UPPER BOUNDS FOR THE DECAY RATE IN A NONLOCAL $p-$LAPLACIAN EVOLUTION PROBLEM.

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Abstract. We obtain upper bounds for the decay rate for solutions to the nonlocal problem $\partial_t u(x,t) = \int_{\mathbb{R}^n} J(x,y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))dy$ with an initial condition $u_0 \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and a fixed $p>2$. We assume that the kernel J is symmetric, bounded (and therefore there is no regularizing effect) but with polynomial tails, that is, we assume a lower bounds of the form $J(x,y) \geq c_1|x-y|^{-(n+2\sigma)}$, for $|x-y|>c_2$ and $J(x,y)\geq c_1$, for $|x-y|\leq c_2$. We prove that $\|u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq Ct^{-\frac{n}{(p-2)n+2\sigma}(1-\frac{1}{q})}$ for $q\geq 1$ and t large.

1. Introduction.

In this paper we deal with nonlocal Cauchy problems of the form

$$\partial_t u(x,t) = \int_{\mathbb{R}^n} J(x,y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))dy \tag{1.1}$$

for $t\in\mathbb{R}_+$ and $x\in\mathbb{R}^n$ with $n\geq 2$, a fixed $p>2$ and an initial condition $u(x,0) = u_0(x)$ satisfying $u_0\in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. On the kernel J, we will always assume that it is a bounded and symmetric function defined for $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$ together with the integrability condition $J(\cdot,y)\in L^1(\mathbb{R}^n)$ for all $y\in\mathbb{R}^n$. Under these hypotheses existence and uniqueness of a solution follows from a fixed point argument as in [1].

Nonlocal problems have been recently widely used to model diffusion processes (see [6] and [5] for a general nonlocal vector calculus). Problem (1.1) and its stationary version have been considered recently in connection with real applications, for example to peridynamics or a recent model for elasticity. We quote for instance [2], [11], [12], [13], [14] and the recent book [1].

Our main goal here is to obtain upper bounds for the asymptotic behavior of the solution of (1.1) as $t\to+\infty$. It is expected that the diffusive nature of the equation implies that the solution goes to zero when $t\to+\infty$.

To obtain our results the key assumptions are the following lower bounds for J:

$$\begin{array}{c} J(x,y)\geq c_1|x-y|^{-(n+2\sigma)}, \quad \text{for} \quad |x-y|>c_2, \\ \text{and} \\ J(x,y)\geq c_1, \quad \text{for } |x-y|\leq c_2. \end{array} \tag{1.2}$$

for certain constants $c_1,c_2>0$ and $\sigma\in(0,1)$. For simplicity we will assume $c_2=1$.

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The main result of this paper reads as follows:

Theorem 1.1. *Let $n \geq 2$, $q \in [1, +\infty)$ and $\sigma \in (0,1)$. Let J be a kernel satisfying (1.2). Then the solution of (1.1) associated to an initial condition $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ decays in $L^q(\mathbb{R}^n)$ with the upper bound*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq C t^{-\frac{n}{(p-2)n+2\sigma}(1-\frac{1}{q})}, \tag{1.3}$$

where the constant C depends on u_0, q, σ and n.

Let us end the introduction with some comments on the previous bibliography. For the linear case, $p = 2$, and for smooth kernels J with compact support, it is proven in [8] that the solution u of the equation (1.1) has the decay estimate

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq C t^{-\frac{n}{2}(1-\frac{1}{q})}$$

for any $q \in [1, \infty)$. Note that this decay rate is the same as the one that holds for solutions of the classical Heat equation. In the case of an equation in convolution form, that is when $J(x,y) = K(x-y)$ with K a nonnegative radial function, not necessarily compactly supported, it is proven in [3] that the solutions of equations with the form (1.1) have the decay estimate

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq C t^{-\frac{n}{2\sigma}(1-\frac{1}{q})},$$

provided the function K has a Fourier transform satisfying the expansion $\hat{K}(\xi) = 1 - A|\xi|^{2\sigma} + o(|\xi|^{2\sigma})$, where $A > 0$ is a constant. In this case the decay estimate is analogous to the one for the $\sigma-$order fractional heat equation, $v_t = -(-\Delta)^\sigma v$, with $\sigma \in (0,1)$. We also note that the convolution form of the equation allows the use of Fourier analysis to obtain this result. However, the use of Fourier analysis is not helpful here due to fact that our operator is not in convolution form. Despite of this difficulty, energy methods can be applied, see [8], [4]. We borrow ideas and techniques from these references. In particular we use Proposition 3.2 of [4] (whose proof is included here for completeness). However we have to point out that in [4] only the linear case, that is, $p = 2$, was treated, while here we deal with (1.1) for any $p \geq 2$. For examples of kernels with exponential decay bounds we refer to [9] and [10].

The case $1 \leq p < 2$ remains open as well as the corresponding estimate for the L^∞-norm.

2. Basic Facts and Preliminaries.

First, we need to introduce fractional Sobolev spaces and its seminorms, we refer to [7] for details. For $\sigma \in (0,1)$ and $r \in [1, \infty)$, $W^{\sigma,r}(\mathbb{R}^n)$ is the fractional Sobolev space of all $L^r(\mathbb{R}^n)$ functions with finite fractional seminorm $[v]_{\sigma,r}$, given by

$$[v]^r_{\sigma,r} = \iint_{\mathbb{R}^{2n}} \frac{|v(x+z) - v(x)|^r}{|z|^{n+r\sigma}} dx\, dz. \tag{2.1}$$

Under these definitions, we have the following fractional Sobolev-type inequality, there exists a constant $C > 0$ such that for each $v \in W^{\sigma,r}(\mathbb{R}^n)$ with $\sigma r < n$, it holds

$$||v||^r_{L^s(\mathbb{R}^n)} \leq C[v]^r_{\sigma,r}, \tag{2.2}$$

where $s = nr/(n - \sigma r)$ (see [7]).

First, we consider a positive smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ with the following properties

$$\operatorname{supp}(\psi) \subset B_1, \qquad \text{and} \qquad \int_{\mathbb{R}^n} \psi(x)dx = 1. \tag{2.3}$$

With the aid of this function, we split a function u into two parts. We will denote the "smooth" part of u as v and the remaining as w. We let

$$v(x,t) := \int_{\mathbb{R}^n} \psi(x-z)u(z,t)dz; \qquad w(x,t) := u(x,t) - v(x,t). \tag{2.4}$$

Sometimes, for simplicity in the notation and where the context is clear, we will write u, v and w as functions depending only of x.

As a first property of this decomposition we have that each L^r norm of the functions v and w is controlled by the corresponding norm of u.

Lemma 2.1. *Let v and w be given by (2.4). For each $r \in (1, +\infty)$, there exists $C = C(r, \psi)$ such that*

$$\|v\|_{L^r(\mathbb{R}^n)} \le C\|u\|_{L^r(\mathbb{R}^n)}, \quad \text{and} \quad \|w\|_{L^r(\mathbb{R}^n)} \le C\|u\|_{L^r(\mathbb{R}^n)}.$$

Proof. We start with v. Denoting $r' = r/(r-1)$ the Hölder conjugate of r and using the definition of v, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |v(x)|^r dx &= \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \psi(x-y)u(y)dy \Big|^r dx \\
&= \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \psi(x-y)^{1/r'} \psi(x-y)^{1/r} u(y)dy \Big|^r dx \\
&\le \int_{\mathbb{R}^n} \Big[\Big(\int_{\mathbb{R}^n} \psi(x-y)dy \Big)^{1/r'} \Big(\int_{\mathbb{R}^n} \psi(x-y)|u(y)|^r dy \Big)^{1/r} \Big]^r dx \\
&= C(r,\psi) \int_{\mathbb{R}^n} |u(y)|^r \int_{\mathbb{R}^n} \psi(x-y)dxdy \\
&\le C(r,\psi) \int_{\mathbb{R}^n} |u(y)|^r dy.
\end{aligned}$$

The inequality for w easily follows immediately from the triangular inequality in L^r. □

Now we state a key result to get the desired estimate on the decay rate.

Proposition 2.2. *Let $n \ge 2$ and let $J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a kernel satisfying (1.2), ψ satisfying (2.3), $\beta \in (0,1)$ and $r > \max\{1, 2\beta\}$. Then, there exists a constant $C > 0$ such that for all $u \in L^r(\mathbb{R}^n)$ and v, w defined in (2.4), we have*

$$[v]^r_{2\beta r^{-1}, r} + \|w\|^r_{L^r(\mathbb{R}^n)} \le C \iint_{\mathbb{R}^{2n}} J(x,y)|u(x) - u(y)|^r dx\, dy. \tag{2.5}$$

The constant C depends on ψ, β, r and n.

Proof. For the estimate concerning w, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |w(x)|^r dx &= \int_{\mathbb{R}^n} |u(x) - v(x)|^r dx \\ &= \int_{\mathbb{R}^n} \Big| u(x) - \int_{\mathbb{R}^n} \psi(x-z)u(z)dz \Big|^r dx \\ &= \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \psi(x-z)(u(x)-u(z))dz \Big|^r dx \\ &= \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \psi(x-z)^{1/r'} \psi(x-z)^{1/r} (u(x)-u(z))dz \Big|^r dx.\end{aligned}$$

Applying Holder's inequality, we get

$$\begin{aligned}&\int_{\mathbb{R}^n} |w(x)|^r dx \\ &\leq \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \psi(x-z)dz \Big)^{r/r'} \Big(\int_{\mathbb{R}^n} \psi(x-z)|u(x)-u(z)|^r dz \Big) dx \\ &\leq C(r, r', \psi) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(x-z)|u(x)-u(z)|^r dz\, dx,\end{aligned}$$

where $r' = r/(r-1)$.

Since ψ is supported in B_1, we have that $\psi(x-z) \leq J(x,z)$ for all $|x-z| \geq 1$ and since J verifies $J(x,z) \geq c_1$ for $|x-z| \leq 1$, there exists a constant C depending only on $|\psi|_\infty$ such that $\psi(x-z) \leq CJ(x,z)$. Then

$$\|w\|^r_{L^r(\mathbb{R}^n)} \leq C \iint_{\mathbb{R}^{2n}} J(x,y)|u(x)-u(y)|^r dx\, dy.$$

Now we deal with the term with v. We split the fractional seminorm as

$$\begin{aligned}[v]^r_{2\beta r^{-1}, r} &= \iint_{|x-y|>1} \frac{|v(x)-v(y)|^r}{|x-y|^{n+2\beta}} dxdy + \iint_{|x-y|\leq 1} \frac{|v(x)-v(y)|^r}{|x-y|^{n+2\beta}} dxdy \\ &=: I_{ext} + I_{int}\end{aligned}$$

and look at these integrals separately. For I_{ext}, using the definition of v we have

$$I_{ext} = \iint_{|x-y|>1} \Big| \int_{\mathbb{R}^n} (u(x-z) - u(y-z))\psi(z)dz \Big|^r |x-y|^{-(n+2\beta)} dxdy.$$

Now, we can look at the measure $\mu(dz) = \psi(z)dz$ as a probability measure (because of (2.3)) and since the function $t \mapsto |t|^r$ is convex in $\mathbb{R}$, we can apply Jensen's inequality on the $dz-$integral in right-hand side of the last expression to obtain

$$I_{ext} \leq \iint_{|x-y|>1} \int_{\mathbb{R}^n} |u(x-z) - u(y-z)|^r \psi(z)dz|x-y|^{-(n+2\beta)} dxdy,$$

which, after an application of Fubini's Theorem, gives

$$I_{ext} \leq \int_{\mathbb{R}^n} \psi(z) \Big(\iint_{|x-y|>1} |u(x-z) - u(y-z)|^r |x-y|^{-(n+2\beta)} dxdy \Big) dz,$$

Then, applying the change $\tilde{x} = x - z, \tilde{y} = y - z$ in the $dxdy$ integral and using (2.3), we conclude

$$\begin{aligned} I_{ext} &\leq \int_{\mathbb{R}^n} \psi(z) \Big(\iint_{|\tilde{x}-\tilde{y}|>1} |u(\tilde{x}) - u(\tilde{y})|^r |\tilde{x} - \tilde{y}|^{-(n+2\beta)} d\tilde{x} d\tilde{y} \Big) dz \\ &= \iint_{|\tilde{x}-\tilde{y}|>1} |u(\tilde{x}) - u(\tilde{y})|^r |\tilde{x} - \tilde{y}|^{-(n+2\beta)} d\tilde{x} d\tilde{y}. \end{aligned}$$

Using this last expression, we obtain from the assumption (1.2) that

$$I_{ext} \leq C \iint_{\mathbb{R}^{2n}} J(x,y) |u(x) - u(y)|^r dx\, dy. \tag{2.6}$$

Now we deal with I_{int}. In this case, using the definition of v, we can write

$$I_{int} = \iint_{|x-y|<1} \Big| \int_{\mathbb{R}^n} u(z)(\psi(x-z) - \psi(y-z)) dz \Big|^r |x-y|^{-(n+2\beta)} dxdy. \tag{2.7}$$

Note that by using (2.3), we have for all $x, y \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} u(x)(\psi(x-z) - \psi(y-z)) dz = u(x) \Big(\int_{\mathbb{R}^n} \psi(x-z) dz - \int_{\mathbb{R}^n} \psi(y-z) dz \Big) = 0,$$

and then

$$\int_{\mathbb{R}^n} u(z)(\psi(x-z) - \psi(y-z)) dz = \int_{\mathbb{R}^n} (u(z) - u(x))(\psi(x-z) - \psi(y-z)) dz.$$

Thus, using this equality into (2.7), we get

$$I_{int} = \iint_{|x-y|<1} \Big| \int_{\mathbb{R}^n} (u(z) - u(x))(\psi(x-z) - \psi(y-z)) dz \Big|^r |x-y|^{-(n+2\beta)} dxdy.$$

However, note that if $|x - z| \geq 2$ in the dz integral, since $|x - y| < 1$ necessarily $|y - z| > 1$. Then, due to the fact that ψ is supported in the unit ball, the contribution of the integrand when $|x - z| \geq 2$ is null in the dz integral. Taking this into account, applying Hölder's inequality into the $dz-$integral, we have

$$\begin{aligned} I_{int} &= \iint_{|x-y|<1} \Big| \int_{|x-z|<2} (u(z) - u(x))(\psi(x-z) - \psi(y-z)) dz \Big|^r \\ &\qquad \times |x-y|^{-(n+2\beta)} dxdy \\ &\leq \iint_{|x-y|<1} \Big(\int_{|x-z|<2} |u(z) - u(x)|^r dz \Big) \\ &\qquad \times \Big(\int_{|x-\tilde{z}|<2} |\psi(x-\tilde{z}) - \psi(y-\tilde{z})|^{r'} d\tilde{z} \Big)^{r/r'} |x-y|^{-(n+2\beta)} dxdy. \end{aligned}$$

By Fubini's Theorem we can write

$$I_{int} = \int_{x \in \mathbb{R}^n} \Big(\int_{|x-z|<2} (u(z) - u(x))^r dz \Big) \Psi(x) dx,$$

where

$$\Psi(x) = \int_{|x-y|<1} \Big(\int_{|x-\tilde{z}|<2} |\psi(x-\tilde{z}) - \psi(y-\tilde{z})|^{r'} d\tilde{z} \Big)^{r/r'} |x-y|^{-(n+2\beta)} dy.$$

Using the regularity of ψ, we have

$$\begin{aligned}\Psi(x) \leq & \int_{|x-y|<1} \Big(\int_{|x-\tilde{z}|<2} ||D\psi||_{\infty}^{r'} |x-y|^{r'} d\tilde{z} \Big)^{r/r'} |x-y|^{-(n+2\beta)} dy \\ \leq & ||D\psi||_{\infty}^{r} |B_2|^{r/r'} \int_{|x-y|<1} |x-y|^{r} |x-y|^{-(n+2\beta)} dy,\end{aligned}$$

and since $r > 2\beta$, we conclude that the last integral is convergent, obtaining

$$\Psi(x) \leq C_{n,\beta,r} ||D\psi||_{\infty}^{r} |B_2|^{r/r'},$$

which leads us to the following estimate for I_{int}

$$I_{int} \leq C \int_{x\in\mathbb{R}^n} \int_{|x-z|<2} |u(z)-u(x)|^r dz dx.$$

From this, it is easy to get

$$I_{int} \leq C \int_{|x-z|\leq 2} \frac{|u(z)-u(x)|^r}{(1+|x-z|)^{n+2\beta}} dz dx,$$

which, by the use of (1.2), let us conclude that

$$I_{int} \leq C \iint_{\mathbb{R}^{2n}} J(x,y) |u(x)-u(y)|^r \, dx \, dy.$$

This last estimate together with (2.6) concludes the proof. □

3. Proof of Theorem 1.1

As mentioned in the introduction, existence and uniqueness of solutions to problem (1.1) follows as in [1]. In fact, the symmetry, boundedness and integrability assumptions over J, allows us to perform a fixed point argument to obtain the following result whose proof is omitted.

Theorem 3.1. *Let $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then, there exists a unique solution $u \in C([0,+\infty), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ of equation (1.1). This solution satisfies $||u(\cdot,t)||_{L^1(\mathbb{R}^n)} \leq ||u_0||_{L^1(\mathbb{R}^n)}$ and $||u(\cdot,t)||_{L^\infty(\mathbb{R}^n)} \leq ||u_0||_{L^\infty(\mathbb{R}^n)}$ for all $t \geq 0$.*

Now, let us introduce the main idea behind the energy methods. To clarify the exposition, let us perform these computations in the local case and next see how we can adapt them to our nonlocal problem with the help of Proposition 2.2. Let us describe briefly how the energy method can be applied to obtain decay estimates for local problems. Let us begin with the simpler case of the estimate for solutions to the $p-$Lapacian evolution equation in L^2-norm. Let u be a solution to

$$\partial_t u = \Delta_p u.$$

If we multiply the equation by u and integrate in $\mathbb{R}^n$, we obtain

$$\partial_t \int_{\mathbb{R}^n} \frac{1}{2} u^2(x,t) dx = - \int_{\mathbb{R}^n} |\nabla u(x,t)|^p \, dx.$$

Now we use Sobolev's inequality

$$\int_{\mathbb{R}^n} |\nabla u(x,t)|^p \, dx \geq C \left(\int_{\mathbb{R}^n} |u(x,t)|^{p^*} \, dx \right)^{p/p^*}$$

with $p^* = pn/(n-p)$ to obtain

$$\partial_t \int_{\mathbb{R}^n} u^2(x,t)\,dx \le -C \left(\int_{\mathbb{R}^n} |u(x,t)|^{p^*}\,dx \right)^{p/p^*}.$$

If we use interpolation and that $\|u(\cdot,t)\|_{L^1(\mathbb{R}^n)} \le C(u_0)$ for any $t>0$, we have

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le \|u(\cdot,t)\|^{\alpha}_{L^1(\mathbb{R}^n)} \|u(\cdot,t)\|^{1-\alpha}_{L^{p^*}(\mathbb{R}^n)} \le C\|u(\cdot,t)\|^{1-\alpha}_{L^{p^*}(\mathbb{R}^n)}$$

with α determined by

$$\frac{1}{2} = \alpha + \frac{1-\alpha}{p^*}, \qquad \text{that is,} \qquad \alpha = \left(\frac{1}{2} - \frac{1}{p^*}\right)\frac{p^*}{(p^*-1)}.$$

Hence we get

$$\partial_t \int_{\mathbb{R}^n} u^2(x,t)\,dx \le -C \left(\int_{\mathbb{R}^n} u^2(x,t)\,dx \right)^{\frac{1}{1-\alpha}}$$

from where the decay estimate

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le C\,t^{-\frac{1}{2}\left(\frac{n}{n(p-2)+2}\right)}, \qquad t>0,$$

follows. To obtain a decay bound for $\|u(\cdot,t)\|_{L^q(\mathbb{R}^n)}$ we can use the same idea multiplying by u^{q-1} at the beginning.

Now we are ready to proceed with the proof of our main result.

Proof of Theorem 1.1. The symmetry assumption on J allows us to mimic this idea and use an energy approach in order to get Theorem 1.1. Roughly speaking, this assumption allows us to "integrate by parts" equation (1.1). For $q=1$ the proof is finished by Theorem 3.1. For $q>1$ we multiply the equation by $q|u|^{q-2}u$ and integrate, obtaining the identity

$$\begin{aligned}\partial_t \int_{\mathbb{R}^n} |u(x)|^q dx = -\frac{q}{2} \iint_{\mathbb{R}^{2n}} & J(x,y)|u(y)-u(x)|^{p-2}(u(y)-u(x)) \\ & \times \left(|u(y)|^{q-2}u(y) - |u(x)|^{q-2}u(x)\right) dy\,dx,\end{aligned} \tag{3.1}$$

where we omitted the dependence on t of the function u for simplicity.

Now we recall the following inequality (whose proof is straightforward): let $q>1$ and $a,b \ne 0$. Then, there exists a constant C depending only on q, such that

$$(a-b)(|a|^{q-2}a - |b|^{q-2}b) \ge C|a-b|^q.$$

Hence, using this inequality into (3.1), we conclude

$$\partial_t ||u||^q_{L^q(\mathbb{R}^n)} \le -C \iint_{\mathbb{R}^{2n}} J(x,y)|u(y)-u(x)|^{p-2+q} dydx =: -CE(u). \tag{3.2}$$

Note that we get that the L^q-norm of u is decreasing in t. At this point we would like to use Sobolev's inequality, that is not available due to the lack of regularizing effect of our nonlocal operator. Instead we will use Proposition 2.2 that involves a good control of the smooth part v (but we have to take care of the rough part w).

By the definition of v and w in (2.4), we have

$$||u||^q_{L^q(\mathbb{R}^n)} \le 2^{q-1}\left(||v||^q_{L^q(\mathbb{R}^n)} + ||w||^q_{L^q(\mathbb{R}^n)}\right). \tag{3.3}$$

Now we note that v belongs to L^p for all p. Hence, we can interpolate, obtaining

$$||v||^q_{L^q(\mathbb{R}^n)} \leq ||v||^{q\theta}_{L^s(\mathbb{R}^n)} ||v||^{q(1-\theta)}_{L^1(\mathbb{R}^n)},$$

with

$$s = \frac{n(p+q-2)}{n-2\sigma}$$

where θ is given by

$$\frac{1}{q} = \frac{\theta}{s} + (1-\theta), \qquad \text{that is} \qquad \theta = \frac{s(q-1)}{q(s-1)}.$$

Recalling $||v||_{L^1(\mathbb{R}^n)} \leq ||u(\cdot,t)||_{L^1(\mathbb{R}^n)} \leq ||u_0||_{L^1(\mathbb{R}^n)}$ and the Sobolev-type inequality (2.2), we obtain

$$||v||^q_{L^q(\mathbb{R}^n)} \leq C[v]^{q\theta}_{\tilde{\sigma},p+q-2}, \tag{3.4}$$

where $\tilde{\sigma} = 2\sigma(p+q-2)^{-1}$ and the constant C depends on u_0, q, σ and n.

Concerning w we can also interpolate and obtain

$$||w||^q_{L^q(\mathbb{R}^n)} \leq ||w||^{q\gamma}_{L^{p+q-2}(\mathbb{R}^n)} ||w||^{q(1-\gamma)}_{L^1(\mathbb{R}^n)},$$

with γ given by

$$\frac{1}{q} = \frac{\gamma}{p+q-2} + (1-\gamma), \qquad \text{that is} \qquad \gamma = \frac{(p+q-2)(q-1)}{q(p+q-3)}.$$

Note that we are using that $p > 2$ here. Now we use that

$$||w||_{L^1(\mathbb{R}^n)} \leq ||u(\cdot,t)||_{L^1(\mathbb{R}^n)} \leq ||u_0||_{L^1(\mathbb{R}^n)}$$

to get

$$||w||^q_{L^q(\mathbb{R}^n)} \leq C||w||^{q\gamma}_{L^{p+q-2}(\mathbb{R}^n)}, \tag{3.5}$$

with C depending on u_0, q, σ and n.

From (3.3), (3.4) and (3.5) we obtain

$$||u||^q_{L^q(\mathbb{R}^n)} \leq C[v]^{q\theta}_{\tilde{\sigma},p+q-2} + C||w||^{q\gamma}_{L^{p+q-2}(\mathbb{R}^n)}. \tag{3.6}$$

Now we use Proposition 2.2, with $r = p+q-2$ and $\beta = \sigma$, to obtain

$$||v||^{q\theta}_{L^s(\mathbb{R}^n)} \leq C(E(u))^{\frac{q\theta}{p+q-2}} \qquad \text{and} \qquad ||w||^{q\gamma}_{L^{p+q-2}(\mathbb{R}^n)} \leq C(E(u))^{\frac{q\gamma}{p+q-2}}$$

and we conclude that

$$||u||^q_{L^q(\mathbb{R}^n)} \leq C(E(u))^{\frac{q\theta}{p+q-2}} + C(E(u))^{\frac{q\gamma}{p+q-2}}$$

that is,

$$H^{-1}(||u||^q_{L^q(\mathbb{R}^n)}) \leq E(u)$$

with $H(z) = Cz^{\frac{q\theta}{p+q-2}} + Cz^{\frac{q\gamma}{p+q-2}}$. Since $||u||^q_{L^q(\mathbb{R}^n)}(t) \leq ||u_0||^q_{L^q(\mathbb{R}^n)}$ (recall that the L^q-norm of the solution decreases) and $\frac{q\theta}{p+q-2} < \frac{q\gamma}{p+q-2}$ we have

$$H^{-1}(||u||^q_{L^q(\mathbb{R}^n)}) \geq C(||u||^q_{L^q(\mathbb{R}^n)})^{\frac{p+q-2}{q\theta}}.$$

Then, from (3.2), we obtain

$$\partial_t ||u(\cdot,t)||^q_{L^q(\mathbb{R}^n)} \leq -CE(u) \leq -CH^{-1}(||u||^q_{L^q(\mathbb{R}^n)}) \leq -C(||u||^q_{L^q(\mathbb{R}^n)})^{\frac{p+q-2}{q\theta}}$$

from where it follows that

$$||u(\cdot,t)||^q_{L^q(\mathbb{R}^n)} \leq Ct^{-\frac{q\theta}{p+q-2-q\theta}},$$

that is,

$$||u(\cdot,t)||^q_{L^q(\mathbb{R}^n)} \leq Ct^{-\frac{(q-1)n}{q((p-2)n+2\sigma)}},$$

as we wanted to show.

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