} \leq Ch^2.$$

Proof. We fix our attention on the convergence of the method in a time interval of the form $[t_i, t_{i+1}]$, that is between two consecutive refinement times. Next we will show that this is enough for our purposes. Let us begin by $t \in (0, t_1)$.

In this proof we use the notation $L^2 = L^2((0,1))$ that refers to the L^2 norm in the x variable for each t (we will use analogous notations for other norms below) and u' for the derivative respect to time, u_t.

As u is a solution of (1.1) it satisfies

$$\int_0^1 u'v + \int_0^1 u_x v_x = u^p(1,t)v(1,t) \qquad \forall v \in H^1.$$

The numerical scheme is equivalent to

$$\int_0^1 ((u_h)'v)^I + \int_0^1 u_x v_x = u_h^p(1,t)v(1,t) \qquad \forall v \in V_h.$$

Hence we have that $e = u - u_h$ satisfies the following error equation,

$$\int_0^1 (e'v)^I\, dx + \int_0^1 e_x v_x\, dx = (u^p - u_h^p)(1,t)v(1,t) + \int_0^1 ((u'v)^I - u'v)\, dx$$

for all $v \in V_h$. Writing

$$e = u - u_h + u^I - u^I = u - u^I + \eta$$

and using known error estimates for Lagrange interpolation it rest to estimate $\eta = u^I - u_h$.

First, it is easy to see that,

$$\int_0^1 (u - u^I)_x v_x = 0 \qquad \forall v \in V_h,$$

and therefore, replacing in the error equation we have an equation for η,

$$\begin{aligned}\int_0^1 (\eta'v)^I + \int_0^1 \eta_x v_x &= (u^p(1,t) - u_h^p(1,t))v(1,t)\\ &+ \int_0^1 ((u'v)^I - u'v) - \int_0^1 ((u' - (u^I)')v)^I \qquad \forall v \in V_h.\end{aligned}$$

In particular if we choose $v = \eta \in V_h$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\int_0^1 (\eta^2)^I\Big) + \int_0^1 (\eta_x)^2 = \ & (u^p - u_h^p)(1,t)\eta(1,t)\\ &+ \int_0^1 ((u'\eta)^I - u'\eta) = I + II.\end{aligned}$$

First, let us estimate I.

$$|I| = |(u^p(1,t) - u_h^p(1,t))\eta(1,t)| = \left|((u^I)^p(1,t) - u_h^p(1,t))\eta(1,t)\right|.$$

Hence we get that

$$|I| \leq C|\eta(1,t)|^2.$$

Using the well known inequality,

$$|\eta(1,t)|^2 \leq (C + \frac{C}{\varepsilon})\|\eta\|^2_{L^2((0,1))} + \varepsilon\|\eta_x\|^2_{L^2((0,1))},$$

we have that

$$|I| \leq C_\varepsilon\|\eta\|^2_{L^2((0,1))} + \varepsilon\|\eta_x\|^2_{L^2((0,1))}.$$

In order to get a bound for II we decompose it in the following form,

$$II = \int_0^1 ((u'\eta)^I - u'\eta)\,dx = \int_0^1 ((u'\eta)^I - (u')^I\eta)\,dx + \int_0^1 ((u')^I\eta - u'\eta)\,dx.$$

We proceed as before, for each subinterval I_j of the partition we know that,

$$\begin{aligned} \|((u')^I\eta)^I - (u')^I\eta\|_{L^1(I_j)} \leq &\ Ch^2\|((u')^I\eta)_{xx}\|_{L^1(I_j)} \\ \leq &\ Ch^2\|((u')^I)_x\eta_x\|_{L^1(I_j)} \end{aligned}$$

because $(u')^I$ and η are linear over I_j. Hence, summing over all the elements I_j and using that $\|((u')^I)_x\|_{L^2} \leq C\|u'\|_{H^1}$ we obtain,

$$\begin{aligned} \int_0^1 ((u'\eta)^I - &\ (u')^I\eta)\,dx \leq Ch^2\int_0^1 |((u')^I)_x||\eta_x|\,dx \\ \leq &\ Ch^2\|((u')^I)_x\|_{L^2((0,1))}\|\eta_x\|_{L^2} \leq Ch^4\|u'\|^2_{H^1} + \tfrac{1}{4}\|\eta_x\|^2_{L^2}. \end{aligned}$$

It rests to estimate the second summand of II. We have,

$$\begin{aligned} \int_0^1 ((u')^I - &\ u')\eta\,dx \leq \|(u')^I - u'\|_{L^2}\|\eta\|_{L^2} \\ \leq &\ \|(u')^I - u'\|^2_{L^2} + \|\eta\|^2_{L^2} \leq Ch^4\|u'\|^2_{H^2} + \|\eta\|^2_{L^2}. \end{aligned}$$

Collecting all the previous bounds we obtain,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 (\eta^2)^I + &\ \int_0^1 |\eta_x|^2 \leq C_\varepsilon\|\eta\|^2_{L^2((0,1))} + C\varepsilon\|\eta_x\|^2_{L^2((0,1))} \\ + &\ Ch^4\|u'\|^2_{H^2} + \|\eta\|^2_{L^2} + Ch^4\|u'\|^2_{H^1} + \tfrac{1}{4}\|\eta_x\|^2_{L^2}. \end{aligned}$$

We choose ε such that $C\varepsilon = 1/4$ and we obtain,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\eta^2)^I + \frac{1}{2}\int_0^1 |\eta_x|^2 \leq C\|\eta\|^2_{L^2((0,1))} + Ch^4\|u'\|^2_{H^2}.$$

Since $\int_0^1 (\eta^2)^I\,dx \sim \|\eta\|_{L^2}^2$ we can apply Gronwall's Lemma to obtain for $t \in [0, T_1]$,

$$\|\eta(t)\|_{L^2} + \left(\int_0^{t_1} \|\eta_x\|_{L^2}^2\,dt\right)^{1/2} \le Ce^{C(t_1)}h^2.$$

In particular,

$$\|\eta\|_{L^2} \le C(u, t_1)h^2.$$

and hence,

$$\|e\|_{L^2} \le \|u - u^I\|_{L^2} + \|\eta\|_{L^2} \le C(u, t_1)h^2.$$

This proves the result in $[0, t_1]$. Assume that the result is true in $[0, t_i]$. The same arguments used before prove that the bound holds up to t_{i+1}. The observation that there exists i_0 such that $t_{i_0} > T - \tau$ finishes the proof. □

As a corollary of Theorem 3.1 we can prove the following result that gives uniform, L^∞, convergence in an interval of the form $[0, T - \tau]$.

Corollary 3.1. *Let u be the solution of* (1.1) *and u_h its numerical approximation. Given $\tau > 0$, there exists a constant C depending on $\|u\|$ in $C^{2,1}([0,1] \times [0, T - \tau])$ such that, for h small enough:*

$$\|u - u_h\|_{L^\infty([0,1]\times[0,T-\tau])} \le Ch^{\frac{3}{2}}.$$

Proof. It is known that before the blow up time u is regular, more precisely, $u \in C^{2,1}([0,1] \times [0, T - \tau])$. Let $g(u)$ be a globally Lipschitz function which agrees with $f(u) = u^p$ for $u \le 2M$ where $M = \|u\|_{L^\infty([0,1]\times[0,T-\tau])}$. Let $\overline{u}$ and $\overline{u}_h$ be the exact and approximate solutions of a problem like (1.1) with $f(u) = u^p$ replaced by $g(u)$. By uniqueness $u = \overline{u}$ in $[0,1] \times [0, T - \tau]$. A bound for $\|\overline{u} - \overline{u}_h\|_{L^\infty}$ can be obtained from Theorem 3.1. Indeed, it is enough to bound $\|\overline{u}^I - \overline{u}_h\|_{L^\infty}$, and using a standard inverse inequality (see [10]) we have,

$$\begin{aligned}
\|\overline{u}^I - \overline{u}_h\|_{L^\infty} \le\ & Ch^{-\frac{1}{2}}\|\overline{u}^I - \overline{u}_h\|_{L^\infty([0,T-\tau],L^2)} \\
\le\ & Ch^{-\frac{1}{2}}\left\{\|\overline{u}^I - \overline{u}\|_{L^\infty([0,T-\tau],L^2)}\right. \\
& \left. + C\|\overline{u} - \overline{u}_h\|_{L^\infty([0,T-\tau],L^2)}\right\} \le Ch^{\frac{3}{2}}
\end{aligned}$$

with C depending on u and the constant in Theorem 3.1 and so on τ.

Consequently, for h small enough $|\overline{u}_h| \le 2M$. Therefore $u_h^p = f(\overline{u}_h) = g(\overline{u}_h)$ and so $\overline{u}_h$ is the finite element approximation of u and, by uniqueness $\overline{u}_h = u_h$ which concludes the proof. □

To finish this Section, we state a Lemma that says that the maximum of $U(t)$ is attained at the last node, x_{N+1}.

Lemma 3.1. *If $U(0)$ is increasing, then the numerical solution $U(t)$ is increasing for every time t. Hence it satisfies*

$$\max u_h(\cdot,t) = u_{N+1}(t),$$

for every t.

Proof. As $U(0)$ is increasing then $u_{i+1}(0) > u_i(0)$, let us see that this holds for every $t > 0$. Assume not, then there exists a first positive time t_0 and an index i such that $u_{i+1}(t_0) = u_i(t_0)$. From the equations satisfied by u_{i+1} and u_i we have that at that time t_0 it holds $u'_{i+1}(t_0) - u'_i(t_0) > 0$ a contradiction that proves the result. $\square$

4. Numerical blow-up.

In this section we prove that solutions of the numerical problem blow up if and only if $p > 1$ and we find the blow-up rate in this case.

Lemma 4.1. *If $p > 1$ then every positive solution of the numerical scheme blows up. Moreover, if $U(0)$ is increasing, then*

$$\lim_{t \nearrow T_h} (T_h - t)^{\frac{1}{2(p-1)}} \|U(t)\|_\infty = \Gamma,$$

where T_h is the blow-up time of U.

Proof. If $U(0)$ is increasing, by Lemma 3.1 $U(t)$ has its maximum at $u_{N+1}(t)$. At a refinement time t_i we have that $R(h_i, t_i) = c_1$ and we refine the mesh to get

$$c_1 < R(h_{i+1}, t_i) = \frac{h_i}{h_{i+1}} R(h_i, t_i) = \frac{h_i}{h_{i+1}} c_1$$

We obtain that

$$\frac{u'_{N+1}}{u_{N+1}^{2p-1}}(t) \geq c_1,$$

for every t. By integration we get that, as $p > 1$, u_{N+1} (and hence u_h) cannot be global. Moreover, if we call T_h the blow-up time, using that $h_i/h_{i+1} \to 1$ $(i \to \infty)$, we have

$$\lim_{t \to T_h} \frac{u'_{N+1}}{u_{N+1}^{2p-1}}(t) = c_1. \tag{4.1}$$

Integrating (4.1), we get

$$(c_1 - \varepsilon_2(t))(T_h - t) \leq \int_t^{T_h} \frac{u'_{N+1}}{u_{N+1}^{2p-1}}(s)\, ds \leq (c_1 + \varepsilon_1(t))(T_h - t).$$

where $\varepsilon_i(t) \to 0$ as $t \nearrow T_h$. Changing variables we obtain

$$(c_1 - \varepsilon_2(t))(T_h - t) \leq \int_{u_{N+1}(t)}^{+\infty} \frac{1}{s^{2p-1}}\, ds \leq (c_1 + \varepsilon_1(t))(T_h - t).$$

Hence

$$\lim_{t\to T_h} (T_h - t)^{\frac{1}{2(p-1)}} u_{N+1}(t) = (2(p-1)c_1)^{\frac{-1}{2(p-1)}} = \Gamma,$$

as expected from the description of the method. $\square$

We remark that this rate coincides with the blow-up rate of the continuous problem.

5. Numerical blow-up set.

Now we turn our interest to the blow-up set of the numerical solution. We want to look at the set of points, x, such that $u_h(x,t) \to +\infty$ as $t \nearrow T_h$.

In [3] and [16] it is proved that for a fixed mesh the numerical blow-up set is given by $B(U) = [1 - Kh, 1]$, where K depends only on p. Notice that as K does not depend on h, the blow-up set converges to the continuous blow-up set $B(u) = \{1\}$.

We remark that in our method we refine near $x = 1$, therefore the parameter h_i goes to zero and formally the blow-up set for our method must be $B(U) = \{1\}$. In order to prove this, we take a point $\bar{x} < 1$ and we claim that the numerical solution is bounded in $[0, \bar{x}]$.

Lemma 5.1. *Let $\bar{x} < 1$ then for all nodes $x_i < \bar{x}$ the numerical solution is bounded, i.e., there exists $C > 0$ such that*

$$u_h(x,t) \leq C,$$

for all $x \in [0, \bar{x}]$, $t \in [0, T_h)$.

Proof. Let K be the first integer such that $K - 1/2(p-1) > -1$, this implies that the function $g(t) = t^{-1/2(p-1)}$ is K times integrable.

From Lemma 4.1 we can suppose that

$$\max u_h(\cdot, t) = u_{N+1}(t) \sim C(T_h - t)^{-\frac{1}{2(p-1)}}.$$

Since we refine near $x = 1$ adding points when the solution gets large, we can consider a node $x_{i_K} > \bar{x}$ such that at time t_{i_k} there exits $K+1$ nodes between $\bar{x}$ and x_{i_K}.

On the other hand, as we refine between x_N and $x_{N+1} = 1$, we obtain that for the nodes x_i, $i = 1, \cdots, N-1$ the mesh is fixed. Therefore from the equation satisfied by these nodes we have that for $t > t_{i_k}$

$$u_i'(t) \leq C \max\,(u_{i+1},\, u_{i-1}) \leq C(T_h - t)^{-\frac{1}{2(p-1)}},$$

and then for $i = 1, \cdots, N-2$ we get

$$u_i(t) \le C(T_h - t)^{-\frac{1}{2(p-1)}+1}.$$

This bound implies that

$$u_i'(t) \le C \max\,(u_{i+1},\, u_{i-1}) \le C(T_h - t)^{-\frac{1}{2(p-1)}+1},$$

for $i = 1, \cdots, N-2$. Integrating again we get

$$u_i(t) \le C(T_h - t)^{-\frac{1}{2(p-1)}+2},$$

for every $i = 1, \cdots, N-3$. Repeating the same argument K times we have that $u_i(t)$ is bounded for all $i = 1, \cdots, N-K-1$.

Hence, as there are K nodes between $\bar{x}$ and x_K, we get that for all nodes $x_i < \bar{x}$, $u_i(t)$ is bounded. □

We remark that this result implies that $B(U) = \{1\}$.

6. Convergence of the numerical blow-up times.

In this section we prove that the numerical blow-up times, T_h, converges to the continuous one, T, when h goes to zero.

Lemma 6.1. *The numerical blow-up times converge to the continuous one when $h \to 0$, in fact there exist $\alpha > 0$ and $C > 0$ such that*

$$|T_h - T| \le Ch^{\alpha}.$$

Proof. The idea of the proof is as follows, first we get bounds for the first time t_0 such that the error verifies

$$\|(u_h - u)(t)\|_{L^\infty([0,1])} \le 1, \qquad \text{for all } t \in [0, t_0], \tag{6.1}$$

then we use this bound to prove our convergence result.

To get a bound for t_0 let us look at our scheme between two consecutive refinement times, from section 2 we have,

$$MU' = -AU + BU^p. \tag{6.2}$$

If we call $z_i(t) = u(x_i, t)$ we get that $Z = (z_1, \ldots, z_{N+1})$ satisfies

$$MZ' \le -AZ + BZ^p + Ch(T-t)^{\theta}, \tag{6.3}$$

where θ depends only on p, in fact $\theta = -2 - 1/2(p-1)$. Let us remark that $C(T-t)^{\theta}$ is a bound for the first four spatial derivatives of $u(x,t)$ a regular solution of (1.1).

Subtracting (6.2) from (6.3) we have, for $E = U - Z$,

$$\begin{aligned} ME' \le&\ -AE + B(Z^p - U^p) + Ch(T-t)^{\theta} \\ \le&\ -AE + Bp\xi^{p-1}E + Ch(T-t)^{\theta}, \end{aligned} \tag{6.4}$$

where ξ is a point between u_{N+1} and z_{N+1}. Using that $t \in [0, t_0]$, (6.1), and the blow-up rate we get

$$\xi^{p-1} \le C(T-t)^{-\frac{1}{2}}.$$

Hence (6.4) gives

$$ME' \le -AE + BC\frac{E}{(T-t)^{\frac{1}{2}}} + Ch(T-t)^{\theta}, \tag{6.5}$$

This inequality is a discretization of the problem

$$\begin{cases} E_t \le E_{xx} + C(T-t)^{\theta} h & (x,t) \in (0,1) \times [0,T), \\ E_x(0,t) \le 0 & t \in [0,T), \\ E_x(1,t) \le C(T-t)^{-1/2} E(1,t) & t \in [0,T), \\ E(x,0) \le Ch & x \in (0,1). \end{cases} \tag{6.6}$$

In the appendix we construct a supersolution for problem (6.6) such that

$$\overline{E}(x,t) \le Ch(T-t)^{-\gamma} \tag{6.7}$$

and with the first four spatial derivatives positives in $[0,1]$.

Now we take $\overline{e}_i(t) = \overline{E}(x_i, t)$ and we get a supersolution for (6.5). We use a comparison argument, see [12], between two consecutive refinement times to obtain that

$$e_i(t) \le \overline{e}_i(t).$$

Therefore, using (6.7) we conclude that

$$e_i(t) \le C(T-t)^{-\gamma} h.$$

The same arguments applied to $E = Z - U$ gives the lower bound

$$e_i(t) \ge C(T-t)^{-\gamma} h.$$

Therefore

$$|e_i(t)| \le C(T-t)^{-\gamma} h.$$

We get a bound for t_0 as follows, let t_* be the first time such that $C(T-t_*)^{-\gamma} h = 1$. That is, $T - t_* = Ch^{1/\gamma}$. As $t_0 \ge t_*$ we have

$$T - t_0 \le Ch^{1/\gamma}.$$

This means that the first time where the error is of size one is less than a power of the parameter h. With this bound for $T - t_0$ we can get the desired result as follows

$$|T_h - T| \le |T - t_0| + |T_h - t_0| \le Ch^{1/\gamma} + |T_h - t_0|.$$

Now we observe that the blow-up rate for u_h gives that

$$\begin{aligned} |T_h - t_0| \leq \; & C(u_{N+1}(t_0))^{-2(p-1)} \leq C(z_{N+1}(t_0)+1)^{-2(p-1)} \\ \leq \; & C(T-t_0) \leq Ch^{\frac{1}{\gamma}}, \end{aligned}$$

and the result follows. □

Let us observe that we can obtain a convergence result using these ideas. However, the methods used in Section 3 provided a better estimate.

7. Numerical experiments.

In this section we present a numerical experiments. Our goal is to show that the results presented in the previous sections can be observed when one perform numerical computations. For the numerical experiment we use an adaptive ODE solver to integrate in time. We take as initial datum $u_0(x) = (x^2+1)/2$. In the figures below we compare our adaptive scheme with one with a fixed mesh of the same size $h = 1/50$, for $p = 2$ and $p = 1.5$. We show the graphic of $\ln(u_h(1,t))$ versus $\ln(T_h - t)$. The slope of the obtained curves measures the blow-up rate. As expected these slopes for $p = 2$ are $1/2$ with our adaptive scheme and 1 with a fixed mesh, while for p=1.5 are 1 and 2 respectively.

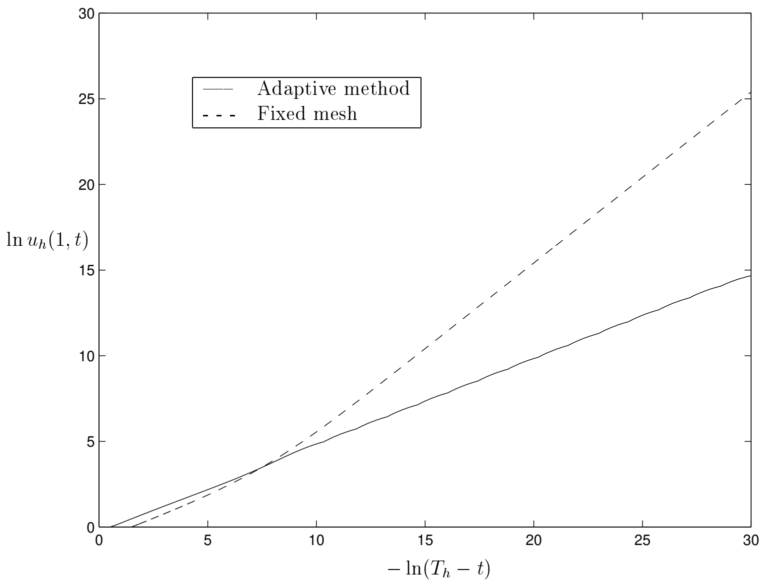

Figure 1. Blow-up rates. $p = 2$.

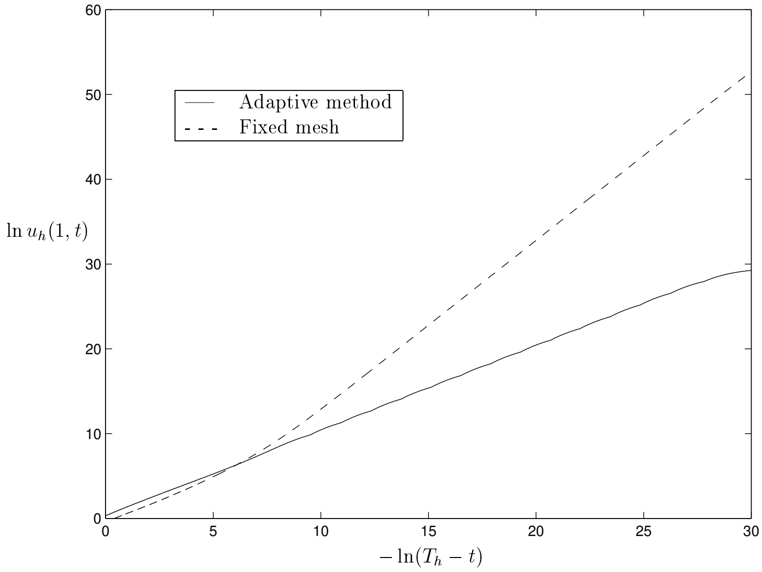

Figure 2. Blow-up rates. $p = 1.5$.

8. Appendix.

In this appendix we find a supersolution for

$$\begin{cases} E_t = E_{xx} + C(T-t)^{\theta} h & (x,t) \in (0,1) \times [0,T), \\ E_x(0,t) = 0 & t \in [0,T), \\ E_x(1,t) = C(T-t)^{-1/2} E(1,t) & t \in [0,T), \\ E(x,0) = Ch & x \in (0,1), \end{cases}$$

such that

$$\overline{E}(x,t) \le Ch(T-t)^{-\gamma}$$

and with the first four spatial derivatives positives in $[0,1]$.

Let us look for a supersolution of the form

$$\overline{E}(x,t) = Ch(T-t)^{\theta} a(x,t),$$

with $a(x,t)$ a solution of

$$\begin{cases} a_t = a_{xx} & (x,t) \in (0,1) \times [0,T), \\ a_x(0,t) = 0 & t \in [0,T), \\ a_x(1,t) = C(T-t)^{-1/2} a(1,t) & t \in [0,T), \\ a(x,0) = a_0(x) & x \in (0,1). \end{cases} \tag{8.1}$$

In order to get a supersolution, $\overline{E}$, with the first four spatial derivatives positives, we impose that $a_0(x)$ is a smooth compatible initial datum with the first four spatial derivatives positives. The positivity of the derivatives is preserved for every $t \in [0,T)$.

Now let us see that there exists r such that

$$a(x,t) \leq \frac{C}{(T-t)^r}. \tag{8.2}$$

To this end we want to construct a supersolution to (8.1) such that (8.2) holds. This can be easily done by the following procedure: take $v(x,t)$ a solution of (1.1) with boundary condition given by $v_x(1,t) = v^q(1,t)$, with q small, and initial datum v_0 such that $v(x,t)$ blows up exactly at time T, see [25] for a proof of the fact that for every time T there exist an initial datum such that $v(x,t)$ blows up at time T. The blow-up rate for solutions of (1.1), see [14], [17], gives that $v(x,t)$ verifies

$$v(1,t) \sim \frac{L}{(T-t)^{\frac{1}{2(q-1)}}},$$

with $L^{q-1} \to +\infty$ as $q \searrow 1$, see [14] for an explicit formula for $L(q)$. Let us fix q such that $L^{q-1}(q) > C$. With this choice, v is a supersolution of (8.1). In fact,

$$v_x(1,t) = v^q(1,t) = v^{q-1}v(1,t) \geq \frac{L^{q-1}}{(T-t)^{\frac{1}{2}}} v(1,t),$$

as we wanted to prove.

Acknowledgements This research was finished during a stay of the second and third authors, (PG) and (JDR), at Universidad Autónoma de Madrid. They are grateful to this institution for its hospitality.

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