Blow-up sets and Fujita type curves for a degenerate parabolic system with nonlinear boundary conditions

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Abstract

We study nonnegative solutions of two porous medium equations with nonlinear coupled boundary conditions and nonnegative nontrivial compactly supported initial data. We describe in terms of the different parameters appearing in the problem when solutions blow up in a finite time and when they exist globally in time. We find three regions, bounded by two curves. One of them is a Fujita type curve. In the first region every solution is global, in the second every nontrivial solution blows up, and in the last one both behaviours coexist. In the blow-up case we find the blow-up rates and the blow-up sets. In particular, we prove that under certain conditions the blow-up sets of the two components of the system are different. This is the first known example of a nontrivially coupled parabolic system showing such a behaviour.

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1 Introduction.

In this paper we deal with the following degenerate parabolic system,

$$\begin{cases} u_t = (u^m)_{xx}, \\ v_t = (v^n)_{xx}, \end{cases} \qquad x > 0,\ 0 < t < T, \tag{1.1}$$

where $m, n > 1$. Thus, we are dealing with two porous medium equations. These equations are complemented with nonlinear coupled boundary conditions,

$$\begin{cases} -(u^m)_x(0,t) = v^p(0,t), \\ -(v^n)_x(0,t) = u^q(0,t), \end{cases} \qquad 0 < t < T, \tag{1.2}$$

where $p, q > 0$. We consider initial data,

$$\begin{cases} u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \end{cases} \qquad x > 0, \tag{1.3}$$

which are assumed to be continuous, nonnegative and compactly supported in $\mathbb{R}_+$.

Parabolic systems like (1.1)–(1.3) appear in several branches of applied mathematics. They have been used to model, for example, chemical reactions, heat transfer or population dynamics. See [22] and the references therein.

As we shall see, under certain conditions the solutions of this problem can become unbounded in a finite time. This phenomenon is known as blow-up, and has been observed for several scalar equations since the pioneering work of Fujita [9]. For further references see the review by Levine, [20]. Blow-up may also happen for systems, see, among others, [1], [5], [7], [8], [23], [24], [27]. Our main interest here will be to determine under which conditions there are solutions of (1.1)–(1.3) that blow up and, in the blow-up case, the speed at which blow-up takes place and the localization of blow-up points in terms of the parameters p, q, m, n.

As a precedent we have the work of Galaktionov and Levine [13], where they study the following scalar problem for the porous medium equation

$$\begin{cases} u_t = (u^m)_{xx} & x > 0,\ 0 < t < T, \\ -(u^m)_x = u^p & x = 0,\ 0 < t < T, \\ u(0,x) = u_0(x) & x > 0, \end{cases} \tag{1.4}$$

where $m > 1$. They show that if $0 < p \le p_0 = \frac{m+1}{2}$, then for arbitrary initial data the solution is global in time, while for $p > \frac{m+1}{2}$ there are solutions with finite time blow-up. Thus, p_0 is the *critical global existence exponent.* Moreover, they prove that $p_c = m + 1$ is a *critical exponent of Fujita type.* By definition, this means that p_c has the following properties:

(i) if $p_0 < p \le p_c$, then $u(x,t) \not\equiv 0$ blows up in a finite time for all nontrivial u_0;

(ii) if $p > p_c$, then $u(x,t)$ is global in time for "small" $u_0 \not\equiv 0$.

For system (1.1)–(1.3), instead of critical exponents there are *critical curves*, one for global existence and another of Fujita type. This is the content of our first theorem.

To state our results we need to introduce the following numbers. Let

$$\alpha_1 = \frac{2p+n+1}{(m+1)(n+1)-4pq}, \qquad \alpha_2 = \frac{2q+m+1}{(m+1)(n+1)-4pq}, \tag{1.5}$$

and

$$\beta_1 = \frac{p(m-1-2q)+(n+1)m}{(m+1)(n+1)-4pq}, \qquad \beta_2 = \frac{q(n-1-2p)+(m+1)n}{(m+1)(n+1)-4pq}. \tag{1.6}$$

Theorem 1.1 I) *If* $pq \le \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right)$, *every nonnegative solution of* (1.1)–(1.3) *is global in time.*
II) *If* $pq > \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right)$, *with* $\alpha_1 + \beta_1 \le 0$ *or* $\alpha_2 + \beta_2 \le 0$, *then every nonnegative, nontrivial solution blows up in finite time.*
III) *If* $pq > \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right)$, *with* $\alpha_1 + \beta_1 > 0$ *and* $\alpha_2 + \beta_2 > 0$, *there exist nonnegative solutions with blow-up and nonnegative solutions that are global.*

Therefore, the critical global existence curve is

$$pq = \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right),$$

and the critical Fujita type curve is

$$\min\{\alpha_1 + \beta_1, \ \alpha_2 + \beta_2\} = 0.$$

The values α_1, α_2, β_1, β_2 are the exponents of self-similar solutions to problem (1.1)–(1.2). Such self-similar solutions are studied in Section 2, and play an important role in the proof of Theorem 1.1.

Let us observe that if we take $m = n = 1$, then the critical curves coincide with those found in [5] for the case in which the equations are linear.

We also remark that Theorem 1.1 contains as a particular case the result of [13]. Indeed, the solution u to (1.4) produces a solution (u,u) to problem (1.1)–(1.3) with $n = m$, $q = p$ and $v_0(x) = u_0(x)$. In this case the critical curves collapse to the critical exponents for the scalar problem, p_0, p_c.

In Figure 1 we represent in terms of p and q, for fixed m and n, the critical curves and the regions I, II and III bounded by them. These regions correspond to the behaviours I, II, III in Theorem 1.1.

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Figure 1. Global existence and Fujita type curves.

In the case blow-up occurs, we would like to know at which speeds the functions u and v approach infinity at time T. These are called the *blow-up rates.* We obtain that, if the solution is increasing in time, then the blow-up rates are the same ones as that of the self-similar solutions.

Theorem 1.2 *Let (u,v) be a solution of* (1.1)–(1.3) *increasing in time ($u_t, v_t \geq 0$) and with a finite blow-up time T. Then there exist positive constants c, C such that,*

$$c(T-t)^{\alpha_1} \leq \|u(\cdot,t)\|_\infty \leq C(T-t)^{\alpha_1}, \qquad c(T-t)^{\alpha_2} \leq \|v(\cdot,t)\|_\infty \leq C(T-t)^{\alpha_2}.$$

Let us remind that in the blow-up case $pq > \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right)$. Hence $\alpha_1 < 0$ and $\alpha_2 < 0$.

Again, if we take $m = n = 1$, we recover the blow-up rates for the linear case found in [5], [27].

With the aid of the blow-up rates we can study the *blow-up sets.* The blow-up set, $B(u)$, of a function u that blows up at time T is defined by

$$B(u) := \{x \in \mathrm{I\!R}_+ : \limsup_{t\to T^-} u(x,t) = \infty\}.$$

Regarding the blow-up sets of problem (1.1)–(1.3) we prove the following theorem.

Theorem 1.3 *Let (u,v) be a solution of* (1.1)–(1.3) *blowing up at time T. Let $B(u)$, $B(v)$ be the corresponding blow-up sets. If*

$$\begin{array}{l} c(T-t)^{\alpha_1} \leq u(0,t) \leq C(T-t)^{\alpha_1}, \\ c(T-t)^{\alpha_2} \leq v(0,t) \leq C(T-t)^{\alpha_2}, \end{array} \tag{1.7}$$

then the blow-up sets are determined by the signs of β_1 *and* β_2 *as follows:*
(a) *if* $\beta_1 > 0$, $B(u) = \{0\}$*;*
(b) *if* $\beta_1 = 0$, $B(u)$ *is a bounded interval containing* 0*;*
(c) *if* $\beta_1 < 0$, $B(u) = \mathbb{R}_+$*;*
(d) *if* $\beta_2 > 0$, $B(v) = \{0\}$*;*
(e) *if* $\beta_2 = 0$, $B(v)$ *is a bounded interval containing* 0*;*
(f) *if* $\beta_2 < 0$, $B(v) = \mathbb{R}_+$.

Remark 1.1 With the help of Theorem 1.2 we can prove that there are nontrivial initial data such that (1.7) holds. The blowing up self-similar solutions of Section 2 also satisfy (1.7).

As a corollary of Theorems 1.2 and 1.3 we obtain new results for the scalar problem (1.4). Namely, the blow-up rate for increasing solutions is $(T-t)^{1/(m+1-2p)}$ and the blow-up set is $\mathbb{R}_+$ if $\frac{m+1}{2} < p < m$, a bounded interval if $p = m$, and $x = 0$ if $p > m$. These problems have not been treated before anywhere else.

In Figure 2 we represent, for m and n fixed, the curves separating the different behaviours of the blow-up sets.

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Figure 2. Blow-up sets regions.

We see that there are nine different regions including six possible behaviours:
(i) both functions blow up only at $x = 0$;
(ii) one function blows up only at $x = 0$ and the other one blows up in a bounded interval;

(iii) one function blows up only at $x = 0$ and the other one blows up in $\mathbb{R}_+$;
(iv) both functions blow up in bounded intervals;
(v) one function blows up in a bounded interval and the other one in $\mathbb{R}_+$;
(vi) both functions blow up in $\mathbb{R}_+$.

The nonlinearity of the diffusion operator plays also an important role. Indeed, if u and v are solutions of the heat equation, that is, if $m = n = 1$, then the blow-up sets of u and v coincide and consist of a single point, $x = 0$, [5].

It is not surprising that the blow-up sets may be a single point, an interval, or $\mathbb{R}_+$, as these types of blow-up sets also occur for the scalar equation

$$u_t = \Delta u^m + u^p.$$

This has been studied in [10], [11], [12], [26], [25] and references therein for spatial dimension $d = 1$ and in [3] for $d > 1$, $p = m$.

What is a novelty is that the blow-up sets of u and v might be different. As far as we know, this is the first example of a nontrivially coupled parabolic system showing different blow-up sets for its components.

Organization of the paper. In Section 2 we study the existence of self-similar solutions of different types. In Section 3 we give some results concerning existence, comparison, monotonicity and uniqueness. In Section 4 we find the critical curves (Theorem 1.1). Section 5 is devoted to the computation of the blow-up rates in the case of solutions which are monotonic in time (Theorem 1.2). Finally, in Section 6 we describe the blow-up sets (Theorem 1.3).

2 Self-similar solutions.

In this section we consider different kinds of self-similar solutions of problem (1.1)–(1.2). We have the following results.

Theorem 2.1 *Let*

$$\begin{array}{ll} u(x,t) = (T-t)^{\alpha_1} f(\xi), & \xi = x/(T-t)^{\beta_1}, \\ v(x,t) = (T-t)^{\alpha_2} g(\zeta), & \zeta = x/(T-t)^{\beta_2}. \end{array} \tag{2.1}$$

If

$$pq > \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right), \tag{2.2}$$

there is a self-similar solution of problem (1.1)–(1.2) *blowing up in a finite time* $T > 0$ *of the form* (2.1). *Moreover, the support of* f *is* $\mathbb{R}_+$ *if* $\beta_1 > 0$ *and a compact set if* $\beta_1 \le 0$. *Analogously, the support of* g *is* $\mathbb{R}_+$ *if* $\beta_2 > 0$ *and a compact set if* $\beta_2 \le 0$.

Remark 2.1 The blow-up set of u (resp. v) depends on the sign of β_1 (resp. β_2). It consists only of the origin if $\beta_1 > 0$ (resp. $\beta_2 > 0$), it is $\mathbb{R}_+$ when $\beta_1 < 0$ ($\beta_2 < 0$), and it is a finite interval $[0, a)$ when $\beta_1 = 0$ ($\beta_2 = 0$). See Section 6 for the details.

Theorem 2.2 *Let*

$$\begin{aligned} u(x,t) &= t^{\alpha_1} f(\xi), \qquad & \xi &= x/t^{\beta_1}, \\ v(x,t) &= t^{\alpha_2} g(\zeta), \qquad & \zeta &= x/t^{\beta_2}. \end{aligned} \tag{2.3}$$

(a) *If*

$$pq < \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right), \tag{2.4}$$

then there exist functions f and g, positive in $\mathbb{R}_+$, such that (u, v) given in (2.3) *is a self-similar solution of problem* (1.1)–(1.2) *global in time. These solutions have $\alpha_1, \alpha_2 > 0$, and thus their initial data are identically zero. They have $\beta_1, \beta_2 < 0$.*

(b) *If*

$$\alpha_1 + \beta_1 > 0 \qquad \textit{and} \qquad \alpha_2 + \beta_2 > 0,$$

then there exist functions f and g, compactly supported in $\mathbb{R}_+$, such that (u, v) given in (2.3) *is a self-similar solution of problem* (1.1)–(1.2) *global in time. These solutions have $\alpha_1, \alpha_2 < 0$, and thus they decay to zero as $t \to \infty$. They have $\beta_1, \beta_2 > 0$, and hence their supports expand as time increases.*

Remark 2.2 If $\alpha_1 + \beta_1 \leq 0$ or $\alpha_2 + \beta_2 \leq 0$, there are no profiles $f, g \in L^1(\mathbb{R}_+)$ such that (u, v) given by (2.3) is a solution. Indeed,

$$\int_0^\infty u(x,t)\,dx = t^{\alpha_1+\beta_1} \int_0^\infty f(\xi)\,d\xi.$$

Then, if $\alpha_1 + \beta_1 \leq 0$, the mass of u would not increase, a contradiction.

Theorem 2.3 *Let*

$$\begin{aligned} u(x,t) &= e^{\alpha_1 t} f(\xi), \qquad & \xi &= x/e^{\beta_1 t}, \\ v(x,t) &= e^{\alpha_2 t} g(\zeta), \qquad & \zeta &= x/e^{\beta_2 t}. \end{aligned} \tag{2.5}$$

If

$$pq = \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right), \tag{2.6}$$

then, for any $\alpha_1 > 0$, there is a self-similar solution of problem (1.1)–(1.2) *global in time of the form* (2.5), *where*

$$\alpha_2 = \left(\frac{m+1}{2p}\right)\alpha_1, \qquad \beta_1 = \left(\frac{m-1}{2}\right)\alpha_1, \qquad \beta_2 = \left(\frac{n-1}{2}\right)\left(\frac{m+1}{2p}\right)\alpha_1. \tag{2.7}$$

Moreover, the supports of f and g are compact.

Remark 2.3 The solutions are in principle only weak. However, if they are positive everywhere, they are also classical.

In order to prove these theorems we will use the following result of Gilding and Peletier, [14], [16], [17]:

Theorem 2.4 *Let* $a, b, V \in \mathbb{R}$ *and* $U \geq 0$. *For fixed* a *and* b, *let* $\mathcal{S}_A$ *denote the set of values* (U, V) *such that there exists a weak, nonnegative, compactly supported solution* f *of*

$$(f^m)''(\eta) + a\eta f'(\eta) = bf(\eta) \qquad \text{for } 0 < \eta < \infty, \tag{2.8}$$

$$f(0) = U, \tag{2.9}$$

$$(f^m)'(0) = V, \tag{2.10}$$

and let $\mathcal{S}_B$ *denote the set of values* (U, V) *for which there exists a bounded, positive, classical solution* f *of* (2.8)–(2.10).

(a) *If* $b < 0$ *and* $2a + b < 0$ *then* $\mathcal{S}_A = \{(0, 0)\}$ *and* $\mathcal{S}_B = \emptyset$.

(b) *If* $b < 0$ *and* $2a + b = 0$ *then* $\mathcal{S}_A = \{(0, V) : \ 0 \leq V < \infty\}$ *and* $\mathcal{S}_B = \emptyset$.

(c) *If* $b \leq 0$ *and* $2a + b > 0$ *then there exists a unique* V_* *such that* $\mathcal{S}_A = \{(U, U^{(m+1)/2}V_*) : 0 \leq U < \infty\}$ *and* $\mathcal{S}_B = \{(U, V) : \ 0 \leq U < \infty, \quad U^{(m+1)/2}V_* < V < \infty\}$, *where* $V_* > 0$ *if* $a + b < 0$, $V_* = 0$ *if* $a + b = 0$ *and* $V_* < 0$ *if* $a + b > 0$.

(d) *If* $b > 0$ *and* $a \geq 0$ *then there exists a unique* $V_* < 0$ *such that* $\mathcal{S}_A = \{(U, U^{(m+1)/2}V_*) : 0 \leq U < \infty\}$ *and* $\mathcal{S}_B = \emptyset$.

(e) *If* $b > 0$ *and* $a < 0$, *or* $b = 0$ *and* $a \leq 0$, *then* $\mathcal{S}_A = \{(0, 0)\}$ *and there exists a unique* V_* *such that* $\mathcal{S}_B = \{(U, U^{(m+1)/2}V_*) : \ 0 \leq U < \infty\}$, *where* $V_* < 0$ *if* $b > 0$ *and* $V_* = 0$ *if* $b = 0$.

Moreover, for each $(U, V) \in \mathcal{S}_A \cup \mathcal{S}_B$ *there exists at most one weak solution of* (2.8)–(2.10).

Remark 2.4 In the case where $a = \left(\frac{m-1}{2}\right) b > 0$ we have $V_* = -1$. This is a consequence of the existence of a self-similar solution of exponential form for the scalar problem (1.4) with $p = \frac{m+1}{2}$, see [13].

Proof of Theorem 2.1: We are considering solutions of the form (2.1). Imposing that the two porous medium equations (1.1) are fulfilled we get the following relations for the exponents:

$$\begin{aligned} \alpha_1 - 1 &= m\alpha_1 - 2\beta_1, \\ \alpha_2 - 1 &= n\alpha_2 - 2\beta_2. \end{aligned} \tag{2.11}$$

On the other hand, the boundary conditions (1.2) imply that

$$\begin{aligned} m\alpha_1 - \beta_1 &= \alpha_2 p, \\ n\alpha_2 - \beta_2 &= \alpha_1 q. \end{aligned} \tag{2.12}$$

Solving the linear system (2.11)–(2.12) we get that α_1, α_2, β_1, β_2 are given by (1.5) and (1.6). Therefore, $\alpha_1, \alpha_2 < 0$ if and only if $pq > \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right)$. On the other hand, the profiles must satisfy

$$\begin{aligned} (f^m)''(\xi) - \beta_1 \xi f'(\xi) &= -\alpha_1 f(\xi), \\ (g^n)''(\zeta) - \beta_2 \zeta g'(\zeta) &= -\alpha_2 g(\zeta), \end{aligned} \tag{2.13}$$

plus the boundary conditions

$$\begin{aligned} -(f^m)'(0) &= g^p(0), \\ -(g^n)'(0) &= f^q(0). \end{aligned} \tag{2.14}$$

Then f and g satisfy (2.8) with coefficients $a_1 = -\beta_1$, $b_1 = -\alpha_1$ and $a_2 = -\beta_2$, $b_2 = -\alpha_2$ respectively. Thus, Theorem 2.4 parts (d) and (e) says that there is a one-parameter family (parameter U_1) of solutions f of (2.8) satisfying

$$f(0) = U_1, \qquad (f^m)'(0) = U_1^{(m+1)/2} V_{*1},$$

where $V_{*1} < 0$ is a constant. The profile f has compact support if $\beta_1 \le 0$ and is positive in $\mathbb{R}_+$ if $\beta_1 > 0$. Analogously, there is a one-parameter family (parameter U_2) of solutions g of (2.8) satisfying

$$g(0) = U_2, \qquad (g^n)'(0) = U_2^{(n+1)/2} V_{*2},$$

where $V_{*2} < 0$ is a constant. The profile g has compact support if $\beta_2 \le 0$ and is positive in $\mathbb{R}_+$ if $\beta_2 > 0$. We choose U_1 and U_2 such that the boundary conditions (2.14) are fulfilled, that is,

$$\left\{ \begin{array}{l} -U_1^{(m+1)/2} V_{*1} = U_2^p, \\ -U_2^{(n+1)/2} V_{*2} = U_1^q. \end{array} \right.$$

Taking logarithms, this is equivalent to

$$\left(\begin{array}{cc} \frac{m+1}{2} & -p \\ -q & \frac{n+1}{2} \end{array} \right) \left(\begin{array}{c} \ln U_1 \\ \ln U_2 \end{array} \right) = \left(\begin{array}{c} -\ln |V_{*1}| \\ -\ln |V_{*2}| \end{array} \right). \tag{2.15}$$

As $pq \neq \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right)$, this system has a unique solution. □

Proof of Theorem 2.2: We are considering solutions of the form (2.3). Imposing that the equations (1.1) and the boundary conditions (1.2) are fulfilled we get again that the exponents should satisfy the relations (2.11)–(2.12). Hence they are given by (1.5)–(1.6). Moreover, the boundary conditions for the profiles are again given by (2.14). However, the equations for the profiles are now different:

$$\begin{aligned} (f^m)''(\xi) + \beta_1 \xi f'(\xi) &= \alpha_1 f(\xi), \\ (g^n)''(\zeta) + \beta_2 \zeta g'(\zeta) &= \alpha_2 g(\zeta). \end{aligned} \tag{2.16}$$

Thus f and g satisfy (2.8) with coefficients $a_1 = \beta_1$, $b_1 = \alpha_1$ and $a_2 = \beta_2$, $b_2 = \alpha_2$ respectively.

(a) If $\alpha_1, \alpha_2 > 0$, that is, if (2.4) holds, then $\beta_1, \beta_2 < 0$ (see Figure 2). Therefore, applying Theorem 2.4 part (d) as in the proof of Theorem 2.1, and taking the solutions U_1, U_2 of (2.15) as values for the parameters, we obtain that there exist positive profiles f and g solving (2.16) and satisfying (2.14).

(b) If $\alpha_1, \alpha_2 < 0$, $\alpha_1 + \beta_1 > 0$ and $\alpha_2 + \beta_2 > 0$, we can apply Theorem 2.4 part (c) as in the proof of Theorem 2.1, and taking the solutions of (2.15) as values for the parameters, we obtain that there exist compactly supported profiles f and g solving (2.16) and satisfying the boundary conditions (2.14). $\Box$

Proof of Theorem 2.3: We are considering solutions of the form (2.5). Though the boundary conditions (1.2) impose again (2.12), now equations (1.1) impose different relations for the exponents. Namely,

$$\begin{aligned} \alpha_1 &= m\alpha_1 - 2\beta_1, \\ \alpha_2 &= n\alpha_2 - 2\beta_2. \end{aligned} \tag{2.17}$$

Thus,

$$\begin{pmatrix} m+1 & -2p \\ -2q & n+1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.18}$$

There is a nontrivial solution (α_1, α_2) of (2.18) if and only if $pq = \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right)$. In this case, α_2, β_1, β_2 are related to α_1 by (2.7).

The boundary conditions for the profiles are again given by (2.14), while the equations for the profiles are given by (2.16). If $\alpha_1 > 0$, then $\alpha_2, \beta_1, \beta_2 > 0$ and $\beta_1 = \left(\frac{m-1}{2}\right)\alpha_1$, $\beta_2 = \left(\frac{n-1}{2}\right)\alpha_2$. Hence, using Remark 2.4 we have solutions of (2.16) with $V_{*1} = V_{*2} = -1$. Choosing one of the solutions of (2.15) with right-hand side zero (again we are using that $4pq = (m+1)(n+1)$) we obtain that there exist compactly supported profiles f and g solving (2.16) and satisfying (2.14). $\Box$

3 Existence and uniqueness.

First, we state a theorem that guarantees the existence of a solution. It can be obtained using a standard monotonicity argument following ideas from [2].

Theorem 3.1 *Given continuous, compactly supported initial data,* (u_0, v_0)*, there exists a local in time continuous weak solution of* (1.1)–(1.3)*. Moreover, if the initial data are smooth and compatible in the sense that*

$$-(u_0^m)_x(0) = v_0^p(0), \qquad -(v_0^n)_x(0) = u_0^q(0),$$

then the solution has continuous time derivatives down to $t = 0$.

Proof: Let us consider the Neumann problem

$$\begin{cases} w_t = (w^r)_{xx} & x > 0,\ 0 < t < \tau, \\ -(w^r)_x(0,t) = h(t) & 0 < t < \tau, \\ w(x,0) = w_0(x) & x > 0, \end{cases} \tag{3.1}$$

with $r > 1$. We define the operator $M_p : C([0,\tau]) \to C([0,\tau])$ as $M_p(h)(t) = w^p(0,t)$, where $w(x,t)$ is the unique solution of (3.1) with $r = m$ and initial condition $w_0(x) = u_0(x)$. In an analogous way, we define $N_q : C([0,\tau]) \to C([0,\tau])$ as $N_q(h)(t) = w^q(0,t)$, where $w(x,t)$ is the unique solution of (3.1) with $r = n$ and initial condition $w_0(x) = v_0(x)$.

It has been proved in [4] that M_p and N_q are continuous and compact. Moreover, they are order preserving.

Now, let $A(h) = N_q \circ M_p(h)$. Using the method of monotone iterations one can prove that there exists $\tau > 0$ such that A has a fixed point in $C([0,\tau])$. This fixed point provides us with a continuous weak solution of (1.1)–(1.3) up to time τ.

In order to obtain the regularity of the solution with compatible initial data we only have to observe that the solution of (3.1) is regular if $-(w_0^r)_x(0) = h(0)$, see [19]. $\square$

Remark 3.1 If the initial data are compactly supported, the solution (u,v) also has compact support as long as it exists.

Remark 3.2 If the initial data are nontrivial, we can assume that they satisfy $u_0(0) > 0$, $v_0(0) > 0$. If not, $u(0,t)$ and $v(0,t)$ eventually become positive (compare with a Barenblatt solution of the corresponding equation).

Next, we define what we call a subsolution and a supersolution for (1.1)–(1.2).

Definition 3.1 The pair $(\underline{u},\underline{v})$ is a subsolution of (1.1)–(1.2) if it satisfies

$$\begin{cases} u_t \le (u^m)_{xx}, \\ v_t \le (v^n)_{xx}, \end{cases} \quad x > 0,\ 0 < t < T, \tag{3.2}$$

$$\begin{cases} -(u^m)_x(0,t) \le v^p(0,t), \\ -(v^n)_x(0,t) \le u^q(0,t), \end{cases} \quad 0 < t < T. \tag{3.3}$$

Definition 3.2 We call $(\overline{u},\overline{v})$ a supersolution of (1.1)–(1.2) if it satisfies (3.2)–(3.3) with the opposite inequalities.

With these definitions of super and subsolutions we can state a comparison lemma.

Lemma 3.1 *Let* $(\overline{u},\overline{v})$ *be a supersolution and* $(\underline{u},\underline{v})$ *be a subsolution. If*

$$\underline{u}(x,0) \le \overline{u}(x,0), \qquad \underline{v}(x,0) \le \overline{v}(x,0),$$

with

$$\underline{u}(0,0) < \overline{u}(0,0) \qquad \textit{and} \qquad \underline{v}(0,0) < \overline{v}(0,0),$$

then

$$\underline{u}(x,t) \le \overline{u}(x,t) \qquad \text{and} \qquad \underline{v}(x,t) \le \overline{v}(x,t),$$

as long as both pairs of functions exist.

Proof: It is standard; therefore we omit the details. Assume that the result is false. Let t_0 be the maximum time such that

$$\underline{u}(x,t) \le \overline{u}(x,t) \qquad \text{and} \qquad \underline{v}(x,t) \le \overline{v}(x,t),$$

up to time t_0. This time t_0 must be positive by continuity. At that time t_0 we must have that $\underline{u}(0,t_0) = \overline{u}(0,t_0)$ or $\underline{v}(0,t_0) = \overline{v}(0,t_0)$. Let us assume that $\underline{u}(0,t_0) = \overline{u}(0,t_0)$. Now the result follows by an application of Hopf's Lemma. Indeed, $\overline{u} - \underline{u}$ satisfies a uniformly parabolic equation in a neighbourhood of $x = 0$, attains a minimum at $(0,t_0)$, and the corresponding flux is greater or equal than zero, a contradiction. □

Now we state a lemma that guarantees that for certain initial data, the solution of (1.1)–(1.3) increases with time.

Lemma 3.2 *Let* (u_0, v_0) *be an initial data for* (1.1)–(1.3) *such that* u_0 *and* v_0 *are smooth, satisfy the compatibility condition at the boundary and* $(u_0^m)_{xx} \ge 0$, $(v_0^n)_{xx} \ge 0$. *Then* u *and* v *increase with time, i.e.,* $u_t \ge 0$, $v_t \ge 0$.

Proof: Set $z = u_t$, $w = v_t$. Then, as the solutions are smooth (Theorem 3.1), we can differentiate to obtain that (z,w) is a solution of

$$\begin{cases} z_t = (mu^{m-1}z)_{xx}, \\ w_t = (nv^{n-1}w)_{xx}, \end{cases} \tag{3.4}$$

$$\begin{cases} -(mu^{m-1}z)_x(0,t) = pv^{p-1}w(0,t), \\ -(nv^{n-1}w)_x(0,t) = qu^{q-1}z(0,t), \end{cases} \tag{3.5}$$

with an initial data satisfying

$$z(x,0) \ge 0, \qquad w(x,0) \ge 0.$$

To conclude the proof we apply the maximum principle. Due to the degeneration of the equations, this cannot be done directly. A standard regularization procedure is needed, see [13] for the details. □

Next, we deal with the problem of uniqueness *versus* non uniqueness for (1.1)–(1.3) in the case of vanishing initial data ($u_0 \equiv v_0 \equiv 0$).

Theorem 3.2 (a) *Let* $pq < (\frac{m+1}{2})(\frac{n+1}{2})$. *Then there exists a nontrivial solution with zero initial data that becomes positive at* $x = 0$ *instantaneously. Then there is no uniqueness for problem* (1.1)–(1.3) *with zero initial data.*
(b) *Let* $pq \ge (\frac{m+1}{2})(\frac{n+1}{2})$. *The solution of* (1.1)–(1.3) *with zero initial data is unique.*

Proof: **(a)** The self-similar solutions constructed in Theorem 2.2 become positive at $x = 0$ instantaneously,

(b) We can construct small supersolutions with the aid of the self-similar ones of exponential form that we found in Theorem 2.3.

First, choose $\tilde{q} \le q$ such that $p\tilde{q} = (\frac{m+1}{2})(\frac{n+1}{2})$, and let

$$\overline{u}(x,t) = e^{\alpha_1(t+\tau)} f\left(x/e^{\beta_1(t+\tau)}\right), \qquad \overline{v}(x,t) = e^{\alpha_2(t+\tau)} g\left(x/e^{\beta_2(t+\tau)}\right),$$

where $\alpha_1 > 0$ is arbitrary and α_2, β_1 and β_2 are given by (2.7). Now we observe that this pair $(\overline{u}, \overline{v})$ is a supersolution of (1.1)–(1.3) as long as $\overline{u}(0,t) \le 1$. By the comparison Lemma 3.1 we obtain that every solution that has initial data identically zero satisfies

$$\overline{u}(x,t) \ge u(x,t) \qquad \text{and} \qquad \overline{v}(x,t) \ge v(x,t).$$

As $(\overline{u}, \overline{v})$ can be chosen as small as we want (by choosing τ negative and large enough) we conclude that $u \equiv v \equiv 0$. □

4 Blow-up versus global existence.

We devote this section to the proof of Theorem 1.1. We borrow some ideas from [13]. However, the fact that we are dealing with a system instead of a single equation forces us to develop a significantly different proof. We will organize the proof in several lemmas.

Our first lemma proves part I) of Theorem 1.1.

Lemma 4.1 *If* $pq \le (\frac{m+1}{2})(\frac{n+1}{2})$ *every nonnegative solution of* (1.1)–(1.3) *is global.*

Proof: It is enough to construct global supersolutions with initial data as large as needed. We achieve this with the aid of the self-similar solutions of exponential form that we found in Theorem 2.3.

First we choose $\tilde{q} \ge q$ such that $p\tilde{q} = (\frac{m+1}{2})(\frac{n+1}{2})$, and we let

$$\overline{u}(x,t) = e^{\alpha_1(t+\tau)} f\left(x/e^{\beta_1(t+\tau)}\right), \qquad \overline{v}(x,t) = e^{\alpha_2(t+\tau)} g\left(x/e^{\beta_2(t+\tau)}\right),$$

where $\alpha_1 > 0$ is arbitrary and α_2, β_1 and β_2 are given by (2.7). Now we observe that this pair $(\overline{u}, \overline{v})$ is a supersolution of (1.1)–(1.3) as long as $\overline{u}(0,t) \ge 1$. This can be done by choosing τ large enough. This also allows to assume that $\overline{u}(x,0) \ge u_0(x)$ and $\overline{v}(x,0) \ge v_0(x)$. Then, by the comparison Lemma 3.1, we obtain that every solution is global. □

Now we construct a pair of subsolutions with finite time blow-up.

Lemma 4.2 *Let* $pq > \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right)$*, then there exists a pair of compactly supported functions* θ_1*,* θ_2*, such that*

$$\begin{array}{ll} \underline{u}(x,t) = (T-t)^{\alpha_1}\theta_1(\xi), & \xi = \frac{x}{(T-t)^{\beta_1}}, \\ \underline{v}(x,t) = (T-t)^{\alpha_2}\theta_2(\zeta), & \zeta = \frac{x}{(T-t)^{\beta_2}}, \end{array}$$

is a subsolution of (1.1)–(1.2).

Proof: To satisfy (3.2) and (3.3) we need that

$$\begin{cases} (\theta_1^m)''(\xi) \geq -\alpha_1\theta_1(\xi) + \beta_1\xi\theta_1'(\xi), \\ (\theta_2^n)''(\zeta) \geq -\alpha_2\theta_2(\zeta) + \beta_2\zeta\theta_2'(\zeta), \end{cases}$$

$$\begin{cases} -(\theta_1^m)'(0) \leq \theta_2^p(0), \\ -(\theta_2^n)'(0) \leq \theta_1^q(0). \end{cases}$$

We choose,

$$\theta_1(\xi) = A_1(a_1-\xi)_+^{\frac{1}{m-1}}.$$

Inserting this in the first equation we get

$$\frac{m}{(m-1)^2}A_1^{m-1} \geq -\alpha_1(a_1-\xi)_+ - \frac{\beta_1}{m-1}\xi \qquad \text{for } 0 \leq \xi \leq a_1.$$

Hence, it is enough to impose

$$\frac{m}{(m-1)^2}A_1^{m-1} \geq -\alpha_1 a_1 + \frac{|\beta_1|}{m-1}a_1,$$

that is

$$C_1A_1^{m-1} \geq a_1. \tag{4.1}$$

Analogously we choose

$$\theta_2(\zeta) = A_2(a_2-\zeta)_+^{\frac{1}{n-1}},$$

and from the second equation we get

$$C_2A_2^{n-1} \geq a_2. \tag{4.2}$$

The boundary conditions impose

$$\begin{array}{c} \frac{m}{m-1}A_1^m a_1^{\frac{1}{m-1}} \geq A_2^p a_2^{\frac{p}{n-1}}, \\ \frac{n}{n-1}A_2^n a_2^{\frac{1}{n-1}} \geq A_1^q a_1^{\frac{q}{m-1}}. \end{array} \tag{4.3}$$

Let

$$b_1 = A_1a_1^{\frac{1}{m-1}}, \qquad b_2 = A_2a_2^{\frac{1}{n-1}}.$$

Then, conditions (4.3) become

$$\begin{array}{l} \frac{m}{m-1}A_1^{m-1}b_1 \geq b_2^p, \\ \frac{n}{n-1}A_2^{n-1}b_2 \geq b_1^q. \end{array} \tag{4.4}$$

We fix $b_1 = b_2 = 1$, and then take A_1, A_2 large enough (and thus a_1 and a_2 small) to satisfy (4.1), (4.2) and (4.4). □

Corollary 4.1 *Let* $pq > \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right)$. *Then there exist solutions of* (1.1)–(1.3) *that blow up in a finite time.*

Proof: We only have to apply Lemma 3.1 to obtain that every solution (u, v) that begins above the subsolutions provided by Lemma 4.2 has finite time blow-up. □

Lemma 4.3 *Let* $pq > \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right)$. *If* $\alpha_1 + \beta_1 \leq 0$ *or* $\alpha_2 + \beta_2 \leq 0$, *then every nontrivial solution of* (1.1)–(1.3) *blows up in finite time.*

Proof: We consider the case $\alpha_1 + \beta_1 \leq 0$. The case $\alpha_2 + \beta_2 \leq 0$ is handled in a completely analogous form.

Assume that there exists a global nonnegative nontrivial solution (u, v) of (1.1)–(1.3). We make the following change of variables

$$\begin{array}{l} \varphi(\xi,\tau) = (1+t)^{-\alpha_1}u(\xi(1+t)^{\beta_1}, t), \\ \psi(\zeta,\tau) = (1+t)^{-\alpha_2}v(\zeta(1+t)^{\beta_2}, t), \\ \tau = \log(1+t). \end{array}$$

These functions satisfy

$$\left\{ \begin{array}{l} \varphi_\tau = (\varphi^m)_{\xi\xi} + \beta_1\xi\varphi_\xi - \alpha_1\varphi, \\ \psi_\tau = (\psi^n)_{\zeta\zeta} + \beta_2\zeta\psi_\zeta - \alpha_2\psi, \end{array} \right. \tag{4.5}$$

$$\left\{ \begin{array}{l} -(\varphi^m)_\xi(0,\tau) = \psi^p(0,\tau), \\ -(\psi^n)_\zeta(0,\tau) = \varphi^q(0,\tau). \end{array} \right. \tag{4.6}$$

As (u, v) is by hypothesis global, the same is true for (φ, ψ). We will construct a solution $(\hat\varphi, \hat\psi)$ to system (4.5)–(4.6) increasing with time, with initial data $(\hat\varphi_0, \hat\psi_0)$ such that $\hat\varphi_0(\xi) \leq u(\xi, 0)$, $\hat\psi_0(\zeta) \leq v(\zeta, 0)$. We will prove that $(\hat\varphi, \hat\psi)$ cannot exist globally, thus contradicting the global existence of (u, v).

In order to achieve our goal we take initial data $(\hat\phi_0, \hat\psi_0)$ satisfying

$$\left\{ \begin{array}{l} (\hat\varphi_0^m)_{\xi\xi} + \beta_1\xi(\hat\varphi_0)_\xi - \alpha_1\hat\varphi_0 \geq 0, \\ (\hat\psi_0^n)_{\zeta\zeta} + \beta_2\zeta(\hat\psi_0)_\zeta - \alpha_2\hat\psi_0 \geq 0, \end{array} \right.$$

and the compatibility condition,

$$\left\{ \begin{array}{l} -(\hat\varphi_0^m)_\xi(0) = \hat\psi^p(0), \\ -(\hat\psi_0^n)_\zeta(0) = \hat\varphi^q(0). \end{array} \right.$$

Hence, arguing as in Lemma 3.2 we have that $\hat{\varphi}_\tau, \hat{\psi}_\tau \geq 0$.

We set

$$\hat{\varphi}_0(\xi) = h(\xi + b),$$

where h is the Barenblatt profile

$$h(\xi) = a_m(c - \xi^2)_+^{1/(m-1)}.$$

Then we have

$$\begin{aligned}(\hat{\varphi}_0^m)_{\xi\xi} + \beta_1\xi(\hat{\varphi}_0)_\xi - \alpha_1\hat{\varphi}_0 = \\ -\tfrac{1}{m+1}bh_\xi(\xi+b) + \left(\beta_1 - \tfrac{1}{m+1}\right)\xi h_\xi(\xi+b) + \left(-\alpha_1 - \tfrac{1}{m+1}\right)h(\xi+b).\end{aligned}$$

The last expression is nonnegative if $\beta_1 - \frac{1}{m+1} \leq 0$ and $-\alpha_1 - \frac{1}{m+1} \geq 0$. But these two conditions are equivalent to $\alpha_1 + \beta_1 \leq 0$.

Now we take $\tilde{\alpha}_2, \tilde{\beta}_2 > 0$ such that $\tilde{\alpha}_2 \geq \alpha_2$, $\tilde{\beta}_2 \geq \beta_2$. We take as $\hat{\psi}_0$ a solution to

$$(\hat{\psi}_0^n)'' = -\tilde{\beta}_2\zeta\hat{\psi}_0' + \tilde{\alpha}_2\hat{\psi}_0.$$

There is a one-parameter family of solutions to this equation (see Theorem 2.4), with $\hat{\psi}_0 \geq 0$, $\hat{\psi}_0' \leq 0$. Hence,

$$(\hat{\psi}_0^n)'' \geq -\beta_2\zeta\hat{\psi}_0' + \alpha_2\hat{\psi}_0.$$

Moreover,

$$\hat{\psi}_0(0) = U, \qquad (\hat{\psi}_0^n)'(0) = U^{(n+1)/2}V_*,$$

where $V_* < 0$ is a constant and U is the free parameter.

We still have to control the boundary conditions. In order to do this we choose the constants c, b and U conveniently. They have to satisfy

$$\begin{cases} \frac{2ma_m}{m-1}b(c-b^2)^{1/(m-1)} = U^p, \qquad b \in (0, c^{1/2}), \\ -V_*U^{(n+1)/2} = a_m^q(c-b^2)^{q/(m-1)}. \end{cases}$$

Thus we choose

$$U = c_1 b^{\frac{2}{2pq-(n+1)}}, \qquad c = b^2 + c_2 U^{\frac{(n+1)(m-1)}{2q}}.$$

Taking b small enough the initial data $(\hat{\varphi}_0, \hat{\psi}_0)$ are below $(u(\xi,0), v(\zeta,0))$. This can be done, as u_0, v_0 can be assumed to be positive at the origin.

To conclude the proof we will prove that $(\hat{\varphi}, \hat{\psi})$ converges to a self-similar profile that does not exist in this range of parameters.

Lemma 4.4 *Either*

$$\lim_{\tau\to\infty} \hat{\varphi}(\xi,\tau) = \overline{\varphi}(\xi) < \infty \qquad \forall \xi > 0, \tag{4.7}$$

or

$$\lim_{\tau\to\infty} \hat{\psi}(\zeta,\tau) = \overline{\psi}(\zeta) < \infty \qquad \forall \zeta > 0. \tag{4.8}$$

Proof: It is clear that $\hat{\varphi}_\xi, \hat{\psi}_\zeta \le 0$. Let us suppose that

$$\begin{array}{ll} \hat{\varphi}(\xi,\tau) \to \infty & \text{uniformly in } (0,\xi_0), \\ \hat{\psi}(\zeta,\tau) \to \infty & \text{uniformly in } (0,\zeta_0). \end{array}$$

In the original variables $(\hat{u},\hat{v})$, we have that for any $M > 0$ there is a value t_0 such that

$$\begin{array}{ll} (1+t_0)^{\alpha_1} M \le \hat{u}(x,t_0) & \text{for } 0 < \frac{x}{(1+t_0)^{\beta_1}} < \xi_0, \\ (1+t_0)^{\alpha_2} M \le \hat{v}(x,t_0) & \text{for } 0 < \frac{x}{(1+t_0)^{\beta_2}} < \zeta_0. \end{array}$$

Now we will check that under these conditions we can put one of the blowing up subsolutions constructed in Lemma 4.2 below these data. This would lead to a contradiction as $(\hat{u},\hat{v})$ is global. In order to be able to do this we need

$$\left\{ \begin{array}{l} (1+t_0)^{\alpha_1} M \ge A_1 a_1^{\frac{1}{m-1}} T^{\alpha_1}, \\ \xi_0 (1+t_0)^{\beta_1} \ge a_1 T^{\beta_1}. \end{array} \right. \tag{4.9}$$

The first equation says that the height at $x = 0$ of $\hat{u}$ is bigger than that of $\underline{u}$ and the second says that the support of $\hat{u}$ is bigger than the support of $\underline{u}$. Imposing analogous conditions for $\hat{v}$ and $\underline{v}$ we get

$$\left\{ \begin{array}{l} (1+t_0)^{\alpha_2} M \ge A_2 a_2^{\frac{1}{n-1}} T^{\alpha_2}, \\ \zeta_0 (1+t_0)^{\beta_2} \ge a_2 T^{\beta_2}. \end{array} \right. \tag{4.10}$$

Taking $T = 1 + t_0$, then a_1, a_2 small enough and A_1, A_2 big enough and then M large, the four conditions (4.9)–(4.10) are fulfilled. □

End of the proof of Lemma 4.3: Let us assume that (4.7) holds. Using standard arguments, see [13], we may pass to the limit to obtain that

$$(\overline{\varphi}^m)_{\xi\xi} + \beta_1 \xi \overline{\varphi}_\xi - \alpha_1 \overline{\varphi} = 0. \tag{4.11}$$

Let $z = \overline{\varphi}^m$. Then

$$z_{\xi\xi} + \frac{\beta_1}{n} \xi z^{\frac{1-n}{n}} z_\xi \le 0.$$

Hence, in $[0,\xi_0]$, $z \ge c > 0$,

$$z_{\xi\xi} \le C z_\xi.$$

We conclude that z and therefore $\overline{\varphi}$ cannot be unbounded at $\xi = 0$. In particular, $0 < \overline{\varphi}(0) \le C$. Then, considering the regularity of $\hat{\varphi}$ in the region where $\overline{\varphi} > 0$, we can pass to the limit in the boundary condition for $(\hat{\varphi}^m)_\xi$ to obtain that

$$-(\overline{\varphi}^m)_\xi(0) = \overline{\psi}^p(0). \tag{4.12}$$

However, as $\alpha_1 + \beta_1 \le 0$, problem (4.11)–(4.12) does not have a nontrivial solution, see Theorem 2.4.

If (4.8) holds, we can proceed as before to obtain that $\overline{\psi}(0) < \infty$. Thus we can pass to the limit in the boundary condition for $\hat{\psi}$, obtaining

$$-(\overline{\psi}^{n})_{\zeta}(0) = \overline{\varphi}^{q}(0).$$

As $\overline{\varphi}(0) \geq \overline{\varphi}(\xi)$, this implies that $\overline{\varphi}$ is finite for every $\xi \geq 0$. We get the same contradiction as before. □

5 Blow-up rates.

In this section we prove Theorem 1.2. We are assuming that the solution (u,v) of (1.1)–(1.3) increases with time and blows up in a finite time T. In particular we have that $pq > \left(\frac{m+1}{2}\right)\left(\frac{n+1}{2}\right)$.

Proof of Theorem 1.2: Let us observe that as $u_t \geq 0$ then $(u^m)_{xx} \geq 0$ and hence $\|u(\cdot,t)\|_\infty = u(0,t)$. In the same way we obtain that $\|v(\cdot,t)\|_\infty = v(0,t)$.

Now let us define

$$M(t) = u(0,t) = \max u(\cdot,t) \quad \text{ and } \quad N(t) = v(0,t) = \max v(\cdot,t).$$

Following ideas from [18] we set, for $t < T$,

$$\begin{array}{ll} \varphi_M(y,s) = \frac{1}{M(t)} u(ay, bs+t), & y > 0,\ -t/b < s < 0, \\ \psi_N(y,s) = \frac{1}{N(t)} v(cy, ds+t), & y > 0,\ -t/d < s < 0. \end{array}$$

This pair of functions (φ_M, ψ_N) satisfies

$$0 \leq \varphi_M, \psi_N \leq 1, \qquad \varphi_M(0,0) = \psi_N(0,0) = 1, \qquad (\varphi_M)_s, (\psi_N)_s \geq 0. \tag{5.1}$$

Choosing

$$a = \frac{M^m}{N^p}, \qquad b = \frac{M^{m+1}}{N^{2p}}, \qquad c = \frac{N^n}{M^q}, \qquad d = \frac{N^{n+1}}{M^{2q}},$$

we have that φ_M and ψ_N are solutions of

$$\left\{ \begin{array}{l} (\varphi_M)_s = (\varphi_M^m)_{yy}, \\ (\psi_N)_s = (\psi_N^n)_{yy}, \end{array} \right. \tag{5.2}$$

$$\left\{ \begin{array}{l} -(\varphi_M^m)_x(0,s) = \psi_N^p(0,s), \\ -(\psi_N^n)_x(0,s) = \varphi_M^q(0,s). \end{array} \right. \tag{5.3}$$

We observe that there exists a number s_* such that that φ_M and ψ_N are well defined for every $(y,s) \in A = \{y > 0,\ s_* < s < 0\}$, if M and N are large enough. Indeed, if not, assume that $-t/b \to 0$. Hence φ_M is a solution of the porous medium equation,

$$(\varphi_M)_s = (\varphi_M^m)_{yy},$$

defined in a small interval of time, $(-t/b, 0)$. The flux is bounded by one,

$$-(\varphi_M^m)(0,s) = \psi_N^p(0,s) \le 1,$$

and the initial data is small,

$$\varphi_M(-t/b, y) = \frac{1}{M} u_0(ay) \le \varepsilon,$$

if M is large enough. But this is a contradiction with the fact that $\varphi_M(0,0) = 1$.

We claim that there exist positive constants such that for every M and N large enough it holds

$$\left\{ \begin{array}{l} c \le (\varphi_M)_s(0,0) \le C, \\ c \le (\psi_N)_s(0,0) \le C. \end{array} \right. \tag{5.4}$$

As we will see later on, the blow-up rates follow from these inequalities. To prove the claim, assume that it were false. ¿From the results for bounded solutions of porous medium equations of [6] Section 6, we have that every sequence, $(\varphi_{M_j}, \psi_{N_j})$ is equicontinuous. Therefore, passing to a subsequence if necessary, we have that

$$\varphi_{M_j} \to \varphi, \qquad \psi_{N_j} \to \psi,$$

uniformly on compact sets of $\{y \ge 0,\ s_* \le s \le 0\}$. These functions φ, ψ are continuous and satisfy $\varphi(0,0) = \psi(0,0) = 1$. Hence, there exists a neighbourhood of $(0,0)$, $U \subset A$, such that $\varphi, \psi > 1/2$ in U. As we have uniform convergence over $\overline{U}$ (we can assume that $\overline{U}$ is compact) for j large enough we have that $1/4 \le \varphi_{M_j}, \psi_{N_j} \le 1$. Therefore, $\varphi_{M_j}, \psi_{N_j}$ are solutions of uniformly parabolic equations in $\overline{U}$. Thus, from the fact that they are uniformly bounded we obtain, using well known Schauder estimates, [21],

$$\|\varphi_{M_j}\|_{C^{2+\alpha,1+\alpha/2}}, \|\psi_{N_j}\|_{C^{2+\alpha,1+\alpha/2}} \le C \qquad \text{in } \overline{U}.$$

We conclude that for M large enough

$$(\varphi_M)_s(0,0) \le C, \qquad (\psi_N)_s(0,0) \le C,$$

proving the first half of our claim.

It remains to prove that $(\varphi_M)_s(0,0) \not\to 0$, $(\psi_N)_s(0,0) \not\to 0$. Assume that one of them, let us say the first one, were false. That is, there is a sequence such that $(\varphi_{M_j})_s(0,0) \to 0$. We proceed just as before to obtain that

$$\varphi_{M_j} \to \varphi, \qquad \psi_{N_j} \to \psi,$$

and

$$\|\varphi_{M_j}\|_{C^{2+\alpha,1+\alpha/2}}, \|\psi_{N_j}\|_{C^{2+\alpha,1+\alpha/2}} \le C \qquad \text{in } \overline{U}.$$

Using that $C^{2+\alpha,1+\alpha/2}$ is compactly included in $C^{2+\beta,1+\beta/2}$, $\beta < \alpha$, we can conclude, refining the sequence if necessary, that

$$\varphi_s(0,0) = 0.$$

But we observe that φ is a weak solution of

$$\begin{cases} \varphi_s = (\varphi^m)_{yy}, \\ -(\varphi^m)_y(0,s) = \psi^p(0,s). \end{cases}$$

Then $w = \varphi_s \geq 0$ satisfies,

$$\begin{cases} w_s = (m\varphi^{m-1}w)_{yy}, \\ -(m\varphi^{m-1}w)_y(0,s) = p\psi^{p-1}\psi_s(0,s) \geq 0. \end{cases}$$

Hence w has a minimum at $(0,0)$ and by Hopf's Lemma, which can be applied whenever $\varphi > 0$, we can conclude that $w \equiv 0$, that is, φ does not depend on s. Hence, $\varphi = \varphi(y)$ is a solution of

$$\begin{cases} 0 = (\varphi^m)_{yy}, \\ -(\varphi^m)_y(0) = \psi^p(0,0) = 1, \end{cases}$$

that satisfies $0 \leq \varphi \leq 1$. This is a contradiction which proves claim (5.4).

Now we will obtain the blow-up rates from the claim. Writing the claim in terms of M and N,

$$c \leq \frac{M^m}{N^{2p}} M'(t) \leq C, \qquad c \leq \frac{N^n}{M^{2q}} N'(t) \leq C.$$

This is equivalent to

$$cN^{2p} \leq M^m M' \leq CN^{2p}, \qquad cM^{2q} \leq N^n N' \leq CM^{2q}.$$

Calling $Z = N^{n+1}$ and $W = M^{m+1}$ we obtain

$$\begin{aligned} cZ^{\frac{2p}{n+1}} &\leq W' \leq CZ^{\frac{2p}{n+1}}, \\ cW^{\frac{2q}{m+1}} &\leq Z' \leq CW^{\frac{2q}{m+1}}. \end{aligned} \tag{5.5}$$

¿From these equations we get

$$CcW^{\frac{2q}{m+1}} Z^{\frac{2p}{n+1}} \leq cZ^{\frac{2p}{n+1}} Z',$$

$$-CcW^{\frac{2q}{m+1}} Z^{\frac{2p}{n+1}} \leq -CW^{\frac{2q}{m+1}} W'.$$

Adding and integrating, we get

$$c_4 W^{\frac{2q}{m+1}+1} \leq c_2 W^{\frac{2q}{m+1}+1} + c_3 \leq c_1 Z^{\frac{2p}{n+1}+1}.$$

Hence

$$Z \geq c_5 W^{\gamma_1}, \qquad \text{where } \gamma_1 = \frac{(2q+m+1)(n+1)}{(m+1)(2p+n+1)}. \tag{5.6}$$

Then

$$W' \geq c_6 W^{\gamma_2}, \qquad \gamma_2 = \frac{2p(2q+m+1)}{(m+1)(2p+n+1)}.$$

As $\gamma_2 > 1$ (this is equivalent to $4pq > (m+1)(n+1)$), we can integrate on (t, T) to obtain

$$W(t) \leq C(T-t)^{\gamma_3}, \qquad \gamma_3 = \frac{(m+1)(2p+n+1)}{(m+1)(2p+n+1) - 2p(2q+m+1)}.$$

Now we recall that $M^{m+1} = W$ and then

$$M(t) \leq C(T-t)^{\alpha_1}.$$

By (5.5) and (5.6) we have that

$$Z' \leq c_7 Z^{\gamma_4}, \qquad \gamma_4 = \frac{2q(2p+n+1)}{(n+1)(2q+m+1)},$$

and hence

$$N(t) \geq c(T-t)^{\alpha_2}.$$

Proceeding in a similar way with the other inequalities we conclude the proof of the theorem. □

6 Blow-up sets.

In this section we prove Theorem 1.3, describing the blow-up sets for u and v in terms of the parameters p, q, m, n. To this end, we use the asymptotic behaviour that we have found in Section 5 and a result from [15].

In [15] the authors study the blow-up set for a porous medium equation,

$$w_t = (w^b)_{xx}, \tag{6.1}$$

with a boundary condition of the type

$$w(0,t) = \psi(t), \tag{6.2}$$

where $\psi(t) \nearrow +\infty$ as $t \nearrow T$, with a nonnegative initial data

$$w(x,0) = w_0(x). \tag{6.3}$$

First they prove that the blow-up set for w is an interval, possibly unbounded, containing $\{x = 0\}$. Next they characterize the blow-up set in terms of the behaviour of ψ close to $t = T$. For the particular case in which $\psi(t) = \psi_0(T-t)^{-a}$, where ψ_0 is a constant, they prove the following result.

Theorem 6.1 *Let w be a solution of* (6.1)–(6.3). *Then,*
(a) *if $a < \frac{1}{b-1}$, the blow-up set of w is $B(w) = \{0\}$;*
(b) *if $a = \frac{1}{b-1}$, the blow-up set of w is an interval. Moreover,*

$$[0, \nu\psi_0^{\frac{(b-1)}{2}}) \subseteq B(w) \subseteq [0, \nu\psi_0^{\frac{(b-1)}{2}}],$$

where

$$\nu = \left(\frac{2b(b+1)}{(b-1)}\right)^{1/2};$$

(c) *if $a > \frac{1}{b-1}$, then $B(w) = \mathbb{R}_+$.*

To prove Theorem 1.3 we only have to apply Theorem 6.1.

Proof of Theorem 1.3: Let $b = m$, $a = -\alpha$. Let w_1 (resp. w_2) be the solution of (6.1)–(6.3) with $\psi_0 = c$ (resp. $\psi_0 = C$) and initial data $u_0(x)$. By a comparison argument we have that

$$w_1(x,t) \le u(x,t) \le w_2(x,t).$$

Hence

$$B(w_1) \subseteq B(u) \subseteq B(w_2).$$

Therefore, if $-\alpha_1 < \frac{1}{m-1}$, the blow-up set for u is a single point, $B(u) = \{0\}$; if $-\alpha_1 = \frac{1}{m-1}$, it is an interval (and we have the bounds $[0, \nu c^{(m-1)/2}) \subseteq B(u) \subseteq [0, \nu C^{(m-1)/2}]$); and if $-\alpha_1 > \frac{1}{m-1}$ then $B(u) = \mathbb{R}_+$. It only remains to observe that the sign of β_1 is the same as the sign of $\alpha_1 + \frac{1}{m-1}$.

Analogously we can find the blow-up set of v. □

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