ASYMPTOTICS OF THE SPECTRAL FUNCTION FOR THE STEKLOV PROBLEM IN A FAMILY OF SETS WITH FRACTAL BOUNDARIES

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Abstract. In this paper we find the asymtotic behaviour of the spectral counting function for the Steklov problem in a family of self similar domains with fractal boundaries. Using renewal theory, we show that the main term in the asymptotics depends on the Minkowski dimension of the boundary. Also, we compute explicitly a three term expansion for a family of self similar sets, and a two term asymptotic expansion for a family of non self similar sets.

1. Introduction.

In this paper we deal with the spectral counting function for the Steklov eigenvalue problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = \lambda u & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Let us define the counting function $N(\lambda, \Omega)$ for problem (1.1) as the number of non zero eigenvalues less than a given λ.

The asymptotic behaviour of $N(\lambda, \Omega)$ was studied by L. Sandgren [14], who showed that for a domain with smooth boundary,

$$N(\lambda, \Omega) = C_N |\partial\Omega|_{N-1} \lambda^{N-1} + o(\lambda^{N-1})$$

as $\lambda \to \infty$.

Our interest here is to find the asymptotic behaviour for domains Ω with fractal boundary.

This problem is motivated for the similar one for the eigenvalues of the Dirichlet laplacian, extensively studied for almost 100 years. We refer the reader to [5], [7] and the references given there for this problem. The classical result for the Dirichlet laplacian on sufficiently smooth domains with some geometric assumptions is the Weyl's formula

$$N(\lambda, \Omega) = C_N \lambda^{N/2} + C_{N-1} |\partial\Omega|_{N-1} \lambda^{(N-1)/2} + o(\lambda^{(N-1)/2})$$

as $\lambda \to \infty$. When the set Ω has a fractal boundary, the best known result is due to Lapidus [7], who proved

$$N(\lambda, \Omega) = C_N \lambda^{N/2} + O(\lambda^{d/2}) \tag{1.2}$$

Key words and phrases. Eigenvalue problems, fractal boundaries, Steklov.

2000 *Mathematics Subject Classification.* 35P20, 35P15, 35P05.

The first author is supported by CONICET and Univ. de San Andres. The second author is supported by Univ. de Buenos Aires grant TX047, ANPCyT PICT No. 03-00000-00137, and Fundación Antorchas.

where d is the interior Minkowski dimension of $\partial\Omega$ (see the definition below). There are many examples where the second term is known, see [4], [9]. For a short proof of (1.2) and other examples see [12].

Here we are interested in domains with fractal boundary. Let us consider a domain Ω of the form $\Omega = \left(\bigcup_{n=1}^{\infty} \bigcup_{\{i_1,\dots,i_n\}} G_{i_1,\dots,i_n}\right) \bigcup G$, where G is a bounded open set with smooth boundary. Here the indices i_k takes the values 1 to M and $G_{i_1,\dots,i_n} = R_{i_1} \cdots R_{i_n}(G)$, with R_{i_j} a similitude with coefficient r_i. We call a mapping R in ^{n} a similitude with coefficient r if it changes the euclidean distances by factor r. In other words, a similitude can be represented as a composition of a homothety, a translation and an orthonormal transformation. As the relative position of the components of Ω is unimportant for our task as long as they are disjoint, we may use different similitudes R_k at different steps, requiring only that all of them has the same coefficient r_k. We must impose the conditions

$$\sum_{j=1}^{M} (r_j)^N < 1 < \sum_{j=1}^{M} (r_j)^{N-1}. \tag{1.3}$$

The first inequality guarantees that we can place all the sets $R_{i_1,\dots,i_n} G$ in ^{N} without overlaps and implies that Ω has a finite volume. We need the second inequality to conclude that the only positive real root d of the equation

$$\sum_{j=1}^{M} (r_j)^d - 1 = 0 \tag{1.4}$$

belongs to the interval $(N-1, N)$.

Removing the second inequality in equation (1.3), we obtain that the only positive root d of the equation (1.4) belongs to the interval $(0, N)$. We will denote it D_s, the "similarity dimension" of $\partial\Omega$.

Next we introduce a geometrical tool to measure the boundary of Ω. We define the interior Minkowski dimension d of $\partial\Omega$ as

$$\begin{aligned} d &= \inf\{h \geq 0 : \lim_{\varepsilon\to 0^+} \varepsilon^{-(N-h)} |\partial\Omega_\varepsilon \cap \Omega|_N = 0\} \\ &= \sup\{h \geq 0 : \lim_{\varepsilon\to 0^+} \varepsilon^{-(N-h)} |\partial\Omega_\varepsilon \cap \Omega|_N = +\infty\}, \end{aligned}$$

where $A_\varepsilon = \{x \in \ : dist(x, A) < \varepsilon\}$. Moreover, we define the interior lower and upper Minkowski content (in dimension d) of $\partial\Omega$ as

$$M_*(\partial\Omega, d) = \liminf_{\varepsilon\to 0^+} \varepsilon^{-(N-h)} |\partial\Omega_\varepsilon \cap \Omega|_N,$$

$$M^*(\partial\Omega, d) = \limsup_{\varepsilon\to 0^+} \varepsilon^{-(N-h)} |\partial\Omega_\varepsilon \cap \Omega|_N.$$

When $M_*(\partial\Omega, d) = M^*(\partial\Omega, d)$, we say that $\partial\Omega$ is internally measurable with interior Minkowski content $M(\partial\Omega, d)$. We stress the fact that the N-dimensional Lebesgue measure of $\partial\Omega$ can be positive and $d < N$. In other words, the interior Minkowski dimension can be smaller than the topological dimension. However, it is always greater (or equal) than $N-1$.

If $\partial\Omega$ is a self similar set and $D_s \geq N-1$, the similarity dimension D_s is the Hausdorff dimension of $\partial\Omega$ and agrees with the Minkowski (or box) dimension and with the interior Minkowski dimension, see [3]. However, when $D_s < N-1$, it is easy to see that the Hausdorff, interior Minkowski and box dimension are all equal to $N-1$. We remark that the interior Minkowski dimension remains unchanged with a rearrangement of the sets, see [2]. For a deeper discussion of the Minkowski dimension we refer the reader to [10] and [11].

We will use the following definition:

Definition 1.1. *A finite set of positive real numbers* $\{a_1, \cdots, a_M\}$ *is called arithmetic if all the ratios* a_j/a_k, *with* $j, k = 1, \cdots, M$ *are rational numbers. The maximal number* a *such that* $a_j/a \in$ *is called the span of* $\{a_1, \cdots, a_M\}$. *If at least one ratio* a_j/a_k *is irrational, then the set is called non-arithmetic.*

Now we can state our main result.

Theorem 1.1. *Let* G *be an open bounded set in* ^{N} *with smooth boundary* ∂G, *and* $\Omega = \left(\bigcup_{n=1}^{\infty} \bigcup_{\{i_1,\ldots,i_n\}} G_{i_1,\ldots,i_n}\right) \bigcup G$, $G_{i_1,\ldots,i_n} = R_{i_1} \cdots R_{i_n}(G)$, *with* R_{i_j} *a similitude of coefficients* r_i *such that* (1.3) *holds. Let* d *be the Minkowski dimension of* Ω, *with* $N \geq d > N-1$. *Then*

1. *If the set* $\{\ln(r_1), \cdots, \ln(r_M)\}$ *is arithmetic we have the following asymptotic development:*
$$N(\lambda, \Omega) = c\lambda^d + o(\lambda^d)$$
where c *is a positive constant.*
2. *If the set* $\{\ln(r_1), \cdots, \ln(r_M)\}$ *is non-arithmetic we have the following asymptotic development:*
$$N(\lambda, \Omega) = f(\ln(\lambda))\lambda^d + o(\lambda^d),$$
where f *is a bounded periodic function. Moreover,* f *is left continuous and the set of points of discontinuity is dense in* .

Now let us look at a concrete example where we can improve the result of Theorem 1.1. If $G = [0,1]^N$, Ω is a disjoint union of scaled copies of the unit cube and we have the following result,

Theorem 1.2. *Let* $G = [0,1]^N$, $N \geq 5$, *and* $\Omega = \left(\bigcup_{n=1}^{\infty} \bigcup_{\{i_1,\ldots,i_n\}} G_{i_1,\ldots,i_n}\right) \bigcup G$. *Let* d *be the similarity dimension of* Ω, *with* $N \geq d > N-3$. *Then*

1. *If* $N \geq d > N-1$, *we have*
$$N(\lambda, \Omega) = f(\ln(\lambda))\lambda^d + o(\lambda^d).$$
2. *If* $N-1 \geq d > N-2$,
$$N(\lambda, \Omega) = c_1\lambda^{N-1} + f(\ln(\lambda))\lambda^d + o(\lambda^d).$$
3. *If* $N-2 \geq d > N-3$,
$$N(\lambda, \Omega) = c_1\lambda^{N-1} + c_2\lambda^{N-2} + f(\ln(\lambda))\lambda^d + o(\lambda^d).$$

where f *is a constant or a bounded periodic function. When* $2 \leq N \leq 4$, *a similar expansion holds changing the lower bound for* d. *When* $N = 4$ *we impose that* $4 \geq d > 3/2$, *for* $N = 3$, $3 \geq d > 2/3$ *and for* $N = 2$, $2 \geq d > 0$.

As long as we know, this is the first example of a three-term asymptotic expansion for the Steklov eigenvalue problem. For the usual Dirichlet eigenvalue problem, a third term asymptotic was known, but only for strong self similar sets, i.e., all the coefficients of the similitudes are equal, see [1]. A similar example for the Dirichlet eigenvalue problem can be found in [13]. We wish to stress the fact that the exponent d yields information about the geometric structure of $\partial\Omega$, but d is not a dimension. When $N-1 > d$, an easy computation shows that $|\partial\Omega|_{N-1} < \infty$, so the Hausdorff, Minkowski, or any other reasonable definition of dimension agrees with the topological dimension $N-1$.

Finally, for strong self similar sets we can formulate Theorem 1.2 as:

Theorem 1.3. *Let $N \geq 5$. Let $G = [0,1]^N$ and $\Omega = \left(\bigcup_{n=1}^{\infty} \bigcup_{\{1,\dots,n\}} G_{1,\dots,n}\right) \bigcup G$. Let d be the similarity dimension of Ω, with $N \geq d > N-3$. Then*

$$N(\lambda,\Omega) = c_1\lambda^{N-1} + c_2\lambda^{N-2} + f(\ln(\lambda))\lambda^d + O(\lambda^{N-3})$$

where f is a constant or a bounded periodic function. A similar statement holds for $4 \geq N \geq 2$.

However, we omit the proof, which can be obtained with slight changes from [1].

2. Proof of Theorem 1.1.

For this proof we will use the following variant of the Renewal Theorem (see [9]):

Theorem 2.1. *Let us consider the renewal equation on*

$$f(z) = \sum_{j=1}^{M} b_j f(z-a_j) + g(z), \tag{2.1}$$

where a_j, b_j are positives, $\sum_{j=1}^{M} b_j = 1$, and $g(z)$ is a piecewise continuous function on satisfying the condition:

$$|g(z)| \leq Ce^{-\tau z} \qquad \forall\, z \in \ . \tag{2.2}$$

Then, there exists a unique solution of (2.1), $f(z)$, which is uniformly bounded on . Moreover, f satisfies,

1. *If the set $\{a_1,\cdots,a_M\}$ is arithmetic with span a, then $f(z)$ has the asymptotic expansion:*
$$f(z) = s(z) + o(1)$$
where $s(z)$ is a bounded a-periodic function.
2. *If the set $\{a_1,\cdots,a_M\}$ is non-arithmetic, $f(z)$ has the asymptotic expansion:*
$$f(z) = c + o(1)$$
where c is a positive constant.
3. *If the function $g(z)$ has the additional property of left- or right-continuity, then the functions $f(z), s(z)$ are left- or right-continuous respectively.*

Now we begin with the proof of Theorem 1.1. Let $N(\lambda, G)$ be the spectral counting function of the Steklov eigenvalue problem in G. Combined with the translation invariance and the scaling properties of the Laplacian, we obtain:

$$N(\lambda,\Omega) = \sum_{j=1}^{M} N(\lambda, r_j\Omega) + N(\lambda,G) = \sum_{j=1}^{M} N(r_j\lambda,\Omega) + N(\lambda,G). \tag{2.3}$$

We propose a solution in the form

$$N(\lambda,\Omega) = f(\lambda)\lambda^d. \tag{2.4}$$

Substituting (2.4) into (2.3), and after the change of variable $z = \ln(\lambda)$, we obtain

$$f(z)e^{zd} = \sum_{j=1}^{M} f(\ln(r_j e^z)(r_j e^z)^d + N(e^z, G).$$

Multiplying both sides by e^{-dz}, and denoting $b_j = (r_j)^d$, $a_j = \ln(1/r_j)$, for $1 \le j \le M$ and

$$\psi(z) = N(e^z, G)e^{-zd},$$

we get the following equation for f,

$$f(z) = \sum_{j=1}^{M} b_j f(z - a_j) + \psi(z) \tag{2.5}$$

As the equation (2.5) has the form (2.1) we only need to show that $\psi(z)$ verifies the condition (2.2) (i.e., that $|\psi(z)| \le Ce^{-\tau|z|}$) in order to apply the Renewal Theorem 2.1. We observe that $N(\lambda, G)$ is zero when $\lambda < \lambda_1$, the first positive Steklov eigenvalue, so the inequality is valid when $z \to -\infty$. When $z \to +\infty$, is a consequence of the asymptotic behaviour $N(\lambda, G) \sim C_n|\partial\Omega|_{n-1}\lambda^{n-1}$.

This completes the proof as we obtain that $N(\lambda, \Omega) \sim f(\ln(\lambda))\lambda^d$ where f is a constant or a periodic function, depending on the sets of coefficients r_j. □

3. A three-term asymptotic expansion.

We begin by looking at the eigenvalues for the Steklov problem in the unit cube,

$$\begin{cases} \Delta u = 0 & \text{in } [0,1]^N \\ \frac{\partial u}{\partial \eta} = \lambda u & \text{on } \partial[0,1]^N \end{cases} \tag{3.1}$$

Let us look for a solution with separate variables, i.e.,

$$u(x_1, \dots, x_N) = \varphi_1(x_1) \cdots \varphi_N(x_N) \tag{3.2}$$

Inserting (3.2) in (3.1) we get

$$\frac{\varphi_1''}{\varphi_1}(x_1) + \cdots + \frac{\varphi_N''}{\varphi_N}(x_N) = 0.$$

Hence,

$$\frac{\varphi_i''}{\varphi_i}(x_i) = \beta_i$$

with

$$\beta_1 + \cdots + \beta_N = 0$$

and the boundary conditions impose that

$$-\varphi_i'(0) = \lambda\varphi(0) \qquad \varphi_i'(1) = \lambda\varphi(1) \tag{3.3}$$

Assume that $\beta_i = 0$, then

$$\varphi_i(x_i) = ax_i + b$$

and the boundary conditions (3.3) impose

$$-a = \lambda b \qquad a = \lambda(a+b)$$

Hence the associated eingenvalue is $\lambda_0 = 0$ or $\lambda_2 = 2$.

If $\beta_i < 0$ then

$$\varphi_i(x_i) = c_1 \sin(\sqrt{|\beta_i|}x_i) + c_2 \cos(\sqrt{|\beta_i|}x_i)$$

and, using the boundary conditions, we get

$$c_1\sqrt{|\beta_i|} = \lambda c_2,$$

$$c_1\sqrt{|\beta_i|}\cos(\sqrt{|\beta_i|}) - c_2\sqrt{|\beta_i|}\sin(\sqrt{|\beta_i|}) = c_1\lambda\sin(\sqrt{|\beta_i|}) + c_2\lambda\cos(\sqrt{|\beta_i|}).$$

To have a nontrivial eigenfunction we need to impose

$$(-|\beta_i| - \lambda^2)\sin(\sqrt{|\beta_i|}) = 0$$

hence

$$\sin(\sqrt{|\beta_i|}) = 0.$$

So, β_i is given by

$$\beta_i = -4\pi^2 n^2 \qquad n \in \quad . \tag{3.4}$$

Now, if $\beta_i > 0$

$$\varphi_i(x_i) = c_1\sinh(\sqrt{\beta_i}x_i) + c_2\cosh(\sqrt{\beta_i}x_i)$$

and, using the boundary conditions (3.3), we get

$$-c_1\sqrt{\beta_i} = \lambda c_2,$$

$$c_1\sqrt{|\beta_i|}\cosh(\sqrt{\beta_i}) + c_2\sqrt{|\beta_i|}\sinh(\sqrt{\beta_i}) = c_1\lambda\sinh(\sqrt{\beta_i}) + c_2\lambda\cosh(\sqrt{\beta_i}).$$

To have a nontrivial eigenfunction we need to impose

$$-\beta_i\sinh(\sqrt{\beta_i}) + 2\lambda\sqrt{\beta_i}\cosh(\sqrt{\beta_i}) - \lambda^2\sinh(\sqrt{\beta_i}) = 0.$$

Hence

$$\lambda_{+/-} = \sqrt{\beta_i}\frac{(\cosh(\sqrt{\beta_i}) +/- 1)}{\sinh(\sqrt{\beta_i})}. \tag{3.5}$$

If we consider $\lambda \neq 0, 2$ we have

$$\beta_{k+1} + \cdots + \beta_N = 4\pi^2(n_1^2 + \cdots + n_k^2)$$

and if $n_1^2 + \cdots + n_k^2$ is large then all the β_i must be the same β, that is given by

$$\beta = \frac{4\pi^2}{N-k}(n_1^2 + \cdots + n_k^2).$$

Clearly, we can associate each eigenvalue to a lattice point $(n_1, \ldots, n_k) \in {}^k$ for $1 \le k \le N-1$. In fact, when λ is large, equation (3.5) gives the lattice points in the sphere with an exponentially decaying error. Moreover, from equation (3.5), it is easy to see that $\lambda \to \sqrt{\beta}$ when $\lambda \to \infty$, so we can compute $N(\lambda, [0,1]^N)$ with the aid of the lattice points problem in spheres. The number of eigenvalues is

$$2\sum_{k=1}^{N-1} c_k \# \left\{ (n_1, \ldots, n_k) \in {}^k \ : \ (n_1, \ldots, n_k) \in B_k\left(\frac{\sqrt{N-k}}{2\pi}\lambda\right)\right\}, \tag{3.6}$$

the factor 2 is due to the term $+/-1$ in (3.5), $B_k(r)$ is the k-dimensional ball of radius r, and $c_k = \begin{pmatrix} N \\ k \end{pmatrix}$, as a consequence of the election of k variables.

We can rewrite it explicitly:

Lemma 3.1. *Let $N \geq 6$. Then,*

$$N(\lambda, [0,1]^N) = C_1\lambda^{N-1} + C_2\lambda^{N-2} + O(\lambda^{N-3}) \tag{3.7}$$

where C_j, $j = 1, 2$ is a constant depending only on N. When $N = 5$, we have the same asymptotic expansion with an error term of the form $O(\lambda^2 \ln^{2/3}(\lambda))$. For $4 \geq N \geq 2$ we have

$$\begin{array}{rcl} N(\lambda, [0,1]^4) & = & C_1\lambda^3 + C_2\lambda^2 + O(\lambda^{3/2}\ln(\lambda)) \\ N(\lambda, [0,1]^3) & = & C_1\lambda^2 + O(\lambda^{2/3}) \\ N(\lambda, [0,1]^2) & = & C_1\lambda + O(1) \end{array} \tag{3.8}$$

Proof. We use the following estimation of the number of lattice points in spheres, which can be found in [6]:

$$\#\{(x_1, \ldots, x_N) \in \;^N : \sum_{j=1}^{N} x_j^2 \leq r\} = V_N(r) + \left\{ \begin{array}{ll} O(r^{N/2-1}) & N \geq 5 \\ O(r\ln^{2/3}(r)) & N = 4 \\ O(r^{3/4}\ln(r)) & N = 3 \\ O(r^{2/3}) & N = 2 \end{array} \right. \tag{3.9}$$

The number of eigenvalues is obtained counting the lattice points with positive coordinates, so we write the formula for the lattice points in spheres as

(3.10)

$$V_N(r) + O(r^{N/2-1}) = 2^N \#\{(x_1, \ldots, x_N) \in \;^N\} + \sum_{j=1}^{N} (-1)^{j+1} \left(\begin{array}{c} N \\ j \end{array} \right) V_{N-j}(r),$$

where $V_k(r)$ is the volume of the k-dimensional sphere of radius r, which gives:

$$\#\{(x_1, \ldots, x_N) \in \;^N\} = 2^{-N} V_N(r) - N V_{N-1}(r) + O(r^{N/2-1}), \tag{3.11}$$

collecting the lower order terms and the remainder terms in $O(r^{N/2-1})$

Now, replacing (3.11) in equation (3.6), and using that $V_k(r) = a_k r^k$, the lemma is proved. □

Let us prove Theorem (1.2). The first part is proved in Theorem (1.1). Now, let us look the functional equation satisfied by $N(\lambda, \Omega)$:

$$N(\lambda, \Omega) = \sum_{j=1}^{M} N(\lambda, r_j\Omega) + N(\lambda, Q) = \sum_{j=1}^{M} N(r_j\lambda, \Omega) + N(\lambda, Q).$$

In the right hand side, $N(\lambda, Q)$ has a term which is a power of λ^{N-1} and another which is a power of λ^{N-2}. The only way to cancel those terms (in $N(\lambda, \Omega)$) is forcing $N(\lambda, \Omega)$ to have such terms, an absurd. So, we introduce the auxiliary function defined as

$$R(\lambda, \Omega) = c_1\lambda^{N-1} + c_2\lambda^{N-2} - N(\lambda, \Omega)$$

for $\lambda \geq 0$, and $R(\lambda, \Omega) = 0$ for $\lambda < 0$. Hence,

$$R(\lambda, \Omega) = \sum_{j=1}^{M} R(\lambda, r_j\Omega) + R(\lambda, Q) = \sum_{j=1}^{M} R(r_j\lambda, \Omega) + R(\lambda, Q).$$

We propose a solution in the form

$$R(\lambda, \Omega) = f(\ln(\lambda))\lambda^d.$$

and the rest of the proof runs as before. □

4. Final remarks.

This family of examples shown the existence of many terms in the asymptotic development of $N(\lambda, \Omega)$. It will be interesting to find, for both smooth and non-smooth boundaries, another cases of a multi-term asymptotic development.

It is natural to wonder if Theorem 1.1 can be extended to other sets with non-smooth or fractal boundaries. Here we deal exclusively with self similar ones, and we are tempted to conjecture:

Conjecture 4.1. *Let Ω be an open set in ^{N}, for which the Steklov's eigenvalue problem* (1.1) *has a discrete sequence of eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots$. Then,*

$$N(\lambda, \Omega) = C(d, N)\lambda^d + o(\lambda^d) \tag{4.12}$$

where $C(d, N)$ is a constant if and only if $\partial\Omega$ is (interior) Minkowski measurable and of (interior) Minkowski dimension d.

Or, may be, a weaker version:

Conjecture 4.2. *Let Ω be an open set in ^{N}, where $\partial\Omega$ has (interior) Minkowski dimension d, for which the Steklov's eigenvalue problem* (1.1) *has a discrete sequence of eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots$. Then,*

$$c_1\lambda^d \leq N(\lambda, \Omega) \leq C_2\lambda^d. \tag{4.13}$$

As in the Dirichlet eigenvalue problem, those conjectures seems very difficult to prove. The problem here is the exponent d. We conjecture the d is the interior Minkowski dimension. In the previous examples, the self-similarity seems to play a crucial role. However, it is not difficult to find a family of non-self similar sets $\{\Omega_d\} \in {}^2$, with d the Minkowski dimension of $\partial\Omega$, and $N(\lambda, \Omega) \sim C\lambda^d$, as the following example in ^{2} shows. Let $\Omega = \cup_{j\geq 1} Q_j$ where Q_j is a translation of $[0, j^{-1/d}]^2$, with $1 < d < 2$. It is easy to see that the interior Minkowski dimension of Ω is d. Moreover, we have

$$N(\lambda, \Omega) = \sum_{n=1}^{\infty} \left[\frac{\lambda}{2\pi n^{1/d}}\right] = \zeta(d)\left(\frac{\lambda}{2\pi}\right)^d + \zeta(1/d)\frac{\lambda}{2\pi} + O(\lambda^{d/(1+d)}),$$

where $\zeta(s)$ is the zeta function of Riemann. A similar result for the eigenvalues of the Laplace operator was obtained (with rather different methods) in [8]. We prove this in the next lemma:

Lemma 4.1.

$$\sum_{j=1}^{\infty} [j^{-1/D}x] = \zeta(1/D)x + \zeta(D)x^D + O(x^{D/(1+D)}) \tag{4.14}$$

Proof. Let us note that the previous equation is the number of lattice points below the function $xt^{-1/D}$. Using the well-known summation formula of Euler-MacLaurin, and the Dirichlet's argument for the lattice points below the hyperbola xt^{-1}, we count the points up to $x^{D/(1+D)}$ below the function $xt^{-1/D}$ and below its inverse $x^D t^{-D}$, and we delete the points in the square, which we counted twice:

$$\sum_{j=1}^{x^{D/(1+D)}} [xj^{-1/D}] + \sum_{j=1}^{x^{D/(1+D)}} [x^D j^{-D}] - [x^{D/(1+D)}]^2 =$$

$$= \quad x(\int_1^{x^{D/(1+D)}} t^{-1/D}dt + A + o(x^{-1/(1+D)}))+$$
$$x^D(\int_1^{x^{D/(1+D)}} t^{-D} + B + o(x^{-D^2/(1+D)})) - x^{2D/(1+D)} + O(x^{D/(1+D)}).$$

Integrating, we have:

$$\frac{D}{D-1}x^{2D/(1+D)} + (A - \frac{D}{D-1})x + \frac{1}{1-D}x^{2D/(1+D)}+$$
$$+(B - \frac{1}{1-D})x^D - x^{2D/(1+D)} + O(x^{D/(1+D)}) =$$
$$= (A - \frac{D}{D-1})x + (B - \frac{1}{1-D})x^D + O(x^{D/(1+D)}).$$

We are left with the task of determining the constants A y B. We call $C(s)$ the function,

$$C(s) = \lim_{x\to\infty}(\sum_{n=1}^{x}[n^{-s}] - \int_1^x t^{-s}dt),$$

and we have the following expression for the zeta function of Riemann (when $Re(s) > 0$)

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty}[n^{-s}] - \int_1^{\infty} t^{-s}dt,$$

which gives the desired result. The previous identity is immediate for $Re(s) > 1$, because we can integrate the power of t, and we obtain the value of the constant; when $0 < Re(s) < 1$, we use analytic continuation, because we know, using again the Euler-MacLaurin formula, that for each s the limit exist. □

We hope that further attempts to prove or disprove these conjectures should lead to address the questions above, which in turn will give new light on the relations between fractal and spectral geometry.

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