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Asymptotic Mean Value Properties for the $p-$Laplacian

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Resumen

The purpose of this article is to obtain mean value characterizations of solutions to some nonlinear PDEs. To motivate the results we review some recent results concerning Tug-of-War games and their relation with PDEs. In particular, we will show that solutions to certain PDEs can be obtained as limits of values of Tug-of-War games when the parameter that controls the length of the possible movements goes to zero. Since the equations under study are nonlinear and not in divergence form we will make extensive use of the concept of viscosity solutions.

Palabras clave : *Mean value properties, Tug-of-War games, viscosity solutions.*
Clasificación por materias AMS : *35J60, 91A05, 49L25, 35J25.*

1. Introduction

The fundamental works of Doob, Hunt, Kakutani, Kolmogorov and many others have shown a deep connection between the classical linear potential theory and the corresponding probability theory. The idea behind this classical interplay is that harmonic functions and martingales have a common origin in mean value properties.

Our main goal in this article is to show that this approach turns out to be useful in the nonlinear theory as well.

First, we explain through an elementary example a way in which the Laplacian arise in Probability. Next, we will enter in what is the main goal of this article, the approximation by means of values of games of solutions to nonlinear problems like $p-$harmonic functions, that is, solutions to the PDE, $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ (including the nowadays popular case $p = \infty$). From this connection we can deduce some asymptotic mean value formulas that characterize solutions to these equations.

Our main result (we include a proof here, see also [24]) reads as follows:

Theorem 1.1 *Let $1 < p \le \infty$ and let u be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. The asymptotic expansion*

$$u(x) = \frac{\alpha}{2}\left\{ \max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u \right\} + \beta \fint_{B_\varepsilon(x)} u(y)\, dy + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

holds for all $x \in \Omega$ in the viscosity sense if and only if

$$\Delta_p u(x) = 0$$

in the viscosity sense. Here α and β are determined by $\alpha = \frac{p-2}{p+N}$, and $\beta = \frac{2+N}{p+N}$.

There is also a parabolic version of this result (see [27] for details).

Theorem 1.2 *Let $1 < p \le \infty$ and let u be a continuous function in $\Omega_T = \Omega \times (0,T)$. The asymptotic mean value formula*

$$u(x,t) = \frac{\alpha}{2}\fint_{t-\varepsilon^2}^{t} \left\{ \max_{y\in \overline{B}_\varepsilon(x)} u(y,s) + \min_{y\in \overline{B}_\varepsilon(x)} u(y,s) \right\} ds$$
$$+ \beta \fint_{t-\varepsilon^2}^{t} \fint_{B_\varepsilon(x)} u(y,s)\, dy\, ds + o(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0,$$

holds for every $(x,t) \in \Omega_T$ in the viscosity sense if and only if u is a viscosity solution to

$$(N+p)u_t(x,t) = |\nabla u|^{2-p}\, \Delta_p u(x,t).$$

Here, as before, $\alpha = \frac{p-2}{p+N}$, and $\beta = \frac{2+N}{p+N}$.

Note that, as the elliptic and the parabolic case involve a nonlinear equation, the asymptotic mean value formulas are nonlinear (they involve a máx and a mín). Also note that the equations involved are $1-$homogeneous as well as the mean value formulas that characterize their solutions.

In this article we assume that the reader is familiar with basic tools from probability theory (like conditional expectations) and with the (not so basic) concept of viscosity solutions for second order elliptic and parabolic PDEs (we refer to the book [8] for this last topic).

2. Linear PDEs and probability

2.1. The probability of reaching the exit and harmonic functions

Let us begin by considering a bounded and smooth two-dimensional domain $\Omega \subset \mathbb{R}^2$ (a room) and assume that the boundary, $\partial\Omega$ is decomposed in two parts, Γ_1 (the exit) and Γ_2 (the wall), that is, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$. We begin with a position $(x,y) \in \Omega$ and ask the following question: assume that you move completely at random beginning at (x,y) (we assume that we are in an homogeneous environment and that we do not privilege any direction, in addition, we assume that every time the particle moves independently of its past history) what is the probability $u(x, y)$ of hitting the first part of the boundary Γ_1 (reaching the exit) the first time that the particle hits the boundary ?.

A simple way to get some insight to solve the question runs as follows: First, we simplify the problem and approximate the movement by random increments of step h in each of the axes directions, with $h > 0$ small. From (x, y) the particle can move to any of the four points, $(x+h, y)$, $(x-h, y)$, $(x, y+h)$, or $(x, y-h)$, each movement being chosen at random with probability $1/4$. Starting at (x, y), let $u_h(x, y)$ be the probability of hitting the exit part $\Gamma_1 + B_\delta(0)$ the first time that $\partial\Omega + B_\delta(0)$ is hitt when we move on the lattice of side h. Observe that we need to enlarge a little the boundary to capture points on the lattice of size h (that do not necessarily lie on $\partial\Omega$).

Applying conditional expectations we get that u_h verifies

$$u_h(x,y) = \frac{1}{4}u_h(x+h,y) + \frac{1}{4}u_h(x-h,y) + \frac{1}{4}u_h(x,y+h) + \frac{1}{4}u_h(x,y-h). \quad (1)$$

That is,

$$\begin{aligned} 0 = \quad & u_h(x+h,y) - 2u_h(x,y) + u_h(x-h,y) \\ & + u_h(x,y+h) - 2u_h(x,y) + u_h(x,y-h). \end{aligned} \quad (2)$$

Now, assume that u_h converges as $h \to 0$ to a function u uniformly in $\overline{\Omega}$. Note that this convergence can be proved rigorously.

Let ϕ be a smooth function such that $u - \phi$ has a strict minimum at $(x_0, y_0) \in \Omega$. By the uniform convergence of u_h to u there are points (x_h, y_h) such that $(u_h - \phi)(x_h, y_h) \le (u_h - \phi)(x, y) + o(h^2)$, $(x, y) \in \Omega$ and $(x_h, y_h) \to (x_0, y_0)$ as $h \to 0$. Note that u_h is not necessarily continuous. Hence, from (2) at the point $(x, y) = (x_h, y_h)$ and using that

$$u_h(x,y) - u_h(x_h,y_h) \ge \phi(x,y) - \phi(x_h,y_h) + o(h^2) \qquad (x,y) \in \Omega,$$

we get

$$\begin{aligned} 0 \ge \quad & \phi(x_h+h,y_h) - 2\phi(x_h,y_h) + \phi(x_h-h,y_h) \\ & + \phi(x_h,y_h+h) - 2\phi(x_h,y_h) + \phi(x_h,y_h-h) + o(h^2). \end{aligned} \quad (3)$$

Now, we just observe that

$$\phi(x_h+h,y_h) - 2\phi(x_h,y_h) + \phi(x_h-h,y_h) = h^2 \frac{\partial^2 \phi}{\partial x^2}(x_h,y_h) + o(h^2)$$

$$\phi(x_h,y_h+h) - 2\phi(x_h,y_h) + \phi(x_h,y_h-h) = h^2 \frac{\partial^2 \phi}{\partial y^2}(x_h,y_h) + o(h^2).$$

Hence, substituting in (3), dividing by h^2 and taking limit as $h \to 0$ we get

$$0 \ge \frac{\partial^2 \phi}{\partial x^2}(x_0,y_0) + \frac{\partial^2 \phi}{\partial y^2}(x_0,y_0).$$

Therefore, a uniform limit of the u_h, u, has the following property: *each time that a smooth function ϕ touches u from below at a point (x_0, y_0) the derivatives of ϕ must satisfy,*

$$0 \geq \frac{\partial^2 \phi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \phi}{\partial y^2}(x_0, y_0).$$

An analogous argument considering ψ a smooth function such that $u - \psi$ has a strict maximum at $(x_0, y_0) \in \Omega$ shows a reverse inequality. Therefore, *each time that a smooth function ψ touches u from above at a point (x_0, y_0) the derivatives of ψ must verify*

$$0 \leq \frac{\partial^2 \psi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \psi}{\partial y^2}(x_0, y_0).$$

But at this point we realize that this is exactly the definition of being u a **viscosity solution to the Laplace equation**

$$-\Delta u(x, y) = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)(x, y) = 0, \qquad (x, y) \in \Omega.$$

Hence, we obtain that the uniform limit of the sequence of solutions to the approximated problems u_h, u is the unique viscosity solution (that is also a classical solution in this case) to the following boundary value problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 1 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2. \end{cases} \tag{4}$$

The boundary conditions can be easily obtained from the fact that $u_h \equiv 1$ in a neighborhood (of width h) of Γ_1 and $u_h \equiv 0$ in a neighborhood of Γ_2.

Note that we have only required *uniform* convergence to get the result, and hence no requirement is made on derivatives of the approximating sequence u_h. Moreover, we do not assume that u_h is continuous.

Notice that in higher dimensions $\Omega \subset \mathbb{R}^N$ nothing changes, the same arguments described above leads in the same simple way to viscosity solutions to the Laplace operator.

Another way (that is closely related to the core of this article) to understand this strong relation between probabilities and the Laplacian is through the *mean value property of harmonic functions*. In the same context of the problem solved above, assume that a closed ball $B_r(x_0, y_0)$ of radius r and centered at a point (x_0, y_0) is contained in Ω. Starting at (x_0, y_0), the probability density of hitting first a given point on the sphere $\partial B_r(x_0, y_0)$ is constant on the sphere, that is, it is uniformly distributed on the sphere. Therefore, the probability $u(x_0, y_0)$ of exiting through Γ_1 starting at (x_0, y_0) is the average of the exit probabilities u on the sphere, here we are using again the formula of conditional probabilities. That is, u satisfies the mean value property on spheres:

$$u(x_0, y_0) = \frac{1}{|\partial B_r(x_0, y_0)|} \int_{\partial B_r(x_0, y_0)} u(x, y)\, dS(x, y)$$

with r small enough. It is well known that this property leads to u being harmonic.

We can also say that, if the movement is completely random and equidistributed in the ball $B_h(x_0, y_0)$, then, by the same conditional expectation argument used before, we have

$$u_h(x_0, y_0) = \frac{1}{|B_h(x_0, y_0)|} \int_{B_h(x_0,y_0)} u_h(x, y)\, dx\, dy.$$

Again one can take the limit as $h \to 0$ and obtain that a uniform the limit of the u_h, u, is harmonic (in the viscosity sense).

3. Tug-of-War games and the $\infty-$Laplacian

In this section we will look for a probabilistic approach to approximate solutions to an elliptic equation called the $\infty-$Laplacian, this is the nonlinear degenerate elliptic operator, usually denoted by Δ_∞, given by,

$$\Delta_\infty u := \big(D^2 u\, \nabla u\big) \cdot \nabla u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i x_j},$$

and arises from taking limit as $p \to \infty$ in the p-Laplacian operator in the viscosity sense, see [3] and [7]. In fact, let us present a formal derivation. First, expand (formally) the $p-$laplacian:

$$\begin{aligned} \Delta_p u &= \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = \\ &= |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4} \sum_{i,j} u_{x_i} u_{x_j} u_{x_i,x_j} = \\ &= (p-2)|\nabla u|^{p-4} \Big\{ \frac{1}{p-2} |\nabla u|^2 \Delta u + \sum_{i,j} u_{x_i} u_{x_j} u_{x_i,x_j} \Big\} \end{aligned}$$

and next, using this formal expansion, pass to the limit in the equation $\Delta_p u = 0$, to obtain $\Delta_\infty u = \sum_{i,j} u_{x_i} u_{x_j} u_{x_i,x_j} = Du \cdot D^2 u \cdot (Du)^t = 0$. Note that this calculation can be made rigorous in the viscosity sense, see [3].

The ∞-laplacian operator appears naturally when one considers absolutely minimizing Lipschitz extensions of a boundary function F; see [14] and also the survey [3]. A fundamental result of Jensen [14] establishes that the Dirichlet problem for Δ_∞ is well posed in the viscosity sense. Solutions to $-\Delta_\infty u = 0$ (that are called infinity harmonic functions) are also used in several applications, for instance, in optimal transportation and image processing. Also the eigenvalue problem related to the ∞-laplacian has been exhaustively studied, see [17], [18], [19].

Let us recall the definition of an absolutely minimizing Lipschitz extension. Let $F : \partial\Omega \to \mathbb{R}$. We denote by $L(F, \partial\Omega)$ the smallest Lipschitz constant of F in $\partial\Omega$, i.e.,

$$L(F, \partial\Omega) := \sup_{x,y\in\partial\Omega} \frac{|F(x) - F(y)|}{|x - y|}.$$

If we are given a Lipschitz function $F : \partial\Omega \to \mathbb{R}$, i.e., $L(F, \partial\Omega) < +\infty$, then it is well-known that there exists a *minimal Lipschitz extension* (MLE for short) of F to Ω, that is, a function $h : \overline{\Omega} \to \mathbb{R}$ such that $h_{|\partial\Omega} = F$ and $L(F, \partial\Omega) = L(h, \overline{\Omega})$. The notion of a minimal Lipschitz extension is not completely satisfactory since it involves only the global Lipschitz constant of the extension and ignore what may happen locally. To solve this problem, in the particular case of the euclidean space $\mathbb{R}^N$, Arosson [1] introduced the concept of *absolutely minimizing Lipschitz extension* (AMLE for short) and proved the existence of AMLE by means of a variant of the Perron's method. The AMLE is given by the following definition. Here we consider the general case of extensions of Lipschitz functions defined on a subset $A \subset \overline{\Omega}$, but the reader may consider $A = \partial\Omega$.

Definition 3.1 Let A be any nonempty subset of $\overline{\Omega}$ and let $F : A \subset \overline{\Omega} \to \mathbb{R}$ be a Lipschitz function. A function $u : \overline{\Omega} \to \mathbb{R}$ is an *absolutely minimizing Lipschitz extension* of F to $\overline{\Omega}$ if

(i) u is an MLE of F to $\overline{\Omega}$,

(ii) whenever $B \subset \overline{\Omega}$ and $g : \overline{\Omega} \to \mathbb{R}$ is and MLE of F to $\overline{\Omega}$ such that $g = u$ in $\overline{\Omega} \setminus B$, then $L(u, B) \leq L(g, B)$.

It turns out (see [3]) that *the unique AMLE of F (defined on $\partial\Omega$) to $\overline{\Omega}$ is the unique solution to*

$$\begin{cases} -\Delta_\infty u(x) = 0 & \textit{in } \Omega, \\ u(x) = F(x) & \textit{on } \partial\Omega. \end{cases}$$

Our main aim in this section is to describe a game that approximates this problem in the same way as problems involving the random walk described in the previous section approximate harmonic functions.

3.1. Description of the game

We follow [31] and [9], but we restrict ourselves to the case of a game in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ (the results presented in [31] are valid in general length spaces).

A Tug-of-War is a two-person, zero-sum game, that is, two players are in contest and the total earnings of one are the losses of the other. Hence, one of them, say Player I, plays trying to maximize his expected outcome, while the other, say Player II is trying to minimize Player I's outcome (or, since the game is zero-sum, to maximize his own outcome).

Let us describe briefly the game introduced in [31] by Y. Peres, O. Schramm, S. Sheffield and D. Wilson. Consider a bounded domain $\Omega \subset \mathbb{R}^N$, and take $\Gamma_D \subset \partial\Omega$ and $\Gamma_N \equiv \partial\Omega \setminus \Gamma_D$. Let $F : \Gamma_D \to \mathbb{R}$ be a Lipschitz continuous function. At an initial time, a token is placed at a point $x_0 \in \overline{\Omega} \setminus \Gamma_D$. Then, a (fair) coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in \overline{B_\epsilon(x_0)} \cap \overline{\Omega}$. At each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in \overline{B_\epsilon(x_{k-1})} \cap \overline{\Omega}$. Once the token has reached some $x_\tau \in \Gamma_D$, the game ends and Player I earns $F(x_\tau)$ (while Player II earns $-F(x_\tau)$). This is the reason why we will refer to F as the *final payoff function.* In more general models, it is considered also a *running payoff* $g(x)$ defined in Ω, which represents the reward (respectively, the cost) at each intermediate state x, and gives rise to nonhomogeneous problems. We will assume here that $g \equiv 0$. This procedure yields a sequence of game states $x_0, x_1, x_2, \dots, x_\tau$, where every x_k except x_0 are random variables, depending on the coin tosses and the strategies adopted by the players.

Now we want to give a precise definition of the *value of the game.* To this end we have to introduce some notation and put the game into its normal or strategic form (see [32] and [28]). The initial state $x_0 \in \overline{\Omega} \setminus \Gamma_D$ is known to both players (public knowledge). Each player i chooses an *action* $a_0^i \in \overline{B_\epsilon(0)}$ which is announced to the other player; this defines an action profile $a_0 = \{a_0^1, a_0^2\} \in \overline{B_\epsilon(0)} \times \overline{B_\epsilon(0)}$. Then, the new state $x_1 \in \overline{B_\epsilon(x_0)}$ (namely, the current state plus the action) is selected according to a probability distribution $p(\cdot|x_0, a_0)$ in $\overline{\Omega}$ which, in our case, is given by the fair coin toss. At stage k, knowing the history $h_k = (x_0, a_0, x_1, a_1, \dots, a_{k-1}, x_k)$, (the sequence of states and actions up to that stage), each player i chooses an action a_k^i. If the game ends at time $j < k$, we set $x_m = x_j$ and $a_m = 0$ for $j \le m \le k$. The current state x_k and the profile $a_k = \{a_k^1, a_k^2\}$ determine the distribution $p(\cdot|x_k, a_k)$ (again given by the fair coin toss) of the new state x_{k+1}.

Denote $H_k = (\overline{\Omega} \setminus \Gamma_D) \times \left(\overline{B_\epsilon(0)} \times \overline{B_\epsilon(0)} \times \overline{\Omega}\right)^k$, the set of *histories up to stage* k, and by $H = \bigcup_{k \ge 1} H_k$ the set of all histories. Notice that H_k, as a product space, has a measurable structure. The *complete history space* H_∞ is the set of plays defined as infinite sequences $(x_0, a_0, \dots, a_{k-1}, x_k, \dots)$ endowed with the product topology. Then, the final payoff for Player I, i.e. F, induces a Borel-measurable function on H_∞. A *pure strategy* $S_i = \{S_i^k\}_k$ for Player i, is a sequence of mappings from histories to actions, namely, a mapping from H to $\overline{B_\epsilon(0)}^{\mathbb{N}}$ such that S_i^k is a Borel-measurable mapping from H_k to $\overline{B_\epsilon(0)}$ that maps histories ending with x_k to elements of $\overline{B_\epsilon(0)}$ (roughly speaking, at every stage the strategy gives the next movement for the player, provided he win the coin toss, as a function of the current state and the past history). The initial state x_0 and a profile of strategies $\{S_I, S_{II}\}$ define (by Kolmogorov's extension theorem) a unique probability $\mathbb{P}^{x_0}_{S_I,S_{II}}$ on the space of plays H_∞. We denote by $\mathbb{E}^{x_0}_{S_I,S_{II}}$ the corresponding expectation.

Then, if S_I and S_{II} denote the strategies adopted by Player I and II respectively, we define the expected payoff for player I as

$$V_{x_0,I}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ -\infty, & \text{otherwise.} \end{cases}$$

Analogously, we define the expected payoff for player II as

$$V_{x_0,II}(S_I,S_{II}) = \begin{cases} \mathbb{E}_{S_I,S_{II}}^{x_0}[F(x_\tau)], & \text{if the game terminates a.s.} \\ +\infty, & \text{otherwise.} \end{cases}$$

Finally, we can define the *ϵ-value of the game for Player I* as

$$u_I^\epsilon(x_0) = \sup_{S_I} \inf_{S_{II}} V_{x_0,I}(S_I,S_{II}),$$

while the ϵ-value of the game for Player II is defined as

$$u_{II}^\epsilon(x_0) = \inf_{S_{II}} \sup_{S_I} V_{x_0,II}(S_I,S_{II}).$$

In some sense, $u_I^\epsilon(x_0), u_{II}^\epsilon(x_0)$ are the least possible outcomes that each player expects to get when the ϵ-game starts at x_0. Notice that, as in [31], we penalize severely the games that never end.

If $u_I^\epsilon = u_{II}^\epsilon := u_\epsilon$, we say that *the game has a value*. In [31] it is shown that, under very general hypotheses, that are fulfilled in the present setting, the ϵ-Tug-of-War game has a value.

The value of the game verifies a *Dynamic Programming Principle*, that in our case reads as follows: the value of the game u_ϵ verifies

$$u_\epsilon(x) = \frac{1}{2} \sup_{y\in\overline{B_\epsilon(x)\cap\bar{\Omega}}} u_\epsilon(y) + \frac{1}{2} \inf_{y\in\overline{B_\epsilon(x)\cap\bar{\Omega}}} u_\epsilon(y),$$

where $B_\epsilon(x)$ denotes the open ball of radius ϵ centered at x, see [25].

All these $\epsilon-$values are Lipschitz functions with respect to the discrete distance d^ϵ given by

$$d_\varepsilon(x,y) = \begin{cases} 0 & \text{if } x=y, \\ \varepsilon\left(\left[\!\left[\frac{|x-y|}{\varepsilon}\right]\!\right]+1\right) & \text{if } x\neq y. \end{cases} \tag{5}$$

where $|.|$ is the Euclidean norm and $[\![r]\!]$ is defined for $r>0$ by $[\![r]\!] := n$, if $n<r\leq n+1,\ n=0,1,2,\dots$, that is,

$$d_\varepsilon(x,y) = \begin{cases} 0 & \text{if } \ x=y, \\ \varepsilon & \text{if } \ 0<|x-y|\leq\varepsilon, \\ 2\varepsilon & \text{if } \ \varepsilon<|x-y|\leq 2\varepsilon \\ \vdots \end{cases}$$

see [31]. Note that, Lipschitz functions with respect to d^ϵ are not necessarily continuous.

Let us present a simple example where we can compute explicitly the value of the game.

3.2. The $1-d$ game

Let us analyze in detail the one-dimensional game and its limit as $\epsilon \to 0$. We set $\Omega = (0,1)$ and play the $\epsilon-$game described before. To simplify we assume that $\epsilon = 1/2^n$. Concerning the final payoff, we end the game at $x=0$ (with zero final payoff) and at $x=1$ (with final payoff equals to one). Note that, the general result from [31] applies and hence we can assert the existence of a value for this game. Nevertheless, in this simple $1-d$ case we can obtain the existence of such value by direct computations. For the moment, let us assume that there exists a value that we call u_ϵ and proceed, in several steps, with the analysis of this sequence of functions u_ϵ for ϵ small. All the calculations below hold both for u_I^ϵ and for u_{II}^ϵ.

Step 1. $u_\epsilon(0) = 0$ and $u_\epsilon(1) = 1$. Moreover, $0 \le u_\epsilon(x) \le 1$ (the value functions are uniformly bounded).

Step 2. u_ϵ is increasing in x and strictly positive in $(0,1]$. Indeed, if $x < y$ then for every pair of strategies S_I, S_{II} for Player I and II beginning at x we can construct strategies beginning at y in such a way that $x_{i,x} \le x_{i,y}$ (here $x_{i,x}$ and $x_{i,y}$ are the positions of the game after i movements beginning at x and y respectively). In fact, just reproduce the movements shifting points by $y-x$ when possible (if not, that is, if the jump is too large and ends outside the interval, just remain at the larger interior position $x=1$). In this way we see that the probability of reaching $x=1$ beginning at y is bigger than the probability of reaching $x=0$ and hence, taking expectations, infimum and supremum, it follows that $u_\epsilon(x) \le u_\epsilon(y)$.

To obtain the strict positivity of u_ϵ, we just observe that there is a positive probability of obtaining a sequence of $1/\epsilon$ consecutive heads, hence the probability of reaching $x=1$ when the first player uses the strategy that points ϵ to the right is strictly positive. Therefore, $u_\epsilon(x) > 0$, for every $x \neq 0$.

Step 3. In this one dimensional case it is easy to identify the optimal strategies for players I and II: to jump ϵ to the right for Player I and to jump ϵ to the left for Player II. That is, if we are at x, the optimal strategies lead to $x \to \text{mín}\{x+\epsilon, 1\}$ for Player I, and to $x \to \text{máx}\{x-\epsilon, 0\}$ for Player II.

This follows from step 2, where we have proved that the function u_ϵ is increasing in x. As a consequence, the optimal strategies follow: for instance, Player I will choose the point where the expected payoff is maximized and this is given by $\text{mín}\{x+\epsilon, 1\}$,

$$\sup_{z\in[x-\epsilon,x+\epsilon]\cap[0,1]} u_\epsilon(z) = \operatorname*{máx}_{z\in[x-\epsilon,x+\epsilon]\cap[0,1]} u_\epsilon(z) = u_\epsilon(\text{mín}\{x+\epsilon,1\}),$$

since u_ϵ is increasing. This is also clear from the following intuitive fact: player I wants to maximize the payoff (reaching $x=1$) and player II wants to minimize the payoff (hence pointing to 0).

Step 4. u_ϵ is constant in every interval of the form $(k\epsilon, (k+1)\epsilon)$ for $k = 1, ..., N$ (we denote by N the total number of such intervals in $(0,1]$). Indeed, from step 3 we know what are the optimal strategies for both players, and hence the result follows noticing that the number of steps that one has to advance to reach $x = 0$ (or $x = 1$) is the same for every point in $(k\epsilon, (k+1)\epsilon)$. Note that u_ϵ is discontinuos at every point of the form $y_k = k\epsilon \in (0,1)$.

Step 5. Let us call $a_k := u_\epsilon \mid_{(k\epsilon,(k+1)\epsilon)}$. Then we have

$$a_0 = 0, \qquad a_k = \frac{1}{2}(a_{k-1} + a_{k+1}),$$

for every $i = 2, ..., n-1$, and

$$a_n = 1.$$

Notice that these identities follow from the Dynamic Programming Principle, using that from step 3 we know the optimal strategies, that from step 4 u_ϵ is constant in every subinterval of the form $(k\epsilon, (k+1)\epsilon)$, we immediately get the conclusion.

Note the similarity with a finite difference scheme used to solve $u_{xx} = 0$ in $(0,1)$ with boundary conditions $u(0) = 0$ and $u(1) = 1$. In fact, a discretization of this problem in a uniform mesh of size ϵ leads to the same formulas obtained in step 5.

Step 6. We have

$$u_\epsilon(x) = \epsilon k, \qquad x \in (k\epsilon, (k+1)\epsilon). \tag{6}$$

Indeed, the constants $a_k = \epsilon k$ are the unique solution to the formulas obtained in step 5. Since formula (6) is in fact valid for u_I^ϵ and u_{II}^ϵ, this proves that the game has a value.

Note that u_ϵ verifies that $0 \leq u_\epsilon(x) - u_\epsilon(y) \leq 2(x-y)$ for every $x > y$ with $x - y > \epsilon$. This is a sort of equicontinuity valid for "far apart points". In this one dimensional case, we can pass to the limit directly, by using the explicit formula for u_ϵ (see Step 7 below). However, in the N-dimensional case there is no explicit formula, and then we will need a compactness result (a sort of Arzela-Ascoli lemma).

Step 7.

$$\lim_{\epsilon \to 0} u_\epsilon(x) = x,$$

uniformly in $[0,1]$. This follows from the explicit formula for u_ϵ in every interval of the form $(k\epsilon, (k+1)\epsilon)$ found in step 6 and from the monotonicity stated in step 2 (to take care of the values of u_ϵ at points of the form $k\epsilon$, we have $a_{k-1} \leq u_\epsilon(k\epsilon) \leq a_k$).

Note that the limit function $u(x) = x$ is the unique viscosity (and classical) solution to $\Delta_\infty u(x) = (u_{xx}(u_x)^2)(x) = 0$ $x \in (0,1)$, with boundary conditions $u(0) = 0$, $u(1) = 1$.

3.3. Mixed boundary conditions for Δ_∞

Now we continue the analysis of the Tug-of-War game described previously. As before we assume that we are in the general case of a bounded domain Ω in $\mathbb{R}^N$. The game ends when the position reaches one part of the boundary Γ_D

(where there is a specified final payoff F) and look for the condition that the limit must verify on the rest of it, $\partial\Omega \setminus \Gamma_D$.

All these $\epsilon-$values are Lipschitz functions with respect to the discrete distance d^ϵ defined in (5), see [31] (but in general they are not continuous as the one-dimensional example shows), which converge uniformly when $\epsilon \to 0$. The uniform limit as $\epsilon \to 0$ of the game values u_ϵ is called *the continuous value* of the game that we will denote by u and it can be seen (see below) that u is a viscosity solution to the problem

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ u(x) = F(x) & \text{on } \Gamma_D. \end{cases} \tag{7}$$

When $\Gamma_D \equiv \partial\Omega$ it is known that this problem has a unique viscosity solution, (as proved in [14]; see also [4], and in a more general framework, [31]).

However, when $\Gamma_D \neq \partial\Omega$ the PDE problem (7) is incomplete, since there is a missing boundary condition on $\Gamma_N = \partial\Omega \setminus \Gamma_D$. Our concern now is to find the boundary condition that completes the problem. Assuming that Γ_N is regular, in the sense that the normal vector field $\vec{n}(x)$ is well defined and continuous for all $x \in \Gamma_N$, it is proved in [9] that it is in fact the homogeneous Neumann boundary condition $\frac{\partial u}{\partial n}(x) = 0$, $x \in \Gamma_N$.

The key point of the proof of the fact that $-\Delta_\infty u = 0$ in Ω, see [31] and [9]) is the *Dynamic Programming Principle*, that in our case reads as follows: the value of the game u_ϵ verifies

$$u_\epsilon(x) = \frac{1}{2} \sup_{y \in \overline{B_\epsilon(x)} \cap \bar{\Omega}} u_\epsilon(y) + \frac{1}{2} \inf_{y \in \overline{B_\epsilon(x)} \cap \bar{\Omega}} u_\epsilon(y) \qquad \forall x \in \bar{\Omega} \setminus \Gamma_D,$$

where $B_\epsilon(x)$ denotes the open ball of radius ϵ centered at x.

This Dynamic Programming Principle, in some sense, plays the role of the mean property for harmonic functions in the infinity-harmonic case. This principle turns out to be an important qualitative property of the approximations of infinity-harmonic functions, and is the main tool to construct convergent numerical methods in this kind of problems; see [29].

We have the following result.

Theorem 3.1 *Let $u(x)$ be the continuous value of the Tug-of-War game described above (as introduced in [31]). Assume that $\partial\Omega = \Gamma_N \cup \Gamma_D$, where Γ_N is of class C^1, and F is a Lipschitz function defined on Γ_D. Then,*

i) $u(x)$ is a viscosity solution to the mixed boundary value problem

$$\begin{cases} -\Delta_\infty u(x) = 0 & \textit{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \textit{on } \Gamma_N, \\ u(x) = F(x) & \textit{on } \Gamma_D. \end{cases} \tag{8}$$

ii) Reciprocally, assume that Ω verifies that for every $z \in \overline{\Omega}$ and every $x^* \in \Gamma_N$ $z \neq x^*$ that

$$\left\langle \frac{x^* - z}{|x^* - z|}; n(x^*) \right\rangle > 0.$$

Then, if $u(x)$ is a viscosity solution to (8), it coincides with the unique continuous value of the game.

The hypothesis imposed on Ω in part ii) holds whenever Ω is strictly convex. The first part of the theorem comes as a consequence of the Dynamic Programming Principle read in the viscosity sense.

The proof of this result is not included in this work. We refer to [9] for details and remark that the proof of the second part uses that the continuous value of the game is determined by the fact that it enjoys comparison with quadratic functions in the sense described in [31].

4. $p-$harmonious functions

The aim of this section is to describe games that approximate the $p-$Laplacian that is given by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. We assume that $2 \leq p < \infty$. The case $p = \infty$ was considered in the previous section.

4.1. $p-$harmonious functions

Definition 4.1 *The function u_ε is p-harmonious in Ω with boundary values a bounded Borel function $F : \Gamma_\varepsilon \to \mathbb{R}$ if*

$$u_\varepsilon(x) = \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x)} u_\varepsilon \, dy \qquad \textit{for every} \quad x \in \Omega, \tag{9}$$

where α, β are defined in (23), and $u_\varepsilon(x) = F(x)$, for every $x \in \Gamma_\varepsilon$.

The reason for using the boundary strip Γ_ε instead of simply using the boundary $\partial\Omega$ is the fact that for $x \in \Omega$ the ball $\overline{B}_\varepsilon(x)$ is not necessarily contained in Ω.

Let us first explain the name p-harmonious. When u is harmonic, then it satisfies the well known mean value property

$$u(x) = \fint_{B_\varepsilon(x)} u \, dy, \tag{10}$$

that is (9) with $\alpha = 0$ and $\beta = 1$. On the other hand, functions satisfying (9) with $\alpha = 1$ and $\beta = 0$

$$u_\varepsilon(x) = \frac{1}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon \right\} \tag{11}$$

are called *harmonious* functions in [12] and [13] and are values of Tug-of-War games like the ones described in the previous section. As we have seen, as ε goes to zero, they approximate solutions to the infinity Laplacian.

Now, recall that the p-Laplacian is given by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}\left\{(p-2)\Delta_\infty u + \Delta u\right\}. \tag{12}$$

Since the p-Laplace operator can be written as a combination of the Laplace operator and the infinity Laplacian, it seems reasonable to expect that the combination (9) of the averages in (10) and (11) give an approximation to a solution to the p-Laplacian. We will see that this is indeed the case. To be more precise, we have that p-harmonious functions are uniquely determined by their boundary values and that they converge uniformly to the p-harmonic function with the given boundary data. Furthermore, we show that p-harmonious functions satisfy the strong maximum and comparison principles. Observe that the validity of the strong comparison principle is an open problem for the solutions of the p-Laplace equation in $\mathbb{R}^N$, $N \geq 3$.

4.2. p-harmonious functions and Tug-of-War games

In this section, we describe the connection between p-harmonious functions and tug-of-war games. Fix $\varepsilon > 0$ and consider the two-player zero-sum-game described before. At the beginning, a token is placed at a point $x_0 \in \Omega$ and the players toss a biased coin with probabilities α and β, $\alpha + \beta = 1$. If they get heads (probability α), they play a tug-of-war, that is, a fair coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in \overline{B}_\varepsilon(x_0)$. On the other hand, if they get tails (probability β), the game state moves according to the uniform probability to a random point in the ball $B_\varepsilon(x_0)$. Then they continue playing the same game from x_1.

This procedure yields a possibly infinite sequence of game states $x_0, x_1, \ldots$ where every x_k is a random variable. We denote by $x_\tau \in \Gamma_\varepsilon$ the first point in Γ_ε in the sequence, where τ refers to the first time we hit Γ_ε. The payoff is $F(x_\tau)$, where $F : \Gamma_\varepsilon \to \mathbb{R}$ is a given *payoff function*. Player I earns $F(x_\tau)$ while Player II earns $-F(x_\tau)$.

Note that, due to the fact that $\beta > 0$, or equivalently $p < \infty$, the game ends almost surely $\mathbb{P}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}(\{\omega \in H^\infty \colon \tau(\omega) < \infty\}) = 1$ for any choice of strategies.

The *value of the game for Player I* is given by

$$u^\varepsilon_{\mathrm{I}}(x_0) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}[F(x_\tau)]$$

while the *value of the game for Player II* is given by

$$u^\varepsilon_{\mathrm{II}}(x_0) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}[F(x_\tau)].$$

The values $u^\varepsilon_{\mathrm{I}}(x_0)$ and $u^\varepsilon_{\mathrm{II}}(x_0)$ are the best expected outcomes each player can guarantee when the game starts at x_0.

We start by the statement of the *Dynamic Programming Principle* (DPP) applied to our game.

Lema 1 (DPP) *The value function for Player I satisfies*

$$\begin{aligned} & u_I^\varepsilon(x_0) = \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x_0)} u_I^\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} u_I^\varepsilon \right\} + \beta \fint_{B_\varepsilon(x_0)} u_I^\varepsilon \, dy, \qquad x_0 \in \Omega, \\ & u_I^\varepsilon(x_0) = F(x_0), \qquad x_0 \in \Gamma_\varepsilon. \end{aligned} \tag{13}$$

The value function for Player II, u_{II}^ε, satisfies the same equation.

Formulas similar to (13) can be found in Chapter 7 of [22]. A detailed proof adapted to our case can also be found in [25].

Let us explain intuitively why the DPP holds by considering the expectation of the payoff at x_0. Whenever the players get heads (probability α) in the first coin toss, they toss a fair coin and play the tug-of-war. If Player I wins the fair coin toss in the tug-of-war (probability 1/2), she steps to a point maximizing the expectation and if Player II wins, he steps to a point minimizing the expectation. Whenever they get tails (probability β) in the first coin toss, the game state moves to a random point according to a uniform probability on $B_\varepsilon(x_0)$. The expectation at x_0 can be obtained by summing up these different alternatives.

We warn the reader that, as happens for the tug-of-war game without noise described previously, the value functions are discontinuous in general.

By adapting the martingale methods used in [31], we prove a comparison principle. This also implies that $u_{\mathrm{I}}^\varepsilon$ and $u_{\mathrm{II}}^\varepsilon$ are respectively the smallest and the largest p-harmonious function.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. If v_ε is a p-harmonious function with boundary values F_v in Γ_ε such that $F_v \geq F_{u_I^\varepsilon}$, then $v \geq u_I^\varepsilon$.*

Proof: We show that by choosing a strategy according to the minimal values of v, Player II can make the process a supermartingale. The optional stopping theorem then implies that the expectation of the process under this strategy is bounded by v. Moreover, this process provides an upper bound for $u_{\mathrm{I}}^\varepsilon$.

Player I follows any strategy and Player II follows a strategy S_{II}^0 such that at $x_{k-1} \in \Omega$ he chooses to step to a point that almost minimizes v, that is, to a point $x_k \in \overline{B}_\varepsilon(x_{k-1})$ such that

$$v(x_k) \leq \inf_{\overline{B}_\varepsilon(x_{k-1})} v + \eta 2^{-k}$$

for some fixed $\eta > 0$. We start from the point x_0. It follows that

$$\begin{aligned} & \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{x_0} [v(x_k) + \eta 2^{-k} \mid x_0, \ldots, x_{k-1}] \\ & \leq \frac{\alpha}{2}\left\{ \inf_{\overline{B}_\varepsilon(x_{k-1})} v + \eta 2^{-k} + \sup_{\overline{B}_\varepsilon(x_{k-1})} v \right\} + \beta \fint_{B_\varepsilon(x_{k-1})} v \, dy + \eta 2^{-k} \\ & \leq v(x_{k-1}) + \eta 2^{-(k-1)}, \end{aligned}$$

where we have estimated the strategy of Player I by sup and used the fact that v is p-harmonious. Thus

$$M_k = v(x_k) + \eta 2^{-k}$$

is a supermartingale. Since $F_v \geq F_{u_{\mathrm{I}}^\varepsilon}$ at Γ_ε, we deduce

$$
\begin{aligned}
u_{\mathrm{I}}^\varepsilon(x_0) &= \sup_{S_{\mathrm{I}}} \operatorname*{ínf}_{S_{\mathrm{II}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{x_0}[F_{u_{\mathrm{I}}^\varepsilon}(x_\tau)] \leq \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}^0}^{x_0}[F_v(x_\tau) + \eta 2^{-\tau}] \\
&\leq \sup_{S_{\mathrm{I}}} \operatorname*{lím\,inf}_{k\to\infty} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}^0}^{x_0}[v(x_{\tau\wedge k}) + \eta 2^{-(\tau\wedge k)}] \\
&\leq \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}^0}[M_0] = v(x_0) + \eta,
\end{aligned}
$$

where $\tau \wedge k = \text{mín}(\tau, k)$, and we used Fatou's lemma as well as the optional stopping theorem for M_k. Since η was arbitrary this proves the claim.

Similarly, we can prove that $u_{\mathrm{II}}^\varepsilon$ is the largest p-harmonious function: Player II follows any strategy and Player I always chooses to step to the point where v is almost maximized. This implies that $v(x_k) - \eta 2^{-k}$ is a submartingale. Fatou's lemma and the optional stopping theorem then prove the claim.

Next we show that the game has a value. This together with the previous comparison principle proves the uniqueness of p-harmonious functions with given boundary values.

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and F a given boundary data in Γ_ε. Then $u_I^\varepsilon = u_{II}^\varepsilon$, that is, the game has a value.*

Proof: Clearly, $u_{\mathrm{I}}^\varepsilon \leq u_{\mathrm{II}}^\varepsilon$ always holds, so we are left with the task of showing that $u_{\mathrm{II}}^\varepsilon \leq u_{\mathrm{I}}^\varepsilon$. To see this we use the same method as in the proof of the previous theorem: Player II follows a strategy S_{II}^0 such that at $x_{k-1} \in \Omega$, he always chooses to step to a point that almost minimizes $u_{\mathrm{I}}^\varepsilon$, that is, to a point x_k such that

$$
u_{\mathrm{I}}^\varepsilon(x_k) \leq \operatorname*{ínf}_{\overline{B}_\varepsilon(x_{k-1})} u_{\mathrm{I}}^\varepsilon + \eta 2^{-k},
$$

for a fixed $\eta > 0$. We start from the point x_0. It follows that from the choice of strategies and the dynamic programming principle for $u_{\mathrm{I}}^\varepsilon$ that

$$
\begin{aligned}
&\mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}^0}^{x_0}[u_{\mathrm{I}}^\varepsilon(x_k) + \eta 2^{-k} \,|\, x_0, \ldots, x_{k-1}] \\
&\qquad \leq \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x_{k-1})} u_{\mathrm{I}}^\varepsilon + \operatorname*{ínf}_{\overline{B}_\varepsilon(x_{k-1})} u_{\mathrm{I}}^\varepsilon + \eta 2^{-k} \right\} + \beta \fint_{B_\varepsilon(x_{k-1})} u_{\mathrm{I}}^\varepsilon \, dy + \eta 2^{-k} \\
&\qquad = u_{\mathrm{I}}^\varepsilon(x_{k-1}) + \eta 2^{-(k-1)}.
\end{aligned}
$$

Thus $M_k = u_{\mathrm{I}}^\varepsilon(x_k) + \eta 2^{-k}$ is a supermartingale. We get by Fatou's lemma and the optional stopping theorem that

$$
\begin{aligned}
u_{\mathrm{II}}^\varepsilon(x_0) &= \operatorname*{ínf}_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}}^{x_0}[F(x_\tau)] \leq \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}^0}^{x_0}[F(x_\tau) + \eta 2^{-\tau}] \\
&\leq \sup_{S_{\mathrm{I}}} \operatorname*{lím\,inf}_{k\to\infty} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}^0}^{x_0}[u_{\mathrm{I}}^\varepsilon(x_{\tau\wedge k}) + \eta 2^{-(\tau\wedge k)}] \\
&\leq \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}},S_{\mathrm{II}}^0}[u_{\mathrm{I}}^\varepsilon(x_0) + \eta] = u_{\mathrm{I}}^\varepsilon(x_0) + \eta.
\end{aligned}
$$

Similarly to the previous theorem, we also used the fact that the game ends almost surely. Since $\eta > 0$ is arbitrary, this completes the proof.

Theorems 4.1 and 4.2 imply that with a fixed boundary data there exists a unique p-harmonious function.

Theorem 4.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there exists a unique p-harmonious function in Ω with given boundary values F.*

Proof: Due to the dynamic programming principle, the value functions of the games are p-harmonious functions. This proves the existence part of Theorem 4.3. Theorems 4.1 and 4.2 imply the uniqueness part of Theorem 4.3.

Corollary 4.1 *The value of the game with pay-off function F coincides with the p-harmonious function with boundary values F.*

4.3. Maximum principles for p-harmonious functions

In this section, we show that the strong maximum and strong comparison principles hold for p-harmonious functions. First, let us state that p-harmonious functions satisfy the *strong maximum principle.*

Theorem 4.4 *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open, and connected set. If u_ε is p-harmonious in Ω with boundary values F, then $\sup_{\Gamma_\varepsilon} F \geq \sup_\Omega u_\varepsilon$. Moreover, if there is a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = \sup_{\Gamma_\varepsilon} F$, then u_ε is constant in Ω.*

Proof: The proof uses the fact that if the maximum is attained inside the domain then all the quantities in the definition of a p-harmonious function must be equal to the maximum. This is possible in a connected domain only if the function is constant.

We begin by observing that a p-harmonious function u_ε with a boundary data F satisfies

$$\sup_\Omega |u_\varepsilon| \leq \sup_{\Gamma_\varepsilon} |F| \, .$$

Assume then that there exists a point $x_0 \in \Omega$ such that

$$u_\varepsilon(x_0) = \sup_\Omega u_\varepsilon = \sup_{\Gamma_\varepsilon} F.$$

Then we employ the definition of a p-harmonious function, Definition 4.1, and obtain

$$u_\varepsilon(x_0) = \frac{\alpha}{2} \left\{ \sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy.$$

Since $u_\varepsilon(x_0)$ is the maximum, the terms

$$\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon, \quad \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon, \quad \text{and} \quad \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy$$

on the right hand side must be smaller than or equal to $u_\varepsilon(x_0)$. On the other hand, when $p > 2$, it follows that $\alpha, \beta > 0$ and thus the terms must equal to $u_\varepsilon(x_0)$. Therefore,

$$u_\varepsilon(x) = u_\varepsilon(x_0) = \sup_\Omega u_\varepsilon \tag{14}$$

for every $x \in B_\varepsilon(x_0)$ when $p > 2$. Now we can repeat the argument for each $x \in B_\varepsilon(x_0)$ and by continuing in this way, we can extend the result to the whole domain because Ω is connected. This implies that u is constant everywhere when $p > 2$.

Finally, if $p = 2$, then (14) holds for almost every $x \in B_\varepsilon(x_0)$ and consequently for almost every x in the whole domain. Then since

$$u(x) = \fint_{B_\varepsilon(x)} u \, dy$$

holds at every point in Ω and u is constant almost everywhere, it follows that u is constant everywhere.

In addition, p-harmonious functions with *continuous* boundary values satisfy the *strong comparison principle*. Note that the validity of the strong comparison principle is not known for the p-harmonic functions in $\mathbb{R}^N$, $N \geq 3$.

Theorem 4.5 *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open and connected set, and let u_ε and v_ε be p-harmonious functions with continuous boundary values $F_u \geq F_v$ in Γ_ε. Then if there exists a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = v_\varepsilon(x_0)$, it follows that $u_\varepsilon = v_\varepsilon$ in Ω, and, moreover, the boundary values satisfy $F_u = F_v$ in Γ_ε.*

Proof: The proof heavily uses the fact that $p < \infty$. Note that it is known that the strong comparison principle does not hold for infinity harmonic functions.

According to Corollary 4.1 and Theorem 4.1, $F_u \geq F_v$ implies $u_\varepsilon \geq v_\varepsilon$. By the definition of a p-harmonious function, we have

$$u_\varepsilon(x_0) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon\right\} + \beta \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy$$

and

$$v_\varepsilon(x_0) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon\right\} + \beta \fint_{B_\varepsilon(x_0)} v_\varepsilon \, dy.$$

Next we compare the right hand sides. Because $u_\varepsilon \geq v_\varepsilon$, it follows that

$$\begin{aligned} \sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon &\leq \sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon, \\ \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon &\leq \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon, \quad \text{and} \\ \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy &\leq \fint_{B_\varepsilon(x_0)} v_\varepsilon \, dy. \end{aligned} \tag{15}$$

Since $u_\varepsilon(x_0) = v_\varepsilon(x_0)$, we must have equalities in (15). In particular, we have equality in the third inequality in (15), and thus $u_\varepsilon = v_\varepsilon$ almost everywhere in $B_\varepsilon(x_0)$. Again, the connectedness of Ω immediately implies that $u_\varepsilon = v_\varepsilon$ almost everywhere in $\Omega \cup \Gamma_\varepsilon$. In particular, $F_u = F_v$ everywhere in Γ_ε since F_u and F_v are continuous. Because the boundary values coincide, the uniqueness result, Theorem 4.3, shows that $u_\varepsilon = v_\varepsilon$ everywhere in Ω.

4.4. Convergence as $\varepsilon \to 0$

In this section, we just state that p-harmonious functions with a fixed boundary datum converge to the unique p-harmonic function. The proof of this fact is rather technical and we refer to [26] for details.

Theorem 4.6 *Let Ω be a bounded smooth domain and F be a continuous function. Consider the unique viscosity solution u to*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u)(x) = 0, & x \in \Omega \\ u(x) = F(x), & x \in \partial\Omega, \end{cases} \tag{16}$$

and let u_ε be the unique p-harmonious function with boundary values F. Then $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$.

The above limit only depends on the values of F on $\partial\Omega$, and therefore any continuous extension of $F|_{\partial\Omega}$ to Γ_{ε_0} gives the same limit.

The proof of this result is based on a variant of the classical Arzela-Ascoli's compactness lemma, see below, Lemma 4.1. The functions u_ε are not continuous, in general, nonetheless, the jumps can be controlled and it can be show that the p-harmonious functions are asymptotically uniformly continuous in the precise sense stated below.

Lemma 4.1 *Let $\{u_\varepsilon : \overline{\Omega} \to \mathbb{R},\ \varepsilon > 0\}$ be a set of functions such that*

1. *there exists $C > 0$ so that $|u_\varepsilon(x)| < C$ for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$,*
2. *given $\eta > 0$ there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $x, y \in \overline{\Omega}$ with $|x - y| < r_0$ it holds $|u_\varepsilon(x) - u_\varepsilon(y)| < \eta$.*

Then, there exists a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ and a subsequence still denoted by $\{u_\varepsilon\}$ such that $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$, as $\varepsilon \to 0$.

Next we include a proof of the fact that the limit u is a solution to (16). The idea is to work in the viscosity setting and to show that the limit is a viscosity sub- and supersolution. To accomplish this, we utilize some ideas from [24], where p-harmonic functions were characterized in terms of asymptotic expansions. We start by recalling the viscosity characterization of p-harmonic functions, see [20].

Definition 4.2 *For $1 < p < \infty$ consider the equation $-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0$.*

1. *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from below with $\nabla\phi(x) \neq 0$, we have $-(p-2)\Delta_\infty \phi(x) - \Delta\phi(x) \geq 0$.*

2. *An upper semi-continuous function u is a subsolution if for every $\phi \in C^2$ such that ϕ touches u at $x \in \Omega$ strictly from above with $\nabla\phi(x) \neq 0$, we have $-(p-2)\Delta_\infty \phi(x) - \Delta\phi(x) \leq 0$.*

3. *Finally, u is a viscosity solution if it is both a sub- and supersolution.*

Theorem 4.7 *Let F and Ω be as in Theorem 4.6. Then the uniform limit u of p-harmonious functions $\{u_\varepsilon\}$ is a viscosity solution to* (16).

Proof: First, note that $u = F$ on $\partial\Omega$, and hence we can focus attention on showing that u is p-harmonic in Ω in the viscosity sense. To this end, we recall from [24] an estimate that involves the regular Laplacian ($p = 2$) and an approximation for the infinity Laplacian ($p = \infty$). Choose a point $x \in \Omega$ and a C^2-function ϕ defined in a neighborhood of x. Let x_1^ε be the point at which ϕ attains its minimum in $\overline{B}_\varepsilon(x)$, $\phi(x_1^\varepsilon) = \text{mín}_{y\in\overline{B}_\varepsilon(x)} \phi(y)$. It follows from the Taylor expansions in [24] that

$$\begin{aligned} &\frac{\alpha}{2}\left\{ \underset{y\in\overline{B}_\varepsilon(x)}{\text{máx}} \phi(y) + \underset{y\in\overline{B}_\varepsilon(x)}{\text{mín}} \phi(y) \right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\,dy - \phi(x) \\ &\geq \frac{\beta\varepsilon^2}{2(n+2)} \left((p-2)\left\langle D^2\phi(x)\left(\frac{x_1^\varepsilon - x}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x}{\varepsilon}\right)\right\rangle + \Delta\phi(x)\right) \\ &\quad + o(\varepsilon^2). \end{aligned} \tag{17}$$

Suppose that ϕ touches u at x strictly from below and that $\nabla\phi(x) \neq 0$. Observe that according to Definition 4.2, it is enough to test with such functions. By the uniform convergence, there exists sequence $\{x_\varepsilon\}$ converging to x such that $u_\varepsilon - \phi$ has an approximate minimum at x_ε, that is, for $\eta_\varepsilon > 0$, there exists x_ε such that $u_\varepsilon(x) - \phi(x) \geq u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) - \eta_\varepsilon$. Moreover, considering $\tilde{\phi} = \phi - u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon)$, we can assume that $\phi(x_\varepsilon) = u_\varepsilon(x_\varepsilon)$. Thus, by recalling the fact that u_ε is p-harmonious, we obtain

$$\eta_\varepsilon \geq -\phi(x_\varepsilon) + \frac{\alpha}{2}\left\{ \underset{\overline{B}_\varepsilon(x_\varepsilon)}{\text{máx}} \phi + \underset{\overline{B}_\varepsilon(x_\varepsilon)}{\text{mín}} \phi \right\} + \beta \fint_{B_\varepsilon(x_\varepsilon)} \phi(y)\,dy,$$

and thus, by (17), and choosing $\eta_\varepsilon = o(\varepsilon^2)$, we have

$$\begin{aligned} 0 \geq &\frac{\beta\varepsilon^2}{2(n+2)}((p-2)\left\langle D^2\phi(x_\varepsilon)\left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right), \left(\frac{x_1^\varepsilon - x_\varepsilon}{\varepsilon}\right)\right\rangle + \Delta\phi(x_\varepsilon)) \\ &+ o(\varepsilon^2). \end{aligned}$$

Since $\nabla\phi(x) \neq 0$, letting $\varepsilon \to 0$, we get

$$0 \geq \frac{\beta}{2(n+2)}\left((p-2)\Delta_\infty\phi(x) + \Delta\phi(x)\right).$$

Therefore u is a viscosity supersolution.

To prove that u is a viscosity subsolution, we use a reverse inequality to (17) by considering the maximum point of the test function and choose a function ϕ that touches u from above.

5. A mean value property that characterizes $p-$harmonic functions

Inspired by the analysis performed in the previous section we can guess a mean value formula for $p-$harmonic functions. In fact, we have proved that $p-$harmonious functions (that can be viewed as solutions to a mean value property) approximate $p-$harmonic functions (solutions to $\Delta_p u = 0$ as $\epsilon \to 0$, hence one may expect that $p-$harmonic functions verify the mean value formula given by the DPP but for a small error. It turns out that this intuitive fact can be proved rigorously, and moreover, it characterizes the fact of being a solution to $\Delta_p u = 0$.

A well known fact that one can find in any elementary PDE textbook states that u is harmonic in a domain $\Omega \subset \mathbb{R}^N$ (that is u satisfies $\Delta u = 0$ in Ω) if and only if it satisfies the mean value property

$$u(x) = \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\, dy,$$

whenever $B_\varepsilon(x) \subset \Omega$. In fact, we can relax this condition by requiring that it holds asymptotically

$$u(x) = \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\, dy + o(\varepsilon^2),$$

as $\varepsilon \to 0$. This follows easily for C^2 functions by using the Taylor expansion and for continuous functions by using the theory of viscosity solutions. Interestingly, a weak asymptotic mean value formula holds in some nonlinear cases as well. Our goal in this paper is to characterize p-harmonic functions, $1 < p \le \infty$, by means of this type of asymptotic mean value properties.

We begin by stating what we mean by weak asymptotic expansions and why is it reasonable to say that our asymptotic expansions hold in "a viscosity sense". As is the case in the theory of viscosity solutions, we test the expansions of a function u against test functions ϕ that touch u from below or above at a particular point.

Select α and β determined by the conditions $\alpha + \beta = 1$ and $\alpha/\beta = (p-2)/(N+2)$. That is, we have

$$\alpha = \frac{p-2}{p+N}, \quad \text{and} \quad \beta = \frac{2+N}{p+N}. \tag{18}$$

Observe that if $p = 2$ above, then $\alpha = 0$ and $\beta = 1$, and if $p = \infty$, then $\alpha = 1$ and $\beta = 0$.

As before we follow the usual convention to denote the mean value of a function

$$\fint_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy.$$

Definition 5.1 *A continuous function u satisfies*

$$u(x) = \frac{\alpha}{2}\left\{ \operatorname*{máx}_{\overline{B_\varepsilon(x)}} u + \operatorname*{mín}_{\overline{B_\varepsilon(x)}} u \right\} + \beta \fint_{B_\varepsilon(x)} u(y)\,dy + o(\varepsilon^2), \quad \textit{as } \varepsilon \to 0, \tag{19}$$

in the viscosity sense *if*

1. *for every $\phi \in C^2$ such that $u-\phi$ has a strict minimum at the point $x \in \overline{\Omega}$ with $u(x) = \phi(x)$, we have*

$$0 \geq -\phi(x) + \frac{\alpha}{2}\left\{ \operatorname*{máx}_{\overline{B_\varepsilon(x)}} \phi + \operatorname*{mín}_{\overline{B_\varepsilon(x)}} \phi \right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\,dy + o(\varepsilon^2).$$

2. *for every $\phi \in C^2$ such that $u-\phi$ has a strict maximum at the point $x \in \overline{\Omega}$ with $u(x) = \phi(x)$, we have*

$$0 \leq -\phi(x) + \frac{\alpha}{2}\left\{ \operatorname*{máx}_{\overline{B_\varepsilon(x)}} \phi + \operatorname*{mín}_{\overline{B_\varepsilon(x)}} \phi \right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\,dy + o(\varepsilon^2).$$

The following theorem states our main result and provides a characterization to the p-harmonic functions.

Theorem 5.1 *Let $1 < p \leq \infty$ and let u be a continuous function in a domain $\Omega \subset \mathbb{R}^N$. The asymptotic expansion*

$$u(x) = \frac{\alpha}{2}\left\{ \operatorname*{máx}_{\overline{B_\varepsilon(x)}} u + \operatorname*{mín}_{\overline{B_\varepsilon(x)}} u \right\} + \beta \fint_{B_\varepsilon(x)} u(y)\,dy + o(\varepsilon^2), \quad \textit{as } \varepsilon \to 0,$$

holds for all $x \in \Omega$ in the viscosity sense if and only if $\Delta_p u(x) = 0$ in the viscosity sense. Here α and β are determined by (18)*.*

We use the notation $\Delta_\infty u = |\nabla u|^{-2}\left\langle D^2u\,\nabla u, \nabla u\right\rangle$ for the 1-homogeneous infinity Laplacian.

We observe that the notions of a viscosity solution and a Sobolev weak solution for the p-Laplace equation agree for $1 < p < \infty$, see Juutinen-Lindqvist-Manfredi [20]. Therefore, Theorem 5.1 characterizes weak solutions when $1 < p < \infty$.

5.1. A heuristic argument

We have that u is a solution to $\Delta_p u = 0$ if and only if

$$(p-2)\Delta_\infty u + \Delta u = 0, \tag{20}$$

because this equivalence can be justified in the viscosity sense even when $\nabla u = 0$ as shown in [20]. Averaging the classical Taylor expansion

$$u(y) = u(x) + \nabla u(x)\cdot(y-x) + \frac{1}{2}\langle D^2u(x)(y-x),(y-x)\rangle + O(|y-x|^3),$$

over $B_\varepsilon(x)$, we obtain

$$u(x) - \fint_{B_\varepsilon(x)} u\,dy = -\frac{\varepsilon^2}{2(n+2)}\Delta u(x) + O(\varepsilon^3), \tag{21}$$

when u is smooth. Here we used the shorthand notation

$$\fint_{B_\varepsilon(x)} u\,dy = \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u\,dy.$$

Then observe that gradient direction is almost the maximizing direction. Thus, summing up the two Taylor expansions roughly gives us

$$\begin{aligned}
& u(x) - \frac{1}{2}\left\{\sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u\right\} \\
&\approx u(x) - \frac{1}{2}\left\{u\left(x+\varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right) + u\left(x-\varepsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right)\right\} \\
&= -\frac{\varepsilon^2}{2}\Delta_\infty u(x) + O(\varepsilon^3).
\end{aligned} \tag{22}$$

Next we multiply (21) and (22) by suitable constants α and β, $\alpha+\beta=1$, and add up the formulas to obtain

$$\begin{aligned}
& u(x) - \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x)} u - \inf_{\overline{B}_\varepsilon(x)} u\right\} + \beta\fint_{B_\varepsilon(x)} u\,dy \\
&= -\alpha\frac{\varepsilon^2}{2}\Delta_\infty u(x) - \beta\frac{\varepsilon^2}{2(n+2)}\Delta u(x) + O(\varepsilon^3)
\end{aligned}$$

Next, we choose α and β so that we have the operator in (20) on the right hand side. This process gives us the choices of the constants

$$\alpha = \frac{p-2}{p+N}, \quad \text{and} \quad \beta = \frac{2+N}{p+N}. \tag{23}$$

and we deduce

$$u(x) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u\right\} + \beta\fint_{B_\varepsilon(x)} u\,dy + O(\varepsilon^3)$$

as $\varepsilon \to 0$.

5.2. Proof of Theorem 5.1

The main idea of the proof of Theorem 5.1 is just to work in the viscosity setting. We start by recalling the viscosity characterization of p-harmonic functions for $p < \infty$, see [20].

Definition 5.2 *For $1 < p < \infty$ consider the equation $-div\left(|\nabla u|^{p-2}\nabla u\right) = 0$.*

1. *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2$ such that $u - \phi$ has a strict minimum at the point $x \in \Omega$ with $\nabla\phi(x) \neq 0$ we have*
$$-(p-2)\Delta_\infty\phi(x) - \Delta\phi(x) \geq 0.$$

2. *An upper semi-continuous function u is a subsolution if for every $\phi \in C^2$ such that $u - \phi$ has a strict maximum at the point $x \in \Omega$ with $\nabla\phi(x) \neq 0$ we have*
$$-(p-2)\Delta_\infty\phi(x) - \Delta\phi(x) \leq 0.$$

3. *Finally, u is a viscosity solution if it is both a supersolution and a subsolution.*

For the case $p = \infty$ we must restrict the class of test functions as in [31]. Let $S(x)$ denote the class of C^2 functions ϕ such that either $\nabla\phi(x) \neq 0$ or $\nabla\phi(x) = 0$ and the limit
$$\lim_{y\to x} \frac{2(\phi(y) - \phi(x))}{|y-x|^2} = \Delta_\infty\phi(x)$$
exists.

Definition 5.3 *Consider the equation $-\Delta_\infty u = 0$.*

1. *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in S(x)$ such that $u-\phi$ has a strict minimum at the point $x \in \Omega$ we have $-\Delta_\infty\phi(x) \geq 0$.*

2. *An upper semi-continuous function u is a subsolution if for every $\phi \in S(x)$ such that $u - \phi$ has a strict maximum at the point $x \in \Omega$ we have $-\Delta_\infty\phi(x) \leq 0$.*

3. *Finally, u is a viscosity solution if it is both a supersolution and a subsolution.*

Proof: We first consider asymptotic expansions for smooth functions that involve the infinity Laplacian ($p = \infty$) and the regular Laplacian ($p = 2$).

Choose a point $x \in \Omega$ and a C^2-function ϕ defined in a neighborhood of x. Let x_1^ε and x_2^ε be the point at which ϕ attains its minimum and maximum in $\overline{B_\varepsilon(x)}$ respectively; that is,
$$\phi(x_1^\varepsilon) = \min_{y\in\overline{B_\varepsilon(x)}} \phi(y) \quad \text{and} \quad \phi(x_2^\varepsilon) = \max_{y\in\overline{B_\varepsilon(x)}} \phi(y).$$

Next, we use some ideas from [9]. Consider the Taylor expansion of the second order of ϕ

$$\phi(y) = \phi(x) + \nabla\phi(x)\cdot(y-x) + \frac{1}{2}\langle D^2\phi(x)(y-x),(y-x)\rangle + o(|y-x|^2)$$

as $|y-x|\to 0$. Evaluating this Taylor expansion of ϕ at the point x with $y = x_1^\varepsilon$ and $y = 2x - x_1^\varepsilon = \tilde{x}_1^\varepsilon$, we get

$$\phi(x_1^\varepsilon) = \phi(x) + \nabla\phi(x)(x_1^\varepsilon - x) + \frac{1}{2}\langle D^2\phi(x)(x_1^\varepsilon - x),(x_1^\varepsilon - x)\rangle + o(\varepsilon^2)$$

and

$$\phi(\tilde{x}_1^\varepsilon) = \phi(x) - \nabla\phi(x)(x_1^\varepsilon - x) + \frac{1}{2}\langle D^2\phi(x)(x_1^\varepsilon - x),(x_1^\varepsilon - x)\rangle + o(\varepsilon^2)$$

as $\varepsilon\to 0$. Adding the expressions, we obtain

$$\phi(\tilde{x}_1^\varepsilon) + \phi(x_1^\varepsilon) - 2\phi(x) = \langle D^2\phi(x)(x_1^\varepsilon - x),(x_1^\varepsilon - x)\rangle + o(\varepsilon^2).$$

Since x_1^ε is the point where the minimum of ϕ is attained, it follows that

$$\phi(\tilde{x}_1^\varepsilon) + \phi(x_1^\varepsilon) - 2\phi(x) \le \max_{y\in \overline{B_\varepsilon(x)}}\phi(y) + \min_{y\in \overline{B_\varepsilon(x)}}\phi(y) - 2\phi(x),$$

and thus

$$\frac{1}{2}\left\{\max_{y\in \overline{B_\varepsilon(x)}}\phi(y) + \min_{y\in \overline{B_\varepsilon(x)}}\phi(y)\right\} - \phi(x) \ge \frac{1}{2}\langle D^2\phi(x)(x_1^\varepsilon - x),(x_1^\varepsilon - x)\rangle + o(\varepsilon^2). \tag{24}$$

Repeating the same process at the point x_2^ε we get instead

$$\frac{1}{2}\left\{\max_{y\in \overline{B_\varepsilon(x)}}\phi(y) + \min_{y\in \overline{B_\varepsilon(x)}}\phi(y)\right\} - \phi(x) \le \frac{1}{2}\langle D^2\phi(x)(x_2^\varepsilon - x),(x_2^\varepsilon - x)\rangle + o(\varepsilon^2). \tag{25}$$

Next we derive a counterpart for the expansion with the usual Laplacian ($p=2$). Averaging both sides of the classical Taylor expansion of ϕ at x we get

$$⨍_{B_\varepsilon(x)}\phi(y)\,dy = \phi(x) + \sum_{i,j=1}^{N}\frac{\partial^2\phi}{\partial x_i^2}(x)⨍_{B_\varepsilon(0)}\frac{1}{2}z_iz_j\,dz + o(\varepsilon^2).$$

The values of the integrals in the sum above are zero when $i\neq j$. Using symmetry, we compute

$$\begin{aligned}⨍_{B_\varepsilon(0)} z_i^2\,dz &= \frac{1}{N}⨍_{B_\varepsilon(0)}|z|^2\,dz\\ &= \frac{1}{N\omega_N\varepsilon^N}\int_0^\varepsilon\int_{\partial B_\rho}\rho^2\,dS\,d\rho = \frac{\sigma_{N-1}\varepsilon^2}{N(N+2)\omega_N} = \frac{\varepsilon^2}{(N+2)}.\end{aligned}$$

We end up with

$$\fint_{B_\varepsilon(x)} \phi(y)\, dy - \phi(x) = \frac{\varepsilon^2}{2(N+2)} \Delta\phi(x) + o(\varepsilon^2). \tag{26}$$

Assume for the moment that $p \geq 2$ so that $\alpha \geq 0$. Multiply (24) by α and (26) by β and add. We arrive at the expansion valid for any smooth function ϕ:

$$\begin{aligned}
&\frac{\alpha}{2} \left\{ \max_{y \in \overline{B_\varepsilon(x)}} \phi(y) + \min_{y \in \overline{B_\varepsilon(x)}} \phi(y) \right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\, dy - \phi(x) \\
&\geq \frac{\beta \varepsilon^2}{2(N+2)} \left((p-2) \left\langle D^2\phi(x) \left(\frac{x_1^\varepsilon - x}{\varepsilon} \right), \left(\frac{x_1^\varepsilon - x}{\varepsilon} \right) \right\rangle + \Delta\phi(x) \right) \\
&\quad + o(\varepsilon^2).
\end{aligned} \tag{27}$$

We remark that $x_1^\varepsilon \in \partial B_\varepsilon(x)$ for $\varepsilon > 0$ small enough whenever $\nabla\phi(x) \neq 0$. In fact, suppose, on the contrary, that there exists a subsequence $x_1^{\varepsilon_j} \in B_{\varepsilon_j}(x)$ of minimum points of ϕ. Then, $\nabla\phi(x_1^{\varepsilon_j}) = 0$ and, since $x_1^{\varepsilon_j} \to x$ as $\varepsilon_j \to 0$, we have by continuity that $\nabla\phi(x) = 0$. A simple argument based on Lagrange multipliers then shows that

$$\lim_{\varepsilon \to 0} \frac{x_1^\varepsilon - x}{\varepsilon} = -\frac{\nabla\phi}{|\nabla\phi|}(x). \tag{28}$$

We are ready to prove that if the asymptotic mean value formula holds for u, then u is a viscosity solution. Suppose that function u satisfies the asymptotic expansion in the viscosity sense according to Definition 5.1. Consider a smooth ϕ such that $u - \phi$ has a strict minimum at x and $\phi \in S(x)$ if $p = \infty$. We obtain

$$0 \geq -\phi(x) + \frac{\alpha}{2} \left\{ \max_{\overline{B_\varepsilon(x)}} \phi + \min_{\overline{B_\varepsilon(x)}} \phi \right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\, dy + o(\varepsilon^2),$$

and thus, by (27),

$$0 \geq \frac{\beta \varepsilon^2}{2(N+2)} \left((p-2) \left\langle D^2\phi(x) \left(\frac{x_1^\varepsilon - x}{\varepsilon} \right), \left(\frac{x_1^\varepsilon - x}{\varepsilon} \right) \right\rangle + \Delta\phi(x) \right) + o(\varepsilon^2).$$

If $\nabla\phi(x) \neq 0$ we take limits as $\varepsilon \to 0$. Taking into consideration (28) we get

$$0 \geq \frac{\beta}{2(N+2)} \left((p-2)\Delta_\infty \phi(x) + \Delta\phi(x) \right).$$

Suppose now that $p = \infty$ and that the limit

$$\lim_{y \to x} \frac{\phi(y) - \phi(x)}{|y - x|^2} = L$$

exists. We need to deduce that $L \leq 0$ from

$$0 \geq \frac{1}{2} \left\{ \max_{\overline{B_\varepsilon(x)}} \phi + \min_{\overline{B_\varepsilon(x)}} \phi \right\} - \phi(x).$$

Let us argue by contradiction. Suppose that $L > 0$ and choose $\eta > 0$ small enough so that $L - \eta > 0$. Use the limit condition to obtain the inequalities

$$(L-\eta)|x-y|^2 \leq \phi(x) - \phi(y) \leq (L+\eta)|x-y|^2,$$

for small $|x-y|$. Therefore, we get

$$\begin{aligned} 0 \geq\ & \frac{1}{2} \max_{B_\varepsilon(x)} (\phi - \phi(x)) + \frac{1}{2} \min_{B_\varepsilon(x)} (\phi - \phi(x)) \\ \geq\ & \frac{1}{2} \max_{B_\varepsilon(x)} (\phi - \phi(x)) \geq (\frac{L-\eta}{2})\varepsilon^2, \end{aligned}$$

which is a contradiction. Thus, we have proved that $L \geq 0$.

To prove that u is a viscosity subsolution, we first derive a reverse inequality to (27) by considering the maximum point of the test function, that is, using (25) and (26), and then choose a function ϕ that touches u from above. We omit the details.

To prove the converse implication, assume that u is a viscosity solution. In particular u is a subsolution. Let ϕ be a smooth test function such that $u - \phi$ has a strict local maximum at $x \in \Omega$. If $p = \infty$, we also assume $\phi \in S(x)$. If $\nabla\phi(x) \neq 0$, we get

$$-(p-2)\Delta_\infty \phi(x) - \Delta\phi(x) \leq 0. \tag{29}$$

The statement to be proven is

$$\liminf_{\varepsilon \to 0+} \frac{1}{\varepsilon^2} \left(-\phi(x) + \frac{\alpha}{2} \left\{ \max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi \right\} + \beta ⨍_{B_\varepsilon(x)} \phi(y)\, dy \right) \geq 0.$$

This again follows from (27). Indeed, divide (27) by ε^2, use (28), and deduce from (29) that the limit on the right hand side is bounded from below by zero.

For the case $p = \infty$ with $\nabla\phi(x) = 0$ we assume the existence of the limit

$$\lim_{y \to x} \frac{\phi(y) - \phi(x)}{|y-x|^2} = L \geq 0$$

and observe that

$$\liminf_{\varepsilon \to 0+} \frac{1}{\varepsilon^2} \left(-\phi(x) + \frac{1}{2} \left\{ \max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi \right\} \right) \geq 0.$$

The argument for the case of supersolutions is analogous.

Finally, we need to address the case $1 < p < 2$. Since $\alpha \leq 0$ we use (25) instead of (24) to get a version of (27) with x_2^ε in place of x_1^ε. The argument then continues in the same way as before.

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